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Necessary and Sufficient Conditions for Farkas' Lemma for Cone Systems and Second-Order Cone Programming Duality*

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We present conditions, which completely characterize Farkas' lemma for cone-convex systems, and obtain strong duality characterizations for convex optimization problems. In particular, we establish a necessary and sufficient closed cone condition for the Farkas lemma. As an application, we obtain necessary and sufficient conditions for the strong duality of convex second-order cone programming problems.

Keywords: Farkas lemma, cone-convex systems, strong duality, closed cone condition, second-order cone programming

1. Introduction

Farkas' lemma states that given any vectors $a_1, a_2, \ldots, a_m$ in $\mathbb{R}^n$ and for each choice of vector $c \in \mathbb{R}^n$, the linear inequality $c^T x \geqq 0$ is a consequence of the linear system $a_i^T x \geqq 0$, $i = 1, 2, \ldots, m$ if and only if $c = \sum_{i=1}^m \lambda_i a_i$, for some multipliers $\lambda_i \geqq 0$. Applications by way of extensions of the celebrated Farkas lemma range from classical nonlinear programming to modern areas of optimization such as nonsmooth optimization and semidefinite programming. For a recent look at its extensions and applications see [3, 7, 12, 18]. However, it is well known that the Farkas lemma for linear systems involving non-polyhedral cones is, in general, not valid without a regularity condition, often called "closed cone condition" (see e.g. [6, 12])

The generalized Farkas lemma for a given cone-convex system $-g(x) \in S$ and for each choice of real-valued convex function f states that $f(x) \geqq 0$ is a consequence of the system $-g(x) \in S$ if and only if there exists $\lambda \in S^+$ such that, for each $x \in \mathbb{R}^n$, $f(x) + (\lambda \circ g)(x) \geqq 0$. Symbolically,

$$[-g(x) \in S \ \Rightarrow f(x) \geqq 0] \Leftrightarrow (\exists \lambda \in S^+)(\forall x \in \mathbb{R}^n)\ f(x) + (\lambda \circ g)(x) \geqq 0, \qquad (1)$$

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where the set $S \subset \mathbb{R}^m$ is a closed convex cone with the dual cone S^+ and $g : \mathbb{R}^n \to \mathbb{R}^m$ is a continuous convex function with respect to S. Various sufficient conditions have been established for this equivalence in (1). For details, see e.g. [6, 10, 11, 12].

The purpose of this paper is to present necessary and sufficient conditions for the Farkas lemma with respect to the given cone-convex system $-g(x) \in S$ and to obtain corresponding duality properties for convex optimization problems. In particular, we establish a necessary and sufficient closed cone condition for the Farkas lemma. As an application, we obtain necessary and sufficient conditions for the strong duality for convex second-order cone programming problems [1, 2, 17], which have received a great deal of attention in recent years.

The outline of the paper is as follows. Section 2 provides background material on convex analysis that will be used later in the paper. Section 3 presents several characterizations of the Farkas lemma, including a necessary and sufficient closed cone (regularity) condition. Section 4 provides an application of the Farkas lemma and establishes characterizations of strong duality properties for second-order cone programming problems and for semidefinite programming problems.

2. Preliminaries

We recall in this section some notations and basic results which will be used in this paper. Let X be a normed space with X^* its dual endowed with weak*-topology. For a subset $D \subset X^*$, the w^*-closure of D will be denoted by $\mathrm{cl}\, D$ and the convex cone generated by D by $\mathrm{cocone}D$.

Let $h : X \to \mathbb{R} \cup \{-\infty, +\infty\}$. The conjugate function of h, $h^* : X^* \to \mathbb{R} \cup \{+\infty\}$, is defined by

$$h^*(v) := \sup\{v(x) - h(x) \mid x \in \mathrm{dom}\, h\},$$

where $\mathrm{dom}\, h := \{x \in X \mid h(x) < +\infty\}$ is the effective domain of h. The function h is said to be proper if h does not take on the value $-\infty$ and $\mathrm{dom}\, h \neq \emptyset$. The **epigraph** of h is defined by

$$\mathrm{epi}\, h := \{(x, r) \in X \times \mathbb{R} \mid x \in \mathrm{dom}\, h, h(x) \leqq r\}.$$

The set (possibly empty)

$$\partial h(a) := \{v \in X^* \mid h(x) - h(a) \geqq v(x - a), \forall x \in \mathrm{dom}\, h\}$$

is the subdifferential of the convex function h at $a \in \mathrm{dom}\, h$. For a closed convex subset D of X, the indicator function δ_D is defined as $\delta_D(x) = 0$ if $x \in D$ and $\delta_D(x) = +\infty$ if $x \notin D$. The **support function** δ_D^* is defined by $\delta_D^*(u) = \sup_{x \in D} u(x)$. Then $\partial \delta_D(x) = N_D(x)$, which is known as the normal cone of D of x. If h is proper lower semicontinuous and sublinear (i.e., convex and positively homogeneous of degree one), then $\mathrm{epi}\, g^* = \partial g(0) \times \mathbb{R}_+$.

For proper lower semicontinuous convex functions $g, h : X \to \mathbb{R} \cup \{+\infty\}$, the infimal convolution of g with h, denoted $g \Box h$, is defined by

$$(g \Box h)(x) = \inf_{x_1 + x_2 = x} \{g(x_1) + h(x_2)\}.$$

The lower semicontinuous envelope and lower semicontinuous convex hull of g are denoted respectively by $\mathrm{cl}\, g$ and $\mathrm{cl}\,\mathrm{co}\, g$. That is, $\mathrm{epi}(\mathrm{cl}\, g) = \mathrm{cl}(\mathrm{epi}\, g)$ and $\mathrm{epi}(\mathrm{cl}\,\mathrm{co}\, g) = \mathrm{cl}\,\mathrm{co}\,(\mathrm{epi}\, g)$. For details, see [21].

Let g, h and g_i, $i \in I$ (where I is an arbitrary index set) be proper lower semicontinuous convex functions. It is well known from the dual operation (see [20], [21]) that if $\mathrm{dom}\, g \cap \mathrm{dom}\, h \neq \emptyset$, then

$$(g \Box h)^* = g^* + h^*, \ (g + h)^* = \mathrm{cl}(g^* \Box h^*)$$

and if $\sup_{i\in I} g_i$ is proper, then

$$(\sup_{i\in I} g_i)^* = \operatorname{cl}\operatorname{co}(\inf_{i\in I} g_i^*).$$

So, one can check that

$$\operatorname{epi}(g+h)^* = \operatorname{cl}(\operatorname{epi} g^* + \operatorname{epi} h^*) \qquad \text{and} \qquad \operatorname{epi}(\sup_{i\in I} g_i)^* = \operatorname{cl}\operatorname{co}(\bigcup_{i\in I} \operatorname{epi} g_i^*).$$

The closure in the first equation is superfluous if one of g and h is continuous at some $x_0 \in \operatorname{dom} g \cap \operatorname{dom} h$ (see [4, 21] for details).

Let Y be another normed linear space with topological dual Y^* and let S be a closed convex cone in Y. Denote by S^+ the dual cone of S, defined as

$$S^+ = \{y^* \in Y^* \mid y^*(y) \geqq 0 \text{ for any } y \in S\}.$$

We say that the map $g : X \to Y$ is $S-$convex if for any $x_1,\ x_2 \in X$ and any $\lambda \in [0,1]$,

$$g(\lambda x_1 + (1-\lambda)x_2) \in \lambda g(x_1) + (1-\lambda)g(x_2) - S$$

and that g is $S-$sublinear if g is $S-$convex and positively homogeneous of degree 1. Note that $g^{-1}(-S) := \{x \in X \mid -g(x) \in S\}$.

3. Necessary and Sufficient Conditions for Farkas' Lemma

In this section we present several characterizations of Farkas lemma for cone-convex systems. In particular, we show that the well known closed convex cone condition for sublinear systems completely characterize the Farkas lemma.

Theorem 3.1. *Suppose that $g : X \to Y$ is a continuous and $S-$convex function with $g^{-1}(-S) \neq \emptyset$. Let $\mathcal{A} = \{l \mid l : X \to \mathbb{R}$ is continuous and convex$\}$. Then the following statements are equivalent:*

(*i*) *For each $f \in \mathcal{A}$,*

$$[-g(x) \in S \ \Rightarrow f(x) \geqq 0] \Leftrightarrow (\exists \lambda \in S^+)(\forall x \in X)\ f(x) + (\lambda \circ g)(x) \geqq 0. \tag{1}$$

(*ii*) *For each $f \in \mathcal{A}$,*

$$[-g(x) \in S \ \Rightarrow f(x) \geqq 0] \Leftrightarrow (\exists \lambda \in S^+)\ (0,0) \in \operatorname{epi} f^* + \operatorname{epi}(\lambda \circ g)^*. \tag{2}$$

(*iii*) *For each $f \in \mathcal{A}$,*

$$(\exists \lambda \in S^+)\ \inf_{x\in X}\{f(x) + (\lambda \circ g)(x)\} = \inf_{x \in g^{-1}(-S)} f(x). \tag{3}$$

(*iv*) *The convex cone,*

$$\bigcup_{\lambda \in S^+} \operatorname{epi}(\lambda \circ g)^* \text{ is } w^* - \text{closed}. \tag{4}$$

Proof. [$(ii) \iff (i)$]. Assume that (ii) holds. Let $f \in \mathcal{A}$. Suppose that $f(x) \geqq 0$ for each $x \in g^{-1}(-S)$. Then, by (ii), there exists $\lambda \in S^+$ such that $(0,0) \in \operatorname{epi} f^* + \operatorname{epi}(\lambda \circ g)^*$. So, there exists $(v,r) \in \operatorname{epi}(\lambda \circ g)^*$ such that $-(v,r) \in \operatorname{epi} f^*$. Now, for each $x \in X$, $v(x) - (\lambda \circ g)(x) \leqq r$, and for each $x \in X$, $-v(x) - f(x) \leqq -r$. Adding the two inequalities, we get that for each $x \in X$, $f(x) + (\lambda \circ g)(x) \geqq 0$. The converse implication in (1) always holds.

Conversely, assume that (i) holds. Let $f \in \mathcal{A}$. Suppose that $f(x) \geqq 0$ for each $x \in g^{-1}(-S)$. Then, by (i), there exists $\lambda \in S^+$ such that for each $x \in X$, $f(x) + (\lambda \circ g)(x) \geqq 0$. So, $(f + \lambda \circ g)^*(0) \leqq 0$ and hence $(0,0) \in \operatorname{epi}(f + \lambda \circ g)^*$. Since f is continuous, $\operatorname{epi}(f + \lambda \circ g)^* = \operatorname{epi} f^* + \operatorname{epi}(\lambda \circ g)^*$, and consequently $(0,0) \in \operatorname{epi} f^* + \operatorname{epi}(\lambda \circ g)^*$.

[$(iii) \iff (i)$]. Suppose that (iii) holds. Let $f \in \mathcal{A}$. If $f(x) \geqq 0$ for each $x \in g^{-1}(-S)$ then $\inf_{x \in g^{-1}(-S)} f(x) \geqq 0$ and so, from (iii), there exists $\lambda \in S^+$ such that

$$\inf_{x \in X} \{f(x) + (\lambda \circ g)(x)\} \geqq 0.$$

Hence (i) holds as the converse implication always holds.

Suppose now that (i) holds. Let $f \in \mathcal{A}$. If $\inf_{x \in g^{-1}(-S)} f(x) = -\infty$, then

$$\inf_{x \in X} \{f(x) + (0 \circ g)(x)\} = \inf_{x \in g^{-1}(-S)} f(x) = -\infty.$$

Since $g^{-1}(-S) \neq \emptyset$, we may assume that $r := \inf_{x \in g^{-1}(-S)} f(x)$ is finite. Since $f(\cdot) - r \in \mathcal{A}$, it follows from (i) that there exists $\lambda \in S^+$such that $f(x) + (\lambda \circ g)(x) \geqq r$ for each $x \in X$. Thus,

$$\inf_{x \in X} \{f(x) + (\lambda \circ g)(x)\} \geqq \inf_{x \in g^{-1}(-S)} f(x).$$

Since $f(x) \geqq f(x) + (\lambda \circ g)(x)$, for each $x \in g^{-1}(-S)$, we have

$$\inf_{x \in g^{-1}(-S)} f(x) \geqq \inf_{x \in X} \{f(x) + (\lambda \circ g)(x)\}.$$

Hence, (iii) holds.

[$(ii) \iff (iv)$]. Suppose that (ii) holds. Let $(v,r) \in \operatorname{cl} \bigcup_{\lambda \in S^+} \operatorname{epi}(\lambda \circ g)^*$. Then there exist nets $\{\lambda_\alpha\} \subset S^+$, $\{(v_\alpha, r_\alpha)\} \subset \operatorname{epi}(\lambda_\alpha \circ g)^*$ such that $(v_\alpha, r_\alpha) \to (v,r)$. We can easily check that $v_\alpha(x) \leqq r_\alpha$ for each $x \in g^{-1}(-S)$. Letting $\alpha \to \infty$, we get that $v(x) \leqq r$, for each $x \in g^{-1}(-S)$. Let $f(x) = -v(x) + r$. Then for each $x \in g^{-1}(-S)$, $f(x) \geqq 0$. Moreover,

$$\operatorname{epi} f^* = \{(-v, -r + \alpha) \mid \alpha \geqq 0\}.$$

So, it follows from (ii) that there exist $\lambda \in S^+$ and $\alpha \geqq 0$ such that $(v, r - \alpha) \in \operatorname{epi}(\lambda \circ g)^*$. This gives us that $(v,r) \in \operatorname{epi}(\lambda \circ g)^*$. Hence $\bigcup_{\lambda \in S^+} \operatorname{epi}(\lambda \circ g)^*$ is w^*-closed.

Conversely, assume that (iv) holds. Since $(S^+)^+ = S$, for each $x \in X$,

$$\sup_{\lambda \in S^+} (\lambda \circ g)(x) = \delta_{g^{-1}(-S)}(x).$$

So,

$$\operatorname{epi} \delta^*_{g^{-1}(-S)} = \operatorname{cl} \operatorname{co} \Big(\bigcup_{\lambda \in S^+} \operatorname{epi}(\lambda \circ g)^* \Big).$$

As the set $\bigcup_{\lambda \in S^+} \operatorname{epi}(\lambda \circ g)^*$ is a convex cone [15], it follows from (iv),

$$\operatorname{epi} \delta^*_{g^{-1}(-S)} = \bigcup_{\lambda \in S^+} \operatorname{epi}(\lambda \circ g)^*.$$

Now,

$$\begin{aligned} & f(x) \geqq 0 \text{ for each } x \in g^{-1}(-S) \\ \Longleftrightarrow \quad & (f + \delta_{g^{-1}(-S)})^*(0) \leqq 0 \\ \Longleftrightarrow \quad & (0,0) \in \operatorname{epi}(f + \delta_{g^{-1}(-S)})^* \\ \Longleftrightarrow \quad & (0,0) \in \operatorname{epi} f^* + \operatorname{epi} \delta^*_{g^{-1}}(-S) \\ \Longleftrightarrow \quad & (0,0) \in \operatorname{epi} f^* + \bigcup_{\lambda \in S^+} \operatorname{epi}(\lambda \circ g)^*. \end{aligned}$$

Hence (ii) holds. □

The following examples illustrate the significance of the closed cone condition (iv) in Theorem 3.1 for Farkas lemma for cone-convex systems.

Example 3.2. Let $g(x) := \max\{0, x\}$, $S = \mathbb{R}_+$ and $\mathcal{A} = \{l \mid l : \mathbb{R} \to \mathbb{R} \text{ is convex}\}$. Then $g : \mathbb{R} \to \mathbb{R}$ is a convex function and $g^{-1}(-S) = (-\infty, 0]$. We can easily check that $\bigcup_{\lambda \in S^+} \operatorname{epi}(\lambda \circ g)^* = \mathbb{R}^2_+$. For each $f \in \mathcal{A}$, we can choose $\lambda \in \mathbb{R}$ such that $\lambda \geqq \max\{-v, 0\}$, where $v \in \partial f(0)$. Now, if, for each $x \in g^{-1}(-S)$, $f(x) \geqq 0$, then for each $x \in \mathbb{R}$, $f(x) + \lambda g(x) \geqq 0$. Indeed, if $x \leqq 0$, then $f(x) + \lambda g(x) = f(x) \geqq 0$; if $x \geqq 0$, then

$$f(x) + \lambda g(x) \geqq vx + \lambda g(x) \geqq (v + \lambda)x \geqq 0.$$

On the other hand, if, for some $\lambda \geqq 0$, $f(x) + \lambda g(x) \geqq 0$, for each $x \in \mathbb{R}$, then, clearly, $f(x) \geqq 0$, for each $x \in g^{-1}(-S)$. Now, let $r := \inf_{x \in g^{-1}(-S)} f(x)$. If r is finite, then $f - r \in \mathcal{A}$ and so, for each $x \in \mathbb{R}$, $f(x) + \lambda g(x) \geqq r$. This gives us that $\inf_{x \in \mathbb{R}} f(x) + \lambda g(x) \geqq r$. By the weak duality, $r = \max_{\mu \in \mathbb{R}^+} \inf_{x \in R} f(x) + \mu g(x)$. If $r = -\infty$ then this equality trivially holds. Hence Theorem 3.1 holds. It is worth noting that the Slater condition that $g(x_0) < 0$ for some $x_0 \in \mathbb{R}$, does not hold for this Example.

Example 3.3. Let $g(x) := [\max\{0, x\}]^2$, $S = \mathbb{R}_+$. Then $g : \mathbb{R} \to \mathbb{R}$ is a convex function and $g^{-1}(-S) = (-\infty, 0]$. Then if $\lambda = 0, \operatorname{epi}(\lambda \circ g)^* = \{0\} \times \mathbb{R}_+$, and if $\lambda > 0, \operatorname{epi}(\lambda \circ g)^* = \{(v, \alpha) \in \mathbb{R}^2 \mid v \geqq 0, \frac{v^2}{4\lambda} \leqq \alpha\}$, and hence $\bigcup_{\lambda \in S^+} \operatorname{epi}(\lambda \circ g)^*$ is not closed. Let $f(x) = -x$. Then $\operatorname{epi} f^* = \{-1\} \times \mathbb{R}_+$, and hence we can not find $\lambda \geqq 0$ such that $(0,0) \in \operatorname{epi} f^* + \operatorname{epi}(\lambda \circ g)^*$. Thus, Theorem 3.1 fails to hold.

The generalized Farkas lemma for cone-sublinear systems has been well known under the closed cone condition that $\bigcup_{\lambda \in S^+} \partial(\lambda \circ g)(0)$ is w^*-closed. For details, see [8, 12] and other references therein. We now show that this closed cone condition completely characterizes the Farkas lemma for the cone-sublinear systems.

Corollary 3.4. *Let $g : X \to Y$ be a continuous and S-sublinear function. Let $\mathcal{B} = \{l \mid l : X \to \mathbb{R}$ is continuous and sublinear$\}$. Then the following statements are equivalent:*

(i) *For each $f \in \mathcal{B}$,*

$$[-g(x) \in S \ \Rightarrow f(x) \geqq 0] \Leftrightarrow (\exists \lambda \in S^+) \ 0 \in \partial f(0) + \partial(\lambda \circ g)(0).$$

(ii) *$\bigcup_{\lambda \in S^+} \partial(\lambda \circ g)(0)$ is w^*-closed.*

Proof. Since g is $S-$sublinear,

$$\bigcup_{\lambda\in S^+} \mathrm{epi}(\lambda\circ g)^* = \bigcup_{\lambda\in S^+} \partial(\lambda\circ g)(0)\times\mathbb{R}^+$$

and hence (iv) of Theorem 3.1 holds if and only if $\bigcup_{\lambda\in S^+}\partial(\lambda\circ g)(0)$ is w^*-closed. Since f is sublinear and g is S-sublinear,

$$f(x)+(\lambda\circ g)(x)\geqq 0\ \forall x\in X \Leftrightarrow 0\in\partial f(0)+\partial(\lambda\circ g)(0).$$

Thus the conclusion follows from Theorem 3.1. □

Corollary 3.5. *Let $A : X \to Y$ be continuous and linear. Then the following statements are equivalent:*

(i) $\forall c\in X^*,\ \left[-Ax\in S\ \Rightarrow c(x)\geqq 0\right]\Leftrightarrow(\exists\lambda\in S^+)\ c+A^T\lambda=0.$

(ii) *$A^T(S^+)$ is w^*-closed.*

Proof. Let $g(x) = Ax$. Then $\bigcup_{\lambda\in S^+}\partial(\lambda\circ g)(0) = A^T(S^+)$. Then, the conclusion follows from Corollary 3.4. □

Remark 3.6. When $Y = \mathbb{R}^m$ and S is a polyhedral convex cone in Y, then $A^T(S^+)$ is a finitely generated cone and hence $A^T(S^+)$ is closed. So, from Corollary 3.5, the original Farkas lemma follows (see [5]).

The following corollary, which can be used to study convex vector optimization problems, provides solvability conditions for systems involving convex vector functions f. See [16, 14] for recent results for convex vector optimization problems.

Corollary 3.7. *Let $g : X \to Y$ be a continuous and S-convex function with $g^{-1}(-S)\neq\emptyset$. Let $\mathcal{C} = \{l \mid l : X\to\mathbb{R}^p$ is continuous and $\mathbb{R}^p_+$- convex $\}$. Let $\Lambda_p = \{x\in\mathbb{R}^p \mid \Sigma_{i=1}^p x_i = 1\}$. Then the following statements are equivalent:*

(i) *For each $f\in\mathcal{C}$,*

$$\begin{aligned}&[-g(x)\in S\ \Rightarrow f(x)\notin -\mathrm{int}\,\mathbb{R}^p_+]\\ \Leftrightarrow\ &(\exists\theta\in\Lambda_p)(\exists\lambda\in S^+)(\forall x\in X)\ \theta^T f(x)+(\lambda\circ g)(x)\geqq 0.\end{aligned}$$

(ii) *For each $f\in\mathcal{C}$,*

$$\begin{aligned}&[-g(x)\in S\ \Rightarrow f(x)\notin -\mathrm{int}\,\mathbb{R}^p_+]\\ \Leftrightarrow\ &(\exists\theta\in\Lambda_p)(\exists\lambda\in S^+)\ (0,0)\in\mathrm{epi}(\theta^T f)^*+\mathrm{epi}(\lambda\circ g)^*.\end{aligned}$$

(iii) *$\bigcup_{\lambda\in S^+}\mathrm{epi}(\lambda\circ g)^*$ is w^*-closed.*

Proof. Let $f\in\mathcal{C}$. The Hahn-Banach separation theorem [20] ensures that

$$[-g(x)\in S\ \Rightarrow f(x)\notin -\mathrm{int}\,\mathbb{R}^p_+]\ \Leftrightarrow\ (\exists\theta\in\Lambda_p)\ [-g(x)\in S\ \Rightarrow\theta^T f(x)\geqq 0].$$

Now, from Theorem 3.1, (i) is equivalent to (ii), and (iii) implies (ii).

To establish that (ii) implies (iii) suppose that (ii) holds. Let $(v,r)\in\bigcup_{\lambda\in S^+}\mathrm{epi}(\lambda\circ g)^*$. Then $v(x)\leqq r$ for each $x\in g^{-1}(-S)$. Let $f(x) = (r-v(x),\cdots,r-v(x))$. Then $f(x)\notin -\mathrm{int}\,\mathbb{R}^p_+$, and from (ii), there exist $\theta\in\Lambda_p$ and $\lambda\in S^+$ such that $(0,0)\in\mathrm{epi}(\theta^T f)^*+\mathrm{epi}(\lambda\circ g)^*$. Noting that $\theta^T f(x) = r - v(x)$, the same arguments of the proof of Theorem 3.1 lead to the condition that $(v,r)\in\mathrm{epi}(\lambda\circ g)^*$. Hence (iii) holds. □

4. Characterizations of Duality in Cone Programming

In this section, we examine second-order cone programming problems which arise in various application areas (see e.g. [2, 17]). As an application of our Farkas lemma, we first present a constraint qualification which completely characterizes the strong duality of a second-order cone programming problem. We then derive a necessary and sufficient condition for the strong min-max duality whenever the given problem attains its infimum.

Consider the following convex second-order cone program:

$$(SOCP) \quad \inf f(x)$$
$$\text{subject to } x \in M := \{x \in \mathbb{R}^n \mid \|Hx + b\| \leqq c^T x + d \},$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function and H is an $(m-1) \times n$ matrix, $b \in \mathbb{R}^{m-1}$, $c \in \mathbb{R}^n$, $d \in \mathbb{R}$ and $\|z\| = \sqrt{z^T z},\ z \in \mathbb{R}^{m-1}$.

Suppose that $M \neq \emptyset$. Let $K = \{(y,t)^T \in \mathbb{R}^{m-1} \times \mathbb{R} \mid \|y\| \leqq t \}$, that is, K is a second-order cone in $\mathbb{R}^m$. Then K is self-dual, that is, $K = K^+$.

Theorem 4.1. *Let $\mathcal{A} = \{l \mid l : \mathbb{R}^n \to \mathbb{R}$ is convex $\}$. Then the following statements are equivalent:*

(*i*) *For each $f \in \mathcal{A}$,*

$$\max_{\lambda \in K} \inf_{x \in \mathbb{R}^n} \left\{ f(x) - \lambda^T \begin{pmatrix} Hx + b \\ c^T x + d \end{pmatrix} \right\} = \inf_{x \in M} f(x).$$

(*ii*) $\bigcup_{\lambda \in K} \left\{ \left(-\begin{pmatrix} H \\ c^T \end{pmatrix}^T \lambda, \begin{pmatrix} b \\ d \end{pmatrix}^T \lambda \right) \right\} + \{0\} \times \mathbb{R}^+$ *is closed.*

Proof. Note that

$$\|Hx + b\| \leqq c^T x + d \ \Leftrightarrow \begin{pmatrix} Hx + b \\ c^T x + d \end{pmatrix} \in K.$$

Let $g(x) = -\begin{pmatrix} Hx + b \\ c^T x + d \end{pmatrix}$. Then we have,

$$\bigcup_{\lambda \in K} (\lambda \circ g)^* = \bigcup_{\lambda \in K} \left\{ \left(-\begin{pmatrix} H \\ c^T \end{pmatrix}^T \lambda, \begin{pmatrix} b \\ d \end{pmatrix}^T \lambda \right) \right\} + \{0\} \times \mathbb{R}^+.$$

Hence the conclusions follow from Theorem 3.1. □

Remark 4.2. In Theorem 4.1, if $b = 0$ and $d = 0$, then (*ii*) is equivalent to the fact that $(H^T, c)(K)$ is closed. Moreover, if $m = n$ and the $n \times n$ matrix (H^T, c) is nonsingular, then the set $(H^T, c)(K)$ is closed.

Now we derive a new condition for the min-max Lagrangian duality for (SOCP), where $\inf_{x \in M} f(x)$ is attained.

Theorem 4.3. *Let $\mathcal{A} = \{l \mid l : \mathbb{R}^n \to \mathbb{R}$ is convex $\}$. Then the following statements are equivalent:*

(*i*) *For each* $x \in M$*,*

$$N_M(x) = \left\{ -\begin{pmatrix} H \\ c^T \end{pmatrix}^T \lambda \mid \lambda \in K,\ \lambda^T \begin{pmatrix} Hx + b \\ c^T x + d \end{pmatrix} = 0 \right\}.$$

(*ii*) *For each* $f \in \mathcal{A}$*,*

$$\min_{x \in M} f(x) = \max_{\lambda \in K} \inf_{x \in \mathbb{R}^n} \left\{ f(x) - \lambda^T \begin{pmatrix} Hx + b \\ c^T x + d \end{pmatrix} \right\}.$$

Proof. Suppose that (i) holds. Let $f \in \mathcal{A}$. Assume that $x \in M$ and that $f(x) = \min_{y \in M} f(y)$. Then, by optimality condition, $0 \in \partial f(x) + N_M(x)$ and so from (i), there exists $\lambda \in K$ such that

$$0 \in \partial f(x) - \begin{pmatrix} H \\ c^T \end{pmatrix}^T \lambda \qquad \text{and} \qquad \lambda^T \begin{pmatrix} Hx + b \\ c^T x + d \end{pmatrix} = 0.$$

This gives us that

$$\inf_{y \in \mathbb{R}^n} \left\{ f(y) - \lambda^T \begin{pmatrix} Hy + b \\ c^T y + d \end{pmatrix} \right\} \geqq f(x).$$

Hence, by weak duality, (ii) holds.

Conversely, assume that (ii) holds. Let Ω be the right side of the equality in (i). Let $x \in M$ and $u \in N_M(x)$. Then $-u^T x = \min_{y \in M}(-u^T y)$. So, it follows from (ii) that there exists $\lambda \in K$ such that, for each $y \in \mathbb{R}^n$

$$-u^T x \leqq -u^T y - \lambda^T \begin{pmatrix} Hy + b \\ c^T y + d \end{pmatrix}.$$

This gives us that

$$\lambda^T \begin{pmatrix} Hx + b \\ c^T x + d \end{pmatrix} = 0 \qquad \text{and} \qquad u = -\begin{pmatrix} H \\ c^T \end{pmatrix}^T \lambda.$$

Thus $u \in \Omega$. On the other hand, the inclusion $\Omega \subset N_M(x)$ always holds. Hence (i) holds. □

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