Journal of Convex Analysis
Volume 15 (2008), No. 2, 201–214

Ordered Non-Convex Quasi-Variational Sweeping Processes

N. V. Chemetov
CMAF and Departamento de Matemática, Faculdade de Ciências da Universidade de Lisboa, Av. Prof. Gama Pinto 2, 1649-003 Lisboa, Portugal
chemetov@ptmat.fc.ul.pt

M. D. P. Monteiro Marques
CMAF and Departamento de Matemática, Faculdade de Ciências da Universidade de Lisboa, Av. Prof. Gama Pinto 2, 1649-003 Lisboa, Portugal
mmarques@ptmat.fc.ul.pt

U. Stefanelli
Istituto di Matematica Applicata e Tecnologie Informatiche – CNR, via Ferrata 1, 27100 Pavia, Italy
ulisse.stefanelli@imati.cnr.it

Received: August 21, 2006
Revised manuscript received: May 01, 2007

This paper addresses the Cauchy problem for the quasi-variational sweeping process in the ordered Hilbert space H

$$-u'(t) \in N_{C(t,u(t))}(u(t)) \quad \text{for a.e. } t \in (0,T),\ \ u(0) = u_0,$$

where the set $C(t,u(t)) \subset H$ is non-convex and $N_{C(t,u(t))}$ denotes its normal cone. We provide an existence result based on the classical implicit time-discretization procedure and on a fixed point argument in ordered spaces.

Keywords: Sweeping process, non-convex sets, orders in Hilbert spaces

1991 Mathematics Subject Classification: 34A60, 34G25, 47J20

1. Introduction

Let H be a separable Hilbert space and $T > 0$ be a reference time. Moreover, let $C : [0,T] \to 2^H$ be a set-valued mapping with non-empty, $\varphi-$convex and closed values for all times $t \in [0,T]$. A precise definition of the class of non-convex sets called φ-convex sets will be given in Section 2. We consider the evolution of a point $u(t) \in H$ remaining in the set $C(t)$ for all time $t \in [0,T]$ and being swept in a normal direction as it touches the boundary of the set. Hence we shall find $u : [0,T] \to H$ such that

$$-u'(t) \in N_{C(t)}(u(t)) \quad \text{for a.e. } t \in (0,T),\ u(0) = u_0. \tag{1}$$

In the above relation, the prime stands for differentiation with respect to time, while $N_{C(t)}(u(t))$ denotes the *normal cone* to the set $C(t)$ at the point $u(t)$ (see Section 2) and $u_0 \in C(0)$ is an initial datum. This differential inclusion is called a sweeping process.

The sweeping process (1) arises in many applications ranging from non-smooth mechanics to mathematical economics, optimization, free boundary models, etc. Problem (1) was

ISSN 0944-6532 / \$ 2.50

introduced and studied extensively in the seventies by Moreau (e.g. in [24] and [25]) and by many other authors thereafter. The set-valued map $t \mapsto C(t)$ was first assumed to be Lipschitz continuous or to have bounded variation and the sets $C(t)$ were convex. The subject has developed in different directions. For instance, the continuity requirements on the set-valued map have been weakened in case the sets have non-empty interior ([6], [20] are two examples) and the non-convex situation, pioneered by Valadier [32] has been much developed ever since.

In the present article we address instead a quasi-variational version of (1), that is the case of a set-valued driving function which depends on the solution u as well. In particular, we assume to be given $C : [0,T] \times H \to 2^H$ with non-empty and $\varphi-$convex values and a point $u_0 \in C(0, u_0)$. We look for a solution to the quasi-variational problem

$$-u'(t) \in N_{C(t,u(t))}(u(t)) \quad \text{for a.e. } t \in (0,T),\ u(0) = u_0. \tag{2}$$

This type of problems appears in quasi-statical evolution problems with friction, micromechanical damage theory and shape memory models.

Let us now comment on the previous contributions on quasi-variational sweeping processes. For convex sets, Kunze & Monteiro Marques [17] proved the existence of solutions when the moving set C depends in a Lipschitz continuous way on the time t and the state u. The Lipschitz continuity constant of C with respect to the dependence on u is asked to be strictly less than 1. In order to solve (2) in a Hilbert space H, some compactness assumption on C is required in [17], [18]. A second existence result for the above-referred Lipschitz continuous case has been provided by Brokate, Krejči, & Schnabel [3] by replacing the compactness assumptions with a non-empty interior condition along with extra continuity properties. In this case, uniqueness of a solution is also achieved.

A third class of existence results has been recently obtained by replacing compactness and Lipschitz continuity by means of ordering-type assumptions and exploiting fixed point devices in ordered spaces. With this approach the existence of suitably weak solutions to the quasi-variational sweeping processes (2) in the convex case has been obtained by Stefanelli [30, 29] and Rossi & Stefanelli [28].

For non-convex quasi-variational sweeping processes (2), an existence result was given by Chemetov & Monteiro Marques in [7], assuming that $C(t, u)$, $t \in [0, T]$ are φ-convex sets and H is either a finite or an infinite-dimensional space. In particular, in the infinite-dimensional situation, a compactness assumption has to be made.

In the present paper, we focus on the situation of a quasi-variational sweeping process in the setting of an ordered Hilbert space H. In particular, we do not assume compactness but instead require some monotonicity along with some continuity for the moving convex sets. By suitably exploiting the techniques developed in [7], [17] and [29], we provide a new existence result for the φ-convex quasi-variational sweeping process (2).

2. Notations and preliminaries

Sets. We shall denote by H a separable Hilbert space endowed with the scalar product $(\cdot, \cdot)$ and by $|\cdot|$ the corresponding norm. For all non-empty sets $A, B \subset H$ and $x \in H$ we let $d(x, A) := \inf_{a\in A} |x - a|$ be the usual distance and $e(A, B) := \sup_{a\in A} d(a, B)$ denote the Hausdorff or excess semi-distance between the sets A, B. Given a final reference time

$T > 0$, a non-empty set-valued function $C : [0,T] \to 2^H$ and $[s,t] \subset [0,T]$, we shall define the *retraction* $\mathrm{ret}(C; s,t)$ of C on $[s,t]$ as

$$\mathrm{ret}(C; s,t) := \sup_p \left\{ \sum_{i=1}^N e(C(t_{i-1}), C(t_i)) \text{ for } p = \{s = t_0 < t_1 < \cdots < t_{N-1} < t_N = t\} \right\},$$

on all partitions.

We finally let $C([0,T]; H)$, $W^{1,\infty}(0,T;H)$, $L^1(0,T;H)$ and $BV(0,T;H)$ denote the space of continuous, Lipschitz continuous, integrable and bounded variation functions of $[0,T]$ with values in H, respectively.

Normals. Let $C \subset H$ be a non-empty and closed set and let $z \in H$. We denote by $\pi_C(z) \subset H$ the metric projection of z into C, defined as

$$\pi_C(z) := \{x \in C \,:\, |x - z| = \inf_{y \in C} |y - z|\}.$$

The latter could of course be empty (see below). It is however non-empty in case C is weakly closed and it is a singleton if C is convex. Given $x \in C$, an element $v \in H$ is said to be a *proximal normal* to C at x if it is of the form $y - x$ with $x \in \pi_C(y)$. The set of all proximal normals to C at x is denoted by $N_C(x)$ and is a convex cone which may not be closed. It is easy to show that if v is a proximal normal to C at x then there exists $\sigma > 0$ such that

$$(v, y - x) \le \sigma |y - x|^2 \quad \forall y \in C.$$

This is actually a characterization since, owing to [8, Prop. 1.5],

$$N_C(x) \equiv \bigcup_{\sigma > 0} \{v \in H \,:\, (v, y - x) \le \sigma |y - x|^2 \quad \forall y \in C\}.$$

$\varphi-$**convexity.** We shall follow Colombo & Goncharov [9] for notation and terminology. The non-empty and closed set $C \subset H$ is said to be $\varphi-$*convex* if there exists a continuous function $\varphi : C \to [0, +\infty)$ such that, for all $x, y \in C$ and $v \in N_C(x)$, one has that

$$(v, y - x) \le \varphi(x)\, |v|\, |y - x|^2.$$

The name $\varphi-$convexity is borrowed from Marino & Tosques [19] and the reader is referred to the pioneering paper by Degiovanni, Marino, & Tosques [10] and the contributions [5, 8, 26] as well (see also [9] for some additional material and a review of results on $\varphi-$convexity of sets). The very same notion is also commonly referred to as *prox-regularity* see, e.g., Rockafellar & Wets [27]. From the geometric point of view, $\varphi-$convexity entails the *external ball property*. Namely, an external tangent ball, with radius strictly less then $1/(2\varphi(x))$ can be rolled around C by touching the set just in x. Examples of $\varphi-$convex sets are of course convex sets and $C^{1,1}$ smooth sets. More generally, whenever for all $x \in C$ there exists a ball B centered in x with either $C \cap B$ convex or $C^{1,1}$ smooth, then the set C is $\varphi-$convex. We will term the $\varphi-$convex set C to be φ_0-*convex* iff the function φ can be chosen to be the constant $\varphi_0 > 0$. In the latter case, let us observe that the metric projection π_C is non-empty, single-valued and Lipschitz continuous with constant 2 whenever restricted to the set $\{u \in H : d(x, C) \le 1/(4\varphi_0)\}$. We shall stress that π_C could be empty outside some suitable neighborhood of C (see [9, Ex. 7.3]).

Orders and fixed point tool. This material follows the discussion by Baiocchi & Capelo [1]. Let $(E, \leq)$ denote a non-empty ordered set and $F \subset E$. We recall that $f \in F$ is a *maximal (minimal) element* of F if, for all $f' \in F$, $f \leq f'$ ($f' \leq f$, respectively) implies $f = f'$. Then, f is the *maximum (minimum)* of F if $f' \leq f$ ($f \leq f'$, respectively) for all $f' \in F$. Moreover, $e \in E$ is an *upper bound (lower bound)* of F if $f \leq e$ ($e \leq f$, respectively) for all $f \in F$ and $e \in E$ is the *supremum* or *least upper bound* (*infimum* or *greatest lower bound*) if e is the minimum (maximum) of the set of upper bounds (lower bounds, respectively) of F. Moreover, we say that F is a *chain* if it is totally ordered and that F is an *interval* if there exist $e_*, e^* \in E$ such that $F \equiv \{e \in E : e_* \leq e \leq e^*\}$. In the latter case we use the notation $F = [e_*, e^*]$. The set $(E, \leq)$ is said to be *completely s-inductive (completely i-inductive)* if every chain of E has a supremum (infimum, respectively). Finally $(E, \leq)$ is said to be *completely inductive* if it is both completely s-inductive and completely i-inductive. We are now in the position of introducing our fixed point device.

Lemma 2.1. *Let $(E, \leq)$ be an ordered set and $I : [u_*, u^*] \subset E$ be completely inductive. Suppose that $S : I \to I$ is non-decreasing. Then, the set $\{u \in I : u = S(u)\}$ is non-empty and has a minimum and a maximum.*

The latter result was announced by Kolodner [16] and turns out to be the main tool in the analysis of Mignot & Puel [21, 22] and Tartar [31]. Its proof is to be found, for instance, in [1, Thm. 9.26, p. 223].

Orders in Hilbert spaces. Assume we are given a non-empty closed, and convex cone $P \subset H$ with $P \cap -P = \{0\}$ and define $u \leq v$ iff $v - u \in P$. The latter is an order relation [27, Prop. 3.38, p. 95] and we shall interpret P as the cone of positive elements. By defining the *polar cone*

$$P^* := \{u \in H \,:\, (u, v) \leq 0 \ \forall v \in P\},$$

we possibly obtain, for all $u \in H$, a (unique) decomposition [23]

$$u = u_1 + u_2 \quad \text{where } u_1 \in P,\ u_2 \in P^* \text{ and } (u_1, u_2) = 0.$$

Indeed the latter elements u_1 and u_2 are exactly the corresponding projections. Owing to these considerations we will use the notation $u_1 = u^+ = \pi_P(u)$ and $u_2 = -u^- = \pi_{P^*}(u)$. These notations are particularly well motivated in the special case of $P^* = -P$ (which entails indeed the closure and strictness of P) where indeed $u^- = \pi_P(-u)$. Moreover, we will use the following notation

$$u \vee v := v + (u - v)^+, \quad u \wedge v := u - (u - v)^+.$$

In the particular case $P^* = -P$ one of course has that $u \vee v = u + (v - u)^+$ and $u \wedge v = v - (v - u)^+$ while this is not true in general. Let us stress that the symbols $\wedge$ and $\vee$ are chosen just for the sake of notational simplicity. Indeed, we are not claiming that one is able to find, for all $u, v \in H$, the element $\inf\{u, v\}$ or $\sup\{u, v\}$ although, whenever they exist, they coincide with $u \wedge v$ and $u \vee v$, respectively. Let us collect here for the reader's convenience some examples of cones P such that $P^* = -P$.

Example 2.2. Our first example for $P^* = -P$ is the n-dimensional non-negative *orthant* $P := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0 \text{ for } i = 1, \ldots, n\}$.

Example 2.3. Let Ω be a measure space, μ be a positive measure on Ω and denote by $L^2(\Omega, \mu)$ the Hilbert space of all square $\mu-$integrable functions on Ω endowed with the standard inner product. Hence we define $P := \{u \in L^2(\Omega, \mu) : u \geq 0 \ \mu\text{−a.e. in } \Omega\}$ and relation $\leq$ turns out to be

$$u \leq v \quad \text{iff } u(x) \leq v(x) \text{ for } \mu\text{−a.e. } x \text{ in } \Omega.$$

Example 2.4. Let H be the space of symmetric $n \times n$ real matrices endowed with the standard contraction product $(A, B) := \operatorname{tr}(AB)$ for all $A, B \in H$, where tr stands for the trace. We define P as the set of positive semidefinite matrices. Again it is a standard matter to check that $-P = P^*$ [15, Cor. 7.5.4, p. 459]. Of course, relation $\leq$ entails that

$$A \leq B \quad \text{iff } B - A \text{ is positive semidefinite.}$$

Example 2.5. We now consider the case of the so-called *second order cone*. Given the space H let us consider the convex cone in $\mathbb{R} \times H$ defined by $P := \{(t, u) \in \mathbb{R} \times H : t \geq |u|\}$. One easily checks that $-P = P^*$ in $\mathbb{R} \times H$. This example in particular shows that there exist cones with $-P = P^*$ in $\mathbb{R}^n$ that are not isometric to the non-negative orthant of Example 2.2. Moreover, the relation $\leq$ reduces in this case to

$$(t, u) \geq (s, v) \quad \text{iff } t - s \geq |u - v|.$$

Example 2.6. Let u_n denote a countable orthonormal basis for the separable Hilbert space H. We denote by P the range of the mapping $u \mapsto \sum_{n \in \mathbb{N}} (u, u_n)^+ u_n$. Namely, P is the set of linear combinations of u_n with non-negative coefficients (*conic combination*). Hence, it is straightforward to check that $-P = P^*$ (cf. Example 2.2).

Before moving on, let us report here a slight generalization of [1, Thm. 19.12, p. 399].

Lemma 2.7. *Let $P \subset H$ be closed and such that $-P \subset P^*$. Then any non-empty interval $[u_*, u^*] \subset H$ is completely inductive.*

Proof. Let us denote by u_α with $\alpha \in A$ a totally ordered subset in $[u_*, u^*]$ and check for complete s-inductiveness (the proof for complete i-inductiveness being completely analogous). We assume without loss of generality and without introducing new notation that the set of indices $(A, \leq)$ is given in such a way that

$$\alpha \leq \beta \text{ in } A \ \Rightarrow \ u_\alpha \leq u_\beta \text{ in } H.$$

By using $-P \subset P^*$, we easily check that the sequence $(u_\alpha, u^* - u_*)$ is non-decreasing as α increases and it is bounded from above. Hence it converges. On the other hand, for all $\alpha, \beta \in A$ such that $\alpha \leq \beta$ we deduce that

$$|u_\alpha - u_\beta|^2 = (u_\beta - u_\alpha, u_\beta - u_\alpha) \leq (u_\beta - u_\alpha, u^* - u_*).$$

Namely, u_α is a Cauchy sequence in H as α increases. Finally, owing to the closure of P, we conclude that $\sup_{\alpha \in A} u_\alpha \in [u_*, u^*]$ as well. $\square$

We now follow for instance [1, 11] and introduce on the set of non-empty subsets of H the relation $\preceq$ as

$$C_1 \preceq C_2 \quad \text{iff} \ \Big(u_1 \in C_1,\ u_2 \in C_2 \ \Rightarrow\ u_1 \wedge u_2 \in C_1,\ u_1 \vee u_2 \in C_2\Big).$$

Here notation is motivated by the fact that relation $\preceq$ turns out to be an order on the non-empty closed intervals of H. However, $\preceq$ is not an order in general, even when restricted to non-empty, convex and closed sets.

Given the ordered Hilbert space (H, P) we will also consider the corresponding pointwise order for functions with values in H without changing symbols. Namely, for all $u, v : [0, T] \to H$, we shall let

$$u \le v \quad \text{iff } v(t) - u(t) \in P \ \forall t \in [0, T].$$

3. The variational problem

Let us start our discussion on (1) by stating our assumptions on the data C and u_0.

(A1) H is a separable Hilbert space.

(A2) $C : [0, T] \to 2^H$ has non-empty, φ_0-convex and closed values. Moreover, there exists $\lambda > 0$ such that, for all $[s, t] \subset [0, T]$ one has that

$$e(C(s), C(t)) \le \lambda(t - s). \tag{3}$$

(A3) $u_0 \in C(0)$.

Condition (3) plays the role of a Lipschitz continuity requirement. In particular, the function $r(t) := \mathrm{ret}(C, 0, t)$ turns out to be Lipschitz continuous. This assumption can be somehow relaxed (see below). However, we prefer to stick to the present Lipschitz continuous situation for the sake of simplicity. Finally, assumption (A3) makes of course sense since $C(0)$ is non-empty by (A2).

The main result of this section states the existence of strong solutions to (1) in the above setting. Namely, we have the following.

Theorem 3.1. *Under assumptions* (A1)-(A3) *there exists* $u \in W^{1,\infty}(0, T; H)$ *fulfilling* (1). *Moreover* $u(t) \in C(t)$ *for all* $t \in [0, T]$, $|u'(t)| \le \lambda$ *for almost every* $t \in (0, T)$, *and, for all solutions* $u_1, u_2 \in W^{1,\infty}(0, T; H)$ *to* (1), *one has that*

$$|u_1(t) - u_2(t)| \le |u_1(0) - u_2(0)| e^{4\varphi_0 \lambda t} \quad \forall t \in [0, T]. \tag{4}$$

In particular, the solution to (1) *is unique.*

We prove the latter theorem in the remainder of this section by means of a classical time discretization procedure. Before moving one, one shall mention the papers by Bounkhel & Thibault [2] and Edmond & Thibault [13] were some strictly related existence result has been proved. In particular, the above-mentioned papers deal with the even more general situation of a *perturbed* sweeping process

$$-u'(t) \in N_{C(t)}(u(t)) + F(t, u(t)) \quad \text{for a.e. } t \in (0, T),\ u(0) = u_0$$

where $F : [0, T] \times H \to H$ may be multivalued and fulfills suitable boundedness and compactness requirements. On the other hand, the continuity assumtpion on C is slightly different. Indeed, they ask for C to be absolutely continuous with respect to the Hausdorff distance whereas here we impose a one-sided Lipschitz estimate on the excess only (see (3)).

The catching-up algorithm. This scheme can be traced back at least to Moreau [24, 25]. For all $n \in \mathbb{N}$ let us prescribe a uniform partition of $[0, T]$ by means of the nodes $t_n^i = iT/2^n$, for $i = 0, \dots, 2^n$. Of course the forthcoming proof does not rely on the fact that the partition is dyadic nor uniform with diameter $h_n := T/2^n$ and one is entitled to transpose with no particular intricacy the whole argument to non-uniform partitions as well. This could be of some interest from the numerical point of view. We however reduce ourselves to the current choice for the sake of notational simplicity. Finally, let I_n^i denote the subinterval $(t_n^{i-1}, t_n^i]$ for $i = 1, \dots, 2^n$.

Let now $u_n^0 = u_0 \in C(0)$. We shall be constructing an approximating solution by successively projecting on the moving closed set C. To this aim, we readily assume to be given n sufficiently large in such a way that

$$2\lambda h_n = \lambda T 2^{-n+1} \leq 1/(4\varphi_0).$$

Owing to (3), in the latter case, given some suitable $u_n^{i-1} \in C(t_n^{i-1})$, there exists a unique projection

$$u_n^i = \pi_{C(t_n^i)}(u_n^{i-1}) \quad \forall i = 1, \dots, 2^n.$$

This of course stems from the fact that

$$d(u_n^{i-1}, C(t_n^i)) \leq e(C(t_n^{i-1}), C(t_n^i)) \leq \lambda h_n \leq 1/(4\varphi_0).$$

We shall denote by $u_n : [0, T] \to H$ the piecewise linear interpolant of the above defined discrete solution. Namely, we let

$$u_n(0) = u_n^0, \quad u_n(t) = u_n^{i-1} + \frac{t - t_n^{i-1}}{h_n}(u_n^i - u_n^{i-i}) \quad \forall t \in I_n^i, \ i = 1, \dots, 2^n.$$

Moreover, we let $\tau_n(t) = t_n^i$, $s_n(t) = t_n^{i-1}$ for $t \in I_n^i$, $i = 1, \dots, 2^n$. Hence we have that $u_n(\tau_n(t)) \in C(\tau_n(t))$ for all $t \in [0, T]$ and

$$-u_n'(t) \in N_{C(\tau_n(t))}(u_n(\tau_n(t))) \quad \text{for a.e. } t \in (0, T). \tag{5}$$

Moreover, we readily prove that

$$|u_n^i - u_n^{i-1}| = d(u_n^{i-1}, C(t_n^i)) \leq e(C(t_n^{i-1}), C(t_n^i)) \leq \lambda h_n, \tag{6}$$

so that $u_n \in W^{1,\infty}(0, T; H)$ and $|u_n'(t)| \leq \lambda$ for almost every $t \in (0, T)$.

Convergence. We shall now prove that u_n is a Cauchy sequence in $C([0, T]; H)$. Indeed, from (5) and for almost every $t \in (0, T)$ we have

$$d(u_n(t), C(t)) \leq |u_n(t) - u_n(s_n(t))| + e(C(s_n(t)), C(t)) \leq 2\lambda h_n, \tag{7}$$

$$(u_n'(t), u_n(\tau_n(t)) - v) \leq \varphi_0 |u_n'(t)| \, |u_n(\tau_n(t)) - v|^2 \quad \forall v \in C(\tau_n(t)). \tag{8}$$

Moreover, notice that for all $m \geq n$,

$$\begin{aligned} & d(u_m(t), C(\tau_n(t))) \\ & \leq |u_m(t) - u_m(\tau_m(t))| + e(C(\tau_m(t)), C(\tau_n(t))) \leq \lambda h_m + \lambda h_n \leq 2\lambda h_n \leq 1/(4\varphi_0), \end{aligned}$$

so that the projection $v_m(t) = \pi_{C(\tau_n(t))}(u_m(t))$ is well defined and for all $t \in [0, T]$

$$|u_m(t) - v_m(t)| \leq 2\lambda h_n.$$

Therefore, inequality (8) gives

$$(u_n'(t), u_n(t) - u_m(t)) \leq 2\varphi_0\lambda|u_n(t) - u_m(t)|^2 + 18\varphi_0\lambda^3 h_n^2 + 3\lambda^2 h_n \text{ for a.e } t \in (0, T). \quad (9)$$

Similarly, $v_n(t) = \pi_{C(\tau_m(t))}(u_n(t))$ exists and for all $t \in [0, T]$

$$\begin{aligned} |u_n(t) - v_n(t)| = d(u_n(t), C(\tau_m(t))) &\leq |u_n(t) - u_n(s_n(t))| + d(u_n(s_n(t)), C(\tau_m(t))) \\ &\leq \lambda h_n + e(C(s_n(t)), C(\tau_m(t))) \leq 2\lambda h_n. \end{aligned}$$

In particular, by exploiting (8) at level m, one obtains that

$$(u_m'(t), u_m(t) - u_n(t)) \leq 2\varphi_0\lambda|u_n(t) - u_m(t)|^2 + 18\varphi_0\lambda^3 h_n^2 + 3\lambda^2 h_n \quad \text{for a.e } t \in (0, T). \quad (10)$$

Hence, taking the integral on $(0, t)$ for $t \in (0, T]$ of the sum of (9) and (10), we deduce by Gronwall's lemma that

$$|u_n(t) - u_m(t)|^2 \leq \left(18\varphi_0\lambda^3 h_n^2 + 3\lambda^2 h_n\right)(e^{8\varphi_0\lambda t} - 1)/2\phi_0\lambda \quad \forall t \in [0, T]. \quad (11)$$

Finally, there exists a (not relabeled) subsequence u_n such that

$$u_n \to u \quad \text{strongly in } C([0, T]; H) \text{ and weakly star in } W^{1,\infty}(0, T; H),$$

and by (6) we readily check that the limit u fulfills $|u'(t)| \leq \lambda$ for almost every $t \in (0, T)$. We shall prove that actually u solves (1). Let us start by observing that $u(0) = u^0$ and, for all $t \in [0, T]$,

$$\begin{aligned} d(u(t), C(t)) &\leq \liminf_{n \to +\infty} \Big(|u(t) - u_n(s_n(t))| + e(C(s_n(t)), C(t)) \Big) \\ &\leq \liminf_{n \to +\infty} \Big(|u(t) - u_n(s_n(t))| + \lambda(t - s_n(t)) \Big) = 0. \end{aligned}$$

Namely $u(t) \in C(t)$ for all $t \in [0, T]$.

Notice that the proof shows that there exists a continuous selection of C (namely $u = u(t)$), such that $u(0) = u^0$. Clearly, we can conclude that for any $t_0 \in [0, T]$ and any $v^0 \in C(t_0)$ there is a continuous selection $v = v(t)$ on some interval $[t_0, t_0 + \delta]$ with $v(t_0) = v^0$.

Next, for almost every fixed $t_0 \in (0, T)$ which is a Lebesgue point of u' and for all $v^0 \in C(t_0)$, we consider a suitable $v \in C([t_0, t_0 + \delta]; H)$ such that $v(t_0) = v^0$ and $v(s) \in C(s)$ for all $s \in [t_0, t_0 + \delta]$. Then, by choosing $v = v_n(s) = \pi_{C(\tau_n(s))}(v(s))$ in (8) and taking the integral on $[t_0, t_0 + \delta]$, one readily gets that

$$\int_{t_0}^{t_0+\delta} (u_n'(s), u_n(\tau_n(s)) - v(s))\, ds \leq \int_{t_0}^{t_0+\delta} \varphi_0 |u_n'(s)| \left(|u_n(\tau_n(s)) - v(s)| + \lambda h_n\right)^2 ds + \delta\lambda^2 h_n.$$

Finally, by passing to the limit with n and taking into account the above proved convergence we readily conclude that

$$\frac{1}{\delta}\int_{t_0}^{t_0+\delta}(u'(s),u(s)-v(s))\,ds \le \frac{1}{\delta}\int_{t_0}^{t_0+\delta}\varphi_0|u'(s)||u(s)-v(s)|^2\,ds.$$

Taking $\delta\to 0$, we obtain

$$(u'(t_0),u(t_0)-v^0)\le \varphi_0|u'(t_0)||u(t_0)-v^0|^2,$$

and (1) easily follows. Finally, since the solution $u\in W^{1,\infty}(0,T;H)$ is unique (see below), one has that the above-stated convergences hold for the whole sequence u_n.

Error control. The above detailed existence proof provides as a by-product an effective approximation technique. In particular, we have an a priori error control of the discretization error of (suboptimal) order 1/2. Indeed, by passing to the limit in m in (11) one readily gets that

$$|u(t)-u_n(t)|^2 \le (18\varphi_0\lambda^3h_n^2+3\lambda^2h_n)(e^{8\varphi_0\lambda t}-1)/2\phi_0\lambda \quad \forall t\in[0,T].$$

Continuous dependence on initial data. Let $u_1,u_2\in W^{1,\infty}(0,T;H)$ fulfill the inclusion in (1) along with the initial conditions $u_1(0)=u_{0,1}$ and $u_2(0)=u_{0,2}$ for some $u_{0,1},u_{0,2}\in C(0)$, respectively. Hence, we readily have that

$$(u_i'(t),u_i(t)-u_j(t))\le\varphi_0|u_i'(t)||u_i(t)-u_j(t)|^2 \quad \text{for a.e } t\in(0,T),\ i=1,2,\ j=3-i.$$

By adding the corresponding relations and taking the integral we readily check that

$$|u_1(t)-u_2(t)|\le|u_{0,1}-u_{0,2}|e^{4\varphi_0\lambda t}\quad\forall t\in[0,T].$$

Continuous sweeping processes with bounded variation. As mentioned above, the convergence of the catching-up algorithm can be proved under some slightly weaker assumptions on C. In particular, the existence and continuous dependence on data of a function $u\in BV(0,T;H)\cap C([0,T];H)$ solving (1) in a suitably weak sense can be obtained whenever $r:t\mapsto \mathrm{ret}(C;0,t)$ is continuous and of bounded variation. The reader is referred to the recent contributions by Edmond & Thibault [14] and Edmond [12] in this direction. Moreover, we shall mention the paper Brokate, Krejčì, & Stefanelli [4] for a BV counterpart of the analysis on the quasi-variational in ordered spaces for functions C with convex values.

4. Ordering solutions

Assume now we are given a separable and ordered Hilbert space (H,P), solve (1) for two sets of data $(C_1,u_{0,1})$ and $(C_2,u_{0,2})$ fulfilling (A2)-(A3), and obtain $u_1,u_2\in W^{1,\infty}(0,T;H)$, respectively.

We shall be interested in establishing some conditions on the data so that the solutions u_1 and u_2 will be ordered for all times. The very same problem is addressed for (convex) variational inequalities by Duvaut & Lions [11, Ch. 1.6, p. 58] and we will adapt the same idea to the current non-convex situation. In particular, we easily prove the following.

Lemma 4.1. *Assume* (H, P) *is a separable and ordered Hilbert space, let* $(C_1, u_{0,1})$ *and* $(C_2, u_{0,2})$ *fulfilling* (A2)-(A3) *be given, and denote by* $u_1, u_2 \in W^{1,\infty}(0, T; H)$ *the corresponding solutions to* (1). *Moreover let* $u_{0,1} \leq u_{0,2}$ *and* $C_1(t) \preceq C_2(t)$ *for almost every* $t \in (0, T)$. *Then* $u_1 \leq u_2$.

Proof. We start from the following

$$(u_i'(t), u_i(t) - v_i(t)) \leq \varphi_0 |u_i'(t)||u_i(t) - v_i(t)|^2 \quad \text{for a.e. } t \in (0, T),\ \forall v(t) \in C_i(t),\ i = 1, 2.$$

Since $u_i(t) \in C_i(t)$ for all $t \in [0, T]$, $i = 1, 2$, and $C_1(t) \preceq C_2(t)$ for almost every $t \in (0, T)$, one readily has that

$$(u_1 \wedge u_2)(t) \in C_1(t) \text{ and } (u_2 \vee u_1)(t) \in C_2(t) \text{ for a.e. } t \in (0, T).$$

By choosing $v_1(t) = (u_1 \wedge u_2)(t)$ and $v_2(t) = (u_2 \vee u_1)(t)$ above, adding the two resulting relations, and taking the integral on $(0, t)$ for $t \in [0, T]$ we get that

$$|(u_1(t) - u_2(t))^+|^2 \leq |(u_{0,1} - u_{0,2})^+|^2 + 4\varphi_0 \lambda \int_0^t |(u_1(s) - u_2(s))^+|^2 ds \quad \forall t \in [0, T],$$

and, since $(u_{0,1} - u_{0,2})^+ = 0$, the result follows from Gronwall's lemma. □

5. The quasi-variational problem

We shall now turn to problem (2). To this aim, we start by reformulating (A1)-(A3) as follows

(B1) (H, P) is a separable and ordered Hilbert space with P closed and $-P \subset P^*$.

(B2) $C : [0, T] \times H \to 2^H$ has φ_0-convex values and it is Lipschitz continuous with respect to the excess semi-distance, i.e.,

$$e(C(s, u), C(t, v)) \leq \nu(t - s) + \mu|u - v| \quad \forall u, v \in H,\ \forall t > s, \tag{12}$$

for some $\mu, \nu \geq 0$.

(B3) $u_0 \in C(0, u_0)$.

Let us stress that, differently from (A3), the possibility of finding u_0 fulfilling (B3) does not follow directly from the corresponding assumption (B2). Indeed, we are asking in (B3) that the viability problem $u \in C(0, u)$ has at least a solution.

Following the general theory, in order to solve the quasi-variational problem (2) we shall be concerned with its *variational section* [1]. Namely, given $\bar{u} \in W^{1,\infty}(0, T; H)$ we shall first solve for $u \in W^{1,\infty}(0, T; H)$ the following

$$-u'(t) \in N_{C(t, \bar{u}(t))}(u(t)) \quad \text{for a.e. } t \in (0, T),\ u(0) = u_0. \tag{13}$$

We have already developed in Section 3 an existence-uniqueness theory for the latter problem. Indeed, owing to Theorem 3.1, we are entitled to define a *solution* operator S that associates to $\bar{u} \in W^{1,\infty}(0, T; H)$ the corresponding unique solution to (13). We recall that the mapping S is generally referred to as the *variational selection* of the quasi-variational problem (2).

The key feature of the present analysis is the choice of exploiting some monotone structure in (2). The latter is encoded in the following assumption

(B4) For all $\bar{u}_1, \bar{u}_2 \in W^{1,\infty}(0,T;H)$ with $\bar{u}_1 \leq \bar{u}_2$ we have that $C(t,\bar{u}_1(t)) \preceq C(t,\bar{u}_2(t))$ for almost every $t \in (0,T)$.

We shall explicitly observe that we are not directly requiring in that the viability problem $u(t) \in C(t,u(t))$ for almost every $t \in [0,T]$ has at least a solution (which is indeed implicit in (2)). This will eventually follow from our overall assumptions (see below).

For all $\lambda > 0$ let us now define the set

$$E_\lambda := \{u \in W^{1,\infty}(0,T;H) \, : \, |u'| \leq \lambda \text{ a.e. in } (0,T)\}.$$

We shall assume the following

(B5) There exists $\lambda > 0$ such that $S(E_\lambda) \subset E_\lambda$.

Assumption (B5) clearly entails some restriction on the choice of C and goes in the direction of possibly considering a fixed point procedure in E_λ. Of course, whenever we have (B2), the function $t \mapsto C(t,\bar{u}(t))$ turns out to fulfill (3) for all $\bar{u} \in W^{1,\infty}(0,T;H)$. Namely, since $S(E_\lambda) \subset W^{1,\infty}(0,T;H)$, assumption (B5) just requires that the size of the Lipschitz constant of $S(\bar{u})$ is controlled by that of $\bar{u}$.

Let us give here our basic examples of maps C fulfilling (B5). At first, we shall consider the following Lipschitz continuity condition

$$e(C(t,u),C(s,v)) \leq \max\{\mu(t-s), |u-v|\} \quad \forall u,v \in H, \; 0 \leq s < t \leq T, \tag{14}$$

for some $\mu > 0$. The latter choice entails of course (B5) with $\lambda \geq \mu$ since one readily checks that, given $0 \leq s < t \leq T$, we have

$$e(C(t,\bar{u}(t)),C(s,\bar{u}(s))) \leq \max\{\lambda(t-s), |\bar{u}(t)-\bar{u}(s)|\} \leq \lambda(t-s).$$

Hence, starting from $\bar{u} \in E_\lambda$, the very same Lipschitz bound on $S(\bar{u})$ follows.

As soon as the Lipschitz dependence on u in C (see (12)) is strictly contracting, we may possibly consider a second class of functions C such that (B5) holds. Indeed, letting $\nu \in [0,1)$ in (12), we readily check that (B5) follows with the choice $\lambda > \mu/(1-\nu)$.

Before moving on, we briefly comment on the latter assumptions with respect to the former contributions on quasi-variational sweeping processes. First of all, we mention that the absolutely continuous existence results of [3, 17, 18] are placed exactly in the framework of (B2), by restricting it indeed to convex values. In all the above mentioned papers, the Lipschitz continuity constant of C with respect to the dependence on u is asked to be strictly less than 1. In this regard, the current frame is slightly more general since we allow $\nu \geq 1$ in (14) (see the Example below). Moreover, the Lipschitz continuity is referred to the excess semi-distance and is one-sided in time (besides dealing with non-convex sets, of course).

Additionally, let us recall that some compactness for C is assumed in [17, 18] while C is asked to fulfill a non-empty interior condition along with extra continuity properties in [3]. On the contrary, by virtue of our additional ordering assumption (B4), no compactness is needed in the present setting and we are able to deal with the case of functions C having values with empty interior.

We say that $v \in E_\lambda$ is a *subsolution* (*supersolution*) of (2) if $v \leq S(v)$ ($v \geq S(v)$, respectively). Then, our existence result for (2) reads as follows.

Theorem 5.1. *Assume* (B1)-(B5) *and that there exist a subsolution* u_* *and a supersolution* u^* *in* E_λ *of* (2) *with* $u_* \le u^*$. *Then, the set of solutions* $u \in E_\lambda$ *to* (2) *such that* $u_* \le u \le u^*$ *is non-empty and has a minimum and a maximum.*

Proof. Owing to (B1)-(B3) and (B5), for all $\bar{u} \in [u_*, u^*] \cap E_\lambda$ there exist a unique solution $u = S(\bar{u}) \in E_\lambda$ of (13). Assumption (B4) and Lemma 4.1 entail at once that, for all $\bar{u}_1, \bar{u}_2 \in [u_*, u^*] \cap E_\lambda$ with $\bar{u}_1 \le \bar{u}_2$ one has that $S(\bar{u}_1) \le S(\bar{u}_2)$. Lemma 2.7 ensures that $[u_*, u^*] \subset L^2(0, T; H)$ is completely inductive. Indeed, the very same argument of Lemma 2.7 ensures that $[u_*, u^*] \cap E_\lambda \subset W^{1,\infty}(0, T; H)$ is completely inductive as well. Hence, the assertion follows by applying Lemma 2.1. □

6. An example

Let us now motivate our analysis by providing an example where assumptions (B1)-(B5) as well as the existence of suitable sub and supersolution can be fulfilled.

In order to keep the notation as simple as possible, we shall restrict ourselves to the finite dimensional case $H = \mathbb{R}^2$. The reader could of course reinterpret the example in an infinite dimensional setting with small modifications. We assume H to be ordered by means of the orthant

$$P = \{(x_1, x_2) \in H \, : \, x_1 \ge 0, \; x_2 \ge 0\}$$

so that $-P^* = P$ and (B1) holds.

For all $s \in \mathbb{R}, u \in H$, and $t \ge 0$ we shall define

$$C(t, u) := K(u) - (t, 0), \tag{15}$$

where

$$K(u) := \Big\{(v_1, v_2) \in H \, : \, v_1 \le f(u_1) \text{ and } v_2 \le \psi(v_1)\Big\}, \tag{16}$$

with $f(s) := (s \wedge 2s) \wedge \big(s/\sqrt{2} + 1\big)$ and $\psi : \mathbb{R} \to [0, 2]$ smooth, such that

$$\begin{cases} \psi(1) = 1; \\ 0 \le \psi' \le 1 \text{ on } \mathbb{R} \quad \text{and} \quad \psi' = 0 \text{ on } (-\infty, 1]; \\ \psi'' \text{ is bounded on } \mathbb{R}. \end{cases}$$

Clearly, the set C fulfills (B2). The Lipschitz continuity of the excess semidistance (14) holds with the choices $\mu = \nu = 1$ (the latter follows from the fact that we have chosen ψ and f in such a way that $\psi' f' \le 1/\sqrt{2}$ almost everywhere). In particular the set C has φ_0-convex values, since C admits an external tangent ball with a fixed radius (recall that $\psi'' > 0$ is allowed). Assumption (B4) can be verified by using the fact that f and ψ are non-decreasing.

Let us consider the corresponding quasi-variational sweeping process (2) on the time interval $(0, 1)$ with the initial value

$$u_0 = (1, 1), \tag{17}$$

which obviously satisfies (B3). The reader can easily check that the constants $u_* = (0, 0)$ and $u^* = (2, 2)$ are suitable sub and supersolutions, respectively. Hence, we are in the position of applying Theorem 5.1 and obtain the existence of a solution to (2) in $[u_*, u^*]$.

We shall explicitly mention that the choices (15)-(16) along with (17) have merely an academical interest. Indeed, one could easily check that the unique solution to the corresponding quasi-variational sweeping process (2) is $u(t) = (1 - t, 1)$ for $t \in [0, 1]$ and that the latter is indeed the solution to the variational problem referred to the moving convex set $K(t) = [0, 1] \times [0, 1] - (t, 0)$. In fact, we are not claiming to be interested in the problem itself but rather in providing an easy example which is a priori not fitting with the former results by Chemetov & Monteiro Marques [7], because of $\mu \geq 1$. On the other hand, let us stress that the present oversimplified situation is still of a quasi-variational nature. Indeed, one could check that the latter variational characterization holds true by virtue of the choice $t \in (0, 1)$ only. In particular, no absolutely continuous solution to (2) exists for $t > 1$ since the region $\{(v_1, v_2) \in H : v_1 < 0\}$ is not accessible for the solution (this construction is inspired by the counterexample to strong solvability by Kunze & Monteiro Marques [18]).

Acknowledgements. The authors were funded by FCT and FEDER through the grant POCTI/ISFL/209 of Centro de Matemática e Aplicações Fundamentais da Universidade de Lisboa (CMAF/UL). Ulisse Stefanelli gratefully acknowledges the hospitality of the CMAF/UL, where most of this research was performed. Finally, the Authors wish to thank an anonymous Referee for some valuable bibliographical information.

References

[1] C. Baiocchi, A. Capelo: Variational and Quasivariational Inequalities. Applications to Free Boundary Problems, Wiley & Sons, New York (1984).

[2] M. Bounkhel, L. Thibault: Nonconvex sweeping process and prox-regularity in Hilbert space, J. Nonlinear Convex Anal. 6(2) (2005) 359–374.

[3] M. Brokate, P. Krejčí, H. Schnabel: On uniqueness in evolution quasivariational inequalities, J. Convex Analysis 11(1) (2004) 111–130.

[4] M. Brokate, P. Krejčí, U. Stefanelli: Ordered sweeping processes in BV, in preparation (2008).

[5] A. Canino: On p-convex sets and geodesics, J. Differ. Equations 75(1) (1988) 118–157.

[6] C. Castaing: A new class of evolution equation in a Hilbert space, in: Multifunctions and Integrands (Catania, 1983), Lecture Notes in Math. 1091, Springer, Berlin (1984) 117–128.

[7] N. V. Chemetov, M. D. P. Monteiro Marques: Non-convex quasi-variational differential inclusions, preprint.

[8] F. H. Clarke, R. J. Stern, P. R. Wolenski: Proximal smoothness and the lower-C^2 property, J. Convex Analysis 2(1-2) (1995) 117–144.

[9] G. Colombo, V. V. Goncharov: Variational inequalities and regularity properties of closed sets in Hilbert spaces, J. Convex Analysis 8(1) (2001) 197–221.

[10] M. Degiovanni, A. Marino, M. Tosques: Evolution equations with lack of convexity, Nonlinear Anal., Theory Methods Appl. 9(12) (1985) 1401–1443.

[11] G. Duvaut, J.-L. Lions: Inequalities in Mechanics and Physics, Springer, Berlin (1976).

[12] J. F. Edmond: Existence of BV solutions for an evolution problem associated with a moving set, Technical Report CMM-B-06/09-168, Departamento de Ingenieria Matemática, Universidad de Chile (2006).

[13] J. F. Edmond, L. Thibault: Relaxation of an optimal control problem involving a perturbed sweeping process, Math. Program., Ser. B 104(2-3) (2005) 347–373.

[14] J. F. Edmond, L. Thibault: BV solutions of nonconvex sweeping process differential inclusion with perturbation, J. Differ. Equations 226(1) (2006) 135–179.

[15] R. A. Horn, C. R. Johnson: Matrix Analysis. Reprinted with Corrections, Cambridge University Press, Cambridge (1990).

[16] I. I. Kolodner: On completeness of partially ordered sets and fixpoint theorems for isotone mappings, Amer. Math. Monthly 75 (1968) 48–49.

[17] M. Kunze, M. D. P. Monteiro Marques: On parabolic quasi-variational inequalities and state-dependent sweeping processes, Topol. Methods Nonlinear Anal. 12(1) (1998) 179–191.

[18] M. Kunze, M. D. P. Monteiro Marques: An introduction to Moreau's sweeping process, in: Impacts in Mechanical Systems (Grenoble, 1999), Lecture Notes in Phys. 551, Springer, Berlin (2000) 1–60.

[19] A. Marino, M. Tosques: Some variational problems with lack of convexity and some partial differential inequalities, in: Methods of Nonconvex Analysis (Varenna, 1989), Lecture Notes in Math. 1446, Springer, Berlin (1990) 58–83.

[20] M. D. P. Monteiro Marques: Rafle par un convexe semi-continu inférieurement d'intérieur non vide en dimension finie, in: Séminaire d'Analyse Convexe 14 (Montpellier, 1984), Exposé 6, Univ. Sci. Tech. Languedoc, Montpellier (1984).

[21] F. Mignot, J.-P. Puel: Solution maximum de certaines inéquations d'évolution paraboliques, et inéquations quasi variationnelles paraboliques, C. R. Acad. Sci., Paris, Sér. A 280(5) (1975) 259–262.

[22] F. Mignot, J.-P. Puel: Inéquations d'évolution paraboliques avec convexes dépendant du temps. Applications aux inéquations quasi variationnelles d'évolution, Arch. Ration. Mech. Anal. 64(1) (1977) 59–91.

[23] J.-J. Moreau: Décomposition orthogonale d'un espace hilbertien selon deux cônes mutuellement polaires, C. R. Acad. Sci., Paris 255 (1962) 238–240.

[24] J.-J. Moreau: Problème d'évolution associé à un convexe mobile d'un espace hilbertien, C. R. Acad. Sci., Paris, Sér. A 276 (1973) 791–794.

[25] J.-J. Moreau: Evolution problem associated with a moving convex set in a Hilbert space, J. Differ. Equations 26(3) (1977) 347–374.

[26] R. A. Poliquin, R. T. Rockafellar, L. Thibault: Local differentiability of distance functions, Trans. Amer. Math. Soc. 352(11) (2000) 5231–5249.

[27] R. T. Rockafellar, R. J.-B Wets: Variational Analysis, Grundlehren der Mathematischen Wissenschaften 317, Springer, Berlin (1998).

[28] R. Rossi, U. Stefanelli: An order approach to a class of quasivariational sweeping processes, Adv. Differ. Equ. 10(5) (2005) 527–552.

[29] U. Stefanelli: Some quasivariational problems with memory, Boll. Unione Mat. Ital., Sez. B, Artic. Ric. Mat. (8) 7 (2004) 319–333.

[30] U. Stefanelli: Nonlocal quasivariational evolution problems, J. Differ. Equations 229 (2006) 204–228.

[31] L. Tartar: Inéquations quasi variationnelles abstraites, C. R. Acad. Sci., Paris, Sér. A 278 (1974) 1193–1196.

[32] M. Valadier: Quelques problèmes d'entraînement unilateral en dimension finie, in: Séminaire d'Analyse Convexe 18 (Montpellier, 1988), Exposé 8, Univ. Sci. Tech. Languedoc, Montpellier (1988).