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The p-Laplace Eigenvalue Problem as $p \to 1$ and Cheeger Sets in a Finsler Metric

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We consider the p–Laplacian operator on a domain equipped with a Finsler metric. After deriving and recalling relevant properties of its first eigenfunction for $p > 1$, we investigate the limit problem as $p \to 1$.

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1. Introduction

Imagine a nonlinear elastic membrane, fixed on a boundary $\partial\Omega$ of a plane domain Ω. If $u(x)$ denotes its vertical displacement, and if its deformation energy is given by $\int_\Omega |\nabla u|^p \, dx$, then a minimizer of the Rayleigh quotient

$$\frac{\int_\Omega |\nabla u|^p \ dx}{\int_\Omega |u|^p \ dx}$$

on $W_0^{1,p}(\Omega)$ satisfies the Euler-Lagrange equation

$$-\Delta_p u = \lambda_p \ |u|^{p-2} u \quad \text{in } \Omega, \tag{1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the well-known p–Laplace operator. This eigenvalue problem has been extensively studied in the literature. As $p \to 1$, formally the limit equation reads

$$\begin{aligned} &-\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = \lambda_1(\Omega) && \text{in } \Omega, \\ &u = 0 && \text{on } \partial\Omega. \end{aligned} \tag{2}$$

For a precise interpretation of (2) see [23] or [33]. Naturally, here $\lambda_1(\Omega) := \lim_{p\to 1+} \lambda_p(\Omega)$. A somewhat surprising recent result is that the family of eigenfunctions $\{u_p\}$ converges

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in $L^1(\Omega)$ cum grano salis to (a multiple of) the characteristic function χ_{C_Ω} of a subset C_Ω of Ω, a so called Cheeger-set, see [21]. A Cheeger set of Ω is characterized as a domain that minimizes

$$h(\Omega) := \inf_D \frac{|\partial D|}{|D|}$$

with D varying over all smooth subdomains of Ω whose boundary ∂D does not touch $\partial\Omega$, and with $|\partial D|$ and $|D|$ denoting $(n-1)$- and n-dimensional Lebesgue measure of ∂D and D. The existence, uniqueness, regularity and construction of such sets is discussed in [21] and [22] (and partly in [34]) and its continuous dependence on Ω in [18]. The paper [26] contains a numerical method for the calculation of n-dimensional Cheeger sets and some three-dimensional examples. Cheeger sets are of significant importance in the modelling of landslides, see [19], [20], or in fracture mechanics, see [24]. Notice that a set $D \subseteq \Omega$ is a Cheeger set if and only if it is a minimizer of

$$|\partial E| - h(\Omega)|E| \quad \text{for } E \subseteq \Omega. \tag{3}$$

Now suppose that the membrane is not isotropic. It is for instance woven out of elastic strings like a piece of material. Then the deformation energy can be anisotropic, see [5]. Another way to describe this effect is by stating that the Euclidean distance in Ω is somehow distorted. It is the purpose of the present paper to generalize the above result on eigenfunctions and their convergence as $p \to 1$ to the situation, where $\Omega \subset \mathbb{R}^n$ is no longer equippped with the Euclidean norm, but instead with a general norm ϕ. In that case a Lipschitz continuous function $u : \Omega \mapsto \mathbb{R}$ (in a convex domain Ω) has Lipschitz constant $L = \sup_{z\in\Omega} \phi^*(\nabla u(z))$, where ϕ^* denotes the dual norm to ϕ. Therefore the Rayleigh quotient studied in this paper is given by

$$R_p(u) := \frac{\int_\Omega (\phi^*(\nabla u))^p \, dx}{\int_\Omega |u|^p \, dx} \tag{4}$$

on $W_0^{1,p}(\Omega)$ and the Cheeger constant by

$$h(\Omega) := \inf_{D\subset\Omega} \frac{P_\phi(D)}{|D|}, \tag{5}$$

with P_ϕ denoting anisotropic perimeter in $\mathbb{R}^n$ (see (10) below). The minimizer u_p of R_p satisfies the Euler-Lagrange inclusion

$$-Q_p u := -\mathrm{div}\left((\phi^*(\nabla u))^{p-2} J(\nabla u)\right) \ni \lambda_p |u|^{p-2} u \quad \text{in } \Omega \tag{6}$$

in the weak sense [8], i.e.

$$\int_\Omega (\phi^*(\nabla u_p))^{p-2} \langle \eta, \nabla v\rangle \, dx = \lambda_p \int_\Omega |u_p|^{p-2} u_p \cdot v \, dx \tag{7}$$

for any $v \in W_0^{1,p}(\Omega)$ and for a measurable selection $\eta \in J(\nabla u_p)$, where the function $J : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is defined as the subdifferential

$$J(\xi) := \partial \left(\frac{\phi^*(\xi)^2}{2} \right). \tag{8}$$

Note that the function J is single-valued iff the norm ϕ is strictly convex, i.e. if its unit sphere $\{x : \phi(x) = 1\}$ contains no nontrivial line segments [39, p. 400]. Note further that $J(0) = 0$ and that for the Euclidean norm the duality map reduces to the identity $J(\nabla u) = \nabla u$.

The paper is organized as follows. In Section 2 we fix some notation. In Section 3 we recall and derive the existence, uniqueness, regularity and log-concavity of solutions for $p > 1$. In Section 4 we derive the limit equation for $p \to 1$. In Section 5, we discuss in detail the two-dimensional case, proving uniqueness of Cheeger sets in the convex case. In Section 6 we provide some instructive examples.

2. Notation

We say that the norm ϕ is *regular* if $\phi^2, (\phi^*)^2 \in C^2(\mathbb{R}^n)$. This includes for instance $\phi(x) = \|x\|_q$ with $q \in (1,\infty)$ but excludes the crystalline cases $q = 1$ or $q = \infty$, see Section 6.

Given $E \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, we set

$$\mathrm{dist}_\phi(x, E) := \inf_{y\in E} \phi(x - y), \qquad d^E_\phi(x) := \mathrm{dist}_\phi(x, E) - \mathrm{dist}_\phi(\mathbb{R}^n \setminus E, x).$$

$d^E_\phi(x)$ indicates the signed distance of x to ∂E and is positive outside E. Notice that, at each point where d^E_ϕ is differentiable, there holds

$$\phi^*(\nabla d^E_\phi) = 1. \tag{9}$$

Let us define the (anisotropic) perimeter of E as

$$P_\phi(E) := \sup\left\{\int_E \mathrm{div}\eta \, dx \mid \eta \in C^1_c(\mathbb{R}^n),\ \phi(\eta) \le 1\right\} = \int_{\partial^* E} \phi^*(\nu^E) d\mathcal{H}^{n-1}, \tag{10}$$

where $\partial^* E$ and ν^E denote the reduced boundary of E and the (Euclidean) unit normal to $\partial^* E$.

Given an open set $\Omega \subseteq \mathbb{R}^n$ we define the BV-seminorm of $v \in BV(\Omega)$ as

$$\int_\Omega \phi^*(Dv) := \sup\left\{\int_\Omega v \,\mathrm{div}\eta \, dx \ \mid \eta \in C^1_c(\mathbb{R}^n),\ \phi(\eta) \le 1\right\}.$$

Given $\delta > 0$, we define

$$\begin{aligned} E^\delta_+ &:= \left\{x \in \mathbb{R}^n \mid d^E_\phi < \delta\right\} = E + \delta W_\phi, \\ E^\delta_- &:= \left\{x \in \mathbb{R}^n \mid d^E_\phi < -\delta\right\}, \\ E^\delta_\pm &:= \left(E^\delta_-\right)^\delta_+ \subseteq E, \end{aligned}$$

where $W_\phi := \{x \mid \phi(x) < 1\}$, also called *Wulff shape*, denotes the unit ball with respect to the norm ϕ.

Given a compact set $E \subset \mathbb{R}^n$ with Lipschitz boundary, we denote by $n_\phi : \partial E \to \mathbb{R}^n$ any Lipschitz vector field satisfying $n_\phi \in J(\nabla d_\phi^E)$ a.e. on ∂E. Moreover, we set

$$\|\kappa_\phi\|_{L^\infty(\partial E)} := \inf_{n_\phi \in J(\nabla d_\phi^E)} \|\mathrm{div}_\tau n_\phi\|_{L^\infty(\partial E)},$$

which represents the L^∞-norm of the ϕ-mean curvature of ∂E. Here div_τ denotes the tangential divergence operator. We make the convention that $\|\kappa_\phi\|_{L^\infty(\partial E)} = +\infty$ if the set E does not admit any Lipschitz vector field $n_\phi \in J(\nabla d_\phi^E)$. We say that E is ϕ-regular if $\|\kappa_\phi\|_{L^\infty(\partial E)} < +\infty$.

Notice that in the Euclidean case E is ϕ-regular iff ∂E is of class $C^{1,1}$. Moreover, the unit ball W_ϕ is always ϕ-regular and $\|\kappa_\phi\|_{L^\infty(\partial W_\phi)} = n-1$. To see this, it is enough to consider the vector field $n_\phi(x) = x/\phi(x)$.

3. Existence, uniqueness, regularity and log-concavity of solutions

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. If we minimize the functional

$$I_p(v) = \int_\Omega \phi^*(\nabla v)^p \; dx \quad \text{on } K := \{\ v \in W_0^{1,p}(\Omega);\ \|v\|_{L^p(\Omega)} = 1\ \}, \tag{11}$$

then via standard arguments (see [6]) a minimizer u_p exists for every $p > 1$ and it is a weak solution to the equation (6), with $\lambda_p = I_p(u_p)$. Note that $\Lambda_p := I_p(u_p)^{1/p}$ is the minimum of the Rayleigh quotient

$$R_p(v) := \frac{\left(\int_\Omega \ (\phi^*(\nabla v))^p \ dx\right)^{1/p}}{\|v\|_p} \tag{12}$$

on $W_0^{1,p}(\Omega) \setminus \{0\}$. Without loss of generality we may assume that u_p is nonnegative. Otherwise we can replace it by its modulus.

Moreover, as shown in [6] any nonnegative weak solution of (6) is necessarily bounded and positive in Ω. If $p > n$, then u_p is also uniformly Hölder continuous because of the Sobolev-embedding theorem and the equivalence of the usual Sobolev norm with

$$\|u\|_{1,p} := \left(\int_\Omega \ |u|^p \ dx\right)^{1/p} + \left(\int_\Omega \ (\phi^*(\nabla u))^p \ dx\right)^{1/p}. \tag{13}$$

If the norm ϕ is regular and $p > 1$, one can even show that $u_p \in C^{1,\alpha}(\Omega)$. Indeed, the function u_p minimizes

$$J_p(v) := \int_\Omega (\phi^*(\nabla v))^p - \lambda_p(\Omega)|u|^p \; dx,$$

and the theory for quasiminima in [17] implies that minimizers are bounded (Thm. 7.5), Hölder continuous (Thm. 7.16) and satisfy a strong maximum principle (Thm. 7.12), because one can easily check that u_p satisfies (7.71) in [17]. Therefore u_p is positive. Once positivity is known, the uniqueness follows from a simple convexity argument, see [4] or [6]. Moreover, from the result in [12] one can conclude that $u_p \in C^{0,\beta}(\Omega)$ for any $\beta \in (0,1)$. Finally, if ϕ is regular, then $u_p \in C^{1,\alpha}(\Omega)$ according to [7], [28], [37], [38] or [13]. Let us summarize these statements.

Theorem 3.1. *For every $p \in (1,\infty)$ the nonnegative minimizer u_p of (11) is positive, unique, belongs to $C^{0,\beta}(\Omega)$ for any $\beta \in (0,1)$ and it solves (6) in the weak sense. Moreover, if the norm ϕ is regular then u_p is of class $C^{1,\alpha}(\Omega)$ for some $\alpha \in (0,1)$. Finally, if Ω is convex, then u_p is log-concave and the level sets set $\{u_p > t\} \subseteq \Omega$ are convex for all $t > 0$.*

Proof. To prove the last statement, we follow Sakaguchi's approach from [31], first for strictly convex Ω and for a smooth norm ϕ. The general case follows then from approximation arguments for Ω and ϕ. Log-concavity of a sequence $u_{p,n}$ is preserved under pointwise limits as $n \to \infty$, because the inequality

$$\log u_{p,n}\left(\frac{x_1+x_2}{2}\right) \geq \frac{1}{2}\log u_{p,n}(x_1) + \frac{1}{2}\log u_{p,n}(x_2) \quad \text{in } \Omega\times\Omega$$

is stable under such limits. If u_p solves (6), then $v_p := \log u_p$ solves

$$-\text{div}\left((\phi^*(\nabla v))^{p-2}\, J(\nabla v)\right) = (p-1)\phi^*(\nabla v)^p + \lambda_p \quad \text{in } \Omega \tag{14}$$

and this degenerate elliptic equation can be approximated by a nondegenerate one

$$\begin{aligned} &-\text{div}\left(\left(\varepsilon + (\phi^*(\nabla v))^2\right)^{\frac{p-2}{2}} J(\nabla v)\right) \\ &= (p-1-\varepsilon)(\phi^*(\nabla v))^2(\varepsilon + (\phi^*(\nabla v))^2)^{\frac{p-2}{2}} + \lambda_p. \end{aligned} \tag{15}$$

Modulo yet another approximation by a right hand side which is strictly monotone in v, equation (15) is now amenable to Korevaar's concavity maximum principle ([25] or [31]) which states that the concavity function

$$C(x_1,x_2) := v\left(\frac{x_1+x_2}{2}\right) - \frac{1}{2}v(x_1) - \frac{1}{2}v(x_2) \quad \text{with } x_1,x_2 \in \Omega\times\Omega$$

can attain a negative minimum only on the boundary of $\Omega\times\Omega$. The latter is ruled out, however, because of the boundary condition. Thus C is nonnegative in $\Omega\times\Omega$, that is v must be concave in Ω. The interested reader is referred to [31] for more details. □

Remark 3.2. We should point out that without uniqueness of u_p the approximation arguments would only yield log-concavity of *a* solution and not *the* solution u_p.

4. The limit problem for $p \to 1$

The following estimate for λ_p is optimal (as $p \to 1$) for any shape of Ω (see [6]).

Theorem 4.1 (Convergence of eigenvalues). *For every $p \in (1,\infty)$ the eigenvalue $\lambda_p(\Omega)$ can be estimated from below as follows:*

$$\lambda_p(\Omega) \geq \left(\frac{h(\Omega)}{p}\right)^p. \tag{16}$$

Here $h(\Omega)$ is the Cheeger constant of Ω as defined in (5). Moreover, as $p \to 1$, the eigenvalue $\lambda_p(\Omega)$ converges to $\lambda_1(\Omega) = h(\Omega)$.

Proof. In the Euclidean case this is Cheeger's original estimate [11] when $p = 2$, and for general p it can be found in [27], [2], [29] and [36]. For a more general ϕ one can easily modify their proofs by using the generalized coarea formula from [15] or [16]. To prove the limiting behaviour of $\lambda_p(\Omega)$ as $p \to 1$ we proceed as in [21] and observe that (16) implies $\liminf_{p\to 1} \lambda_p(\Omega) \geq h(\Omega)$. Therefore it suffices to find a suitable upper bound. Let $\{D_k\}_{k=1,2,...}$ be a sequence of regular domains for which $P_\phi(D_k)/|D_k|$ converges to $h(\Omega)$. We approximate the characteristic function of each D_k by a function w_k with the following properties: $w \equiv 1$ on $\overline{D_k}$, $w \equiv 0$ outside an ε–neighborhood of D_k and $\phi^*(\nabla w_k) = 1/\varepsilon$ in an ε–layer outside D_k. For small ε the function w_k is in $W_0^{1,\infty}(\Omega)$ and provides the upper bound

$$\lambda_p(\Omega) \leq \frac{P_\phi(D_k)}{|D_k|}\, \varepsilon^{1-p} \,. \tag{17}$$

Now one sends first $p \to 1$, then $k \to \infty$ to complete the proof. □

Theorem 4.2 (Convergence of eigenfunctions). *As $p \to 1$, the eigenfunction u_p converges, up to a subsequence, to a limit function $u_1 \in BV(\Omega)$, with $u_1 \geq 0$ and $\|u_1\|_1 = 1$. Moreover, almost all level sets $\Omega_t := \{u_1 > t\}$ of u_1 are Cheeger sets.*

Proof. For every $p > 1$ the function u_p minimizes

$$J_p(v) := \int_\Omega (\phi^*(\nabla v))^p - \lambda_p(\Omega)|v|^p \, dx$$

on $W_0^{1,p}(\Omega)$. If one extends J_p to $L^1(\Omega)$ by setting it $+\infty$ on $L^1(\Omega) \setminus W_0^{1,p}(\Omega)$, the family J_p Γ-converges (see [14]) with respect to the $L^1(\Omega)$-topology to

$$J_1(v) := \begin{cases} \int_\Omega \phi^*(Dv) - h(\Omega)\int_\Omega |v|\, dx & v \in BV(\Omega), \\ +\infty & v \in L^1(\Omega) \setminus BV(\Omega). \end{cases}$$

Indeed, since J_1 is lower semicontinuous on $L^1(\Omega)$, it is enough to prove the Γ-limsup inequality on the subset $C^1(\overline{\Omega}) \subset L^1(\Omega)$ (which is dense both in topology and in energy), where it becomes trivial.

Let us now prove the Γ-liminf inequality. Notice that, if $v_{p_n} \to u$ in $L^1(\Omega)$, then either there exists a subsequence $v_{p_{n_k}}$ which is equibounded in $BV(\Omega)$, or $J_{p_n}(v_{p_n})$ goes to $+\infty$. If $v_k := v_{p_{n_k}}$ is bounded in $BV(\Omega)$, letting $p_k := p_{n_k}$ and $J_k := J_{p_{n_k}}$, we have

$$\begin{aligned} J_1(v_k) &= \int_\Omega \phi^*(\nabla v_k) - h(\Omega)|v_k|\, dx \\ &\leq \left[\int_\Omega (\phi^*(\nabla v_k))^{p_k}\; dx\right]^{\frac{1}{p_k}} |\Omega|^{\frac{p_k-1}{p_k}} - h(\Omega)\int_\Omega |v_k|\, dx \\ &\leq \frac{1}{p_k}\int_\Omega (\phi^*(\nabla v_k))^{p_k}\; dx + \frac{p_k-1}{p_k}|\Omega| - h(\Omega)\int_\Omega |v_k|\, dx \\ &\qquad +\lambda_{p_k}(\Omega)\int_\Omega |v_k|^{p_k}\, dx - \lambda_{p_k}(\Omega)\int_\Omega |v_k|^{p_k}\, dx \\ &\leq J_k(v_k) + \frac{p_k-1}{p_k}|\Omega| + \lambda_{p_k}(\Omega)\int_\Omega |v_k|^{p_k}\, dx - h(\Omega)\int_\Omega |v_k|\, dx \\ &= J_k(v_k) + \omega_k \,, \end{aligned} \tag{18}$$

where $\lim_{k\to\infty} \omega_k = 0$. It then follows

$$J_1(u) \le \liminf_{k\to\infty} J_1(v_k) \le \liminf_{k\to\infty} J_k(v_k).$$

Since $J_p \ge 0$ on $W_0^{1,p}(\Omega)$, we get $J_1 \ge 0$ on $BV(\Omega)$. Moreover u_p forms a minimizing sequence for J_1 since, from the last inequality in (18), we have

$$\int_\Omega \phi^*(\nabla u_p)\, dx \le \frac{p-1}{p}|\Omega| + \lambda_p(\Omega),$$

where we have used the fact that $J_p(u_p) = 0$ and $\|u_p\|_p = 1$. As a consequence, the family $\{u_p\}_{p>1}$ is bounded in $BV(\Omega)$ and, after possibly passing to a subsequence, it converges strongly in $L^1(\Omega)$ to a limit function $u_1 \in BV(\Omega)$ such that $J_1(u_1) = 0$, $u_1 \ge 0$ and $\|u_1\|_1 = 1$. Using the coarea formula as in [21, Eq. (7)] (adapted to the anisotropic setting), one can see that for all $t \in [0, \max_\Omega u_1)$ the level set $\Omega_t := \{u_1 > t\}$ is a Cheeger set. □

Remark 4.3. As a consequence of Theorem 4.2 and the logconcavity of u_p, for every convex set Ω (Theorem 3.1) there exists a convex Cheeger set. Moreover, it follows from the results of [10] that there exists a convex Cheeger set $D \subseteq \Omega$ which is maximal, in the sense that any other Cheeger set of Ω must be contained in D. Recently it was shown in [9] that for convex Ω the Cheeger set is uniquely determined. The uniqueness of Cheeger sets is in general not true for nonconvex domains (see [22]).

5. The planar case

In this section we derive further properties of the function u_1, under the additional assumption $n = 2$. Let us begin with the following theorem, which extends the analogous result in the Euclidean case [22, Thm. 1].

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open convex set. Then, there exists a unique Cheeger set $D \subseteq \Omega$. Moreover, D is convex and we have*

$$h(\Omega) = \frac{1}{t^*}, \qquad D = \Omega_\pm^{t^*}, \tag{19}$$

where $t^* > 0$ is the (unique) value t such that $|\Omega_-^t| = t^2|W_\phi|$.

Remark 5.2. In other words, the theorem states that the Cheeger set is the union of all the Wulff shapes of radius t^* with center in $\Omega_-^{t^*}$, a level set of the distance function to $\partial\Omega$, that was called inner Cheeger set in [22].

Proof. Let D be a Cheeger set of Ω. Notice first that D is a convex set, since otherwise we could replace it by its convex hull and reduce (3) (see [3, Thm. 7.1]). Moreover, from the first variation of (3) it follows that the anisotropic curvature of ∂D is bounded by $h(\Omega)$, and each connected component of $\partial D \cap \Omega$ is contained up to translation in $\frac{1}{h(\Omega)}\partial W_\phi$ (see [30, Thm. 4.5]). Let $\widetilde{D}$ be the open maximal Cheeger set of Ω (recall Remark 4.3), and let $\Gamma \subset \frac{1}{h(\Omega)}\partial W_\phi$ be a connected component of $\partial D \cap \widetilde{D}$. We denote by $x, y \in \Gamma \cap \partial\widetilde{D}$ the extremal points of Γ, and we let Γ' be the arc of $\partial\widetilde{D}$ with extrema x, y and lying

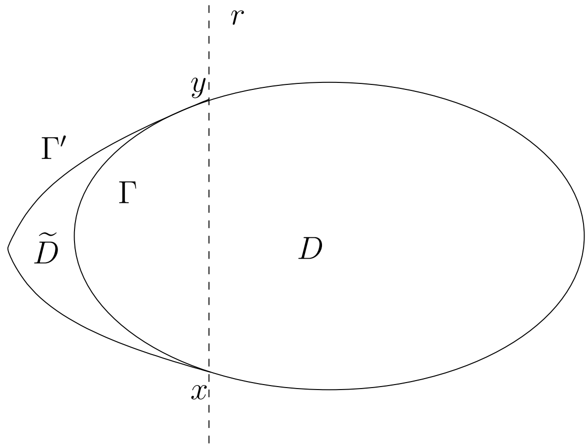

Figure 5.1: The Cheeger sets D, $\widetilde{D}$ of Theorem 5.1.

in the same halfplane of Γ with respect to the straight line r passing through x, y (see Figure 5.1). Reasoning as in [3, Lemma 7.3], it is easy to show that both Γ and Γ' can be written as graphs on r along some directions. More precisely, there exists a vector $v \in \mathbb{R}^2$, with $|v| = 1$, and two functions $f_1, f_2 : r \to \mathbb{R}$ such that $0 \le f_1 \le f_2$ on $[x, y]$, that $\min\{f_2(x), f_2(y)\} = 0$, and that $\Gamma = F_1([x, y])$ and $\Gamma' = F_2([x, y])$, with $F_i(x) := f_i(x)v$, for $i = 1, 2$. Without loss of generality, we shall assume that $v \perp r$. Since D and $\widetilde{D}$ are both minimizers of (3), it follows that both f_1 and f_2 are minimizers of

$$G(f) := \int_{[x,y]} \phi^*(-f'(s), 1) - h(\Omega) f(s)\, ds\,. \tag{20}$$

If ϕ is a regular norm, then the functional G is strictly convex, which implies $f_1 = f_2$, i.e. $D = \widetilde{D}$. For a general norm, one has to be more careful, since the functional G is not strictly convex, but only convex. However, reasoning as in [3, Lemma 8.2], the inclusion $\Gamma \subset \frac{1}{h(\Omega)} \partial W_\phi$ and the inequality $f_1 \le f_2$ imply $\|\kappa_\phi\|_{L^\infty(\Gamma')} \ge h(\Omega)$, with equality iff $\Gamma = \Gamma'$, which proves the uniqueness of the Cheeger set D.

Let us now prove (19), reasoning as in [22, Thm. 1]. It has been proved in [3] that the convex set $D = \Omega_{\pm}^{1/h(\Omega)}$ is a Cheeger set of Ω, hence it is the unique Cheeger set of Ω. Therefore, it remains to prove that $t^* = 1/h(\Omega)$, i.e.

$$\left|\Omega_-^{\frac{1}{h(\Omega)}}\right| = \frac{|W_\phi|}{h(\Omega)^2}.$$

Let us recall from [1, Section 2.7],[32] the following Steiner-type formulae

$$\begin{aligned} |C^\delta| &= |C| + \delta P_\phi(C) + \delta^2 |W_\phi|\,, \\ P_\phi(C^\delta) &= P_\phi(C) + \delta P_\phi(W_\phi)\,. \end{aligned} \tag{21}$$

Incidentally, the second equation follows from the first one and, as in the Euclidean case, $P_\phi(W_\phi) = 2|W_\phi|$. This follows from integrating div$x$ on W_ϕ. Applying (21) to $C = D_-^{1/h(\Omega)}$ and recalling that $h(\Omega) = P_\phi(D)/|D|$, we get

$$|D_-^{1/h(\Omega)}| = \frac{|W_\phi|}{h(\Omega)^2}.$$

The claim now follows if we observe that

$$\Omega_-^{\frac{1}{h(\Omega)}} = D_-^{\frac{1}{h(\Omega)}}.$$

□

Corollary 5.3. *If $n = 2$ and Ω is a bounded convex set, then the sequence of functions u_p converges to a multiple of the characteristic function of D. Moreover, $D = \Omega$ if and only if*

$$\|\kappa_\phi\|_{L^\infty(\partial\Omega)} \le h(\Omega). \tag{22}$$

In particular, (22) *always holds in the case $\Omega = W_\phi$.*

6. Example and concluding remarks

If the norm under consideration for $x \in \Omega$ is the usual l_q- norm, i.e. for $\phi_q(x) = (\sum_{i=1}^n |x_i|^q)^{1/q}$, $q \ge 1$. When $q > 1$, the dual norm of ϕ_q is given by $\phi_q^* = \phi_{q'}$, with $q' = q/(q-1)$, and the duality map according to (8) is

$$J_i(y) = (|y|_{q'})^{2-q'} |y_i|^{q'-2} y_i.$$

Then the p-Laplace operator in this metric is given by (see [6])

$$Q_{p,q}u = \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\phi_{q'}(\nabla u)^{p-q'} \left|\frac{\partial u}{\partial x_i}\right|^{q'-2} \frac{\partial u}{\partial x_i}\right),$$

and for $q = 2 = q'$ the norm $\phi_{q'}$ is just the Euclidean norm and $Q_{p,q}$ reduces to the well-known p-Laplace Operator

$$Q_{p,q}u = \Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) .$$

For general q and $p \to 1$ the operator $Q_{1,q}$ is formally given by

$$Q_{1,q}u = \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\left[\frac{|u_{x_i}|}{\phi_{q'}(\nabla u)}\right]^{q'-2} \frac{u_{x_i}}{\phi_{q'}(\nabla u)}\right).$$

Again for $q = 2 = q'$ this expression shrinks down to the customary

$$Q_{1,2}u = \Delta_1 u = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right).$$

We complete this section with the construction of a particular Cheeger set for a nonregular anisotropy. Let us fix $n = 2$ and consider the norm $\phi = \phi_1$. Notice that in this case the Wulff Shape W_ϕ has the shape of a rhombus. To be precise, it is a square of sidelength $\sqrt{2}$, centered in the origin and rotated by $\pi/2$ with respect to the coordinate axes. Moreover, the dual norm ϕ^* is given by $\phi^*(y) = \max\{|y_1|, |y_2|\}$. To better illustrate the results of Section 5, let us compute the Cheeger set (and Cheeger constant) of a square Q of sidelength 1 (see Figure 6.1).

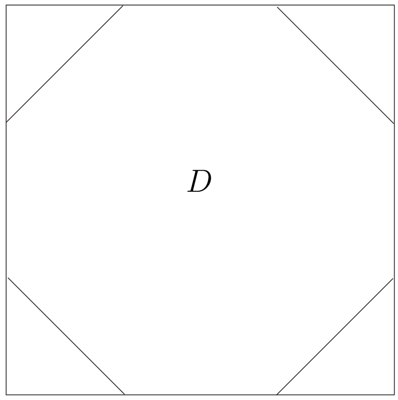

Figure 6.1: The Cheeger set of a square with respect to the norm ϕ_1.

Since in this case $|W_\phi| = 2$ and Q^t_- is a square of sidelength $1 - 2t$, from Theorem 5.1 we get $t^* = 1 - \sqrt{2}/2$ and $h(Q) = 2 + \sqrt{2}$. It is interesting to note that the Cheeger set of Q is a regular octahedron (and the inner Cheeger set $Q^{t^*}_-$ defined in Remark 5.2 is a square). After this manuscript was submitted for publication, we learned that G. Strang independently discusses this example in [35].

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