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Linear Operators on Vector-Valued Function Spaces with Mackey Topologies

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Let E be an ideal of L^0 over a σ-finite measure space (Ω, Σ, μ) and let E' be the Köthe dual of E. Let $(X, \|\cdot\|_X)$ be a real Banach space, and X^* the Banach dual of X. Let $E(X)$ be a subspace of the space $L^0(X)$ of μ-equivalence classes of all strongly Σ-measurable function $f : \Omega \to X$, and consisting of all those $f \in L^0(X)$ for which the scalar function $\widetilde{f}$, defined by $\widetilde{f}(\omega) = \|f(\omega)\|_X$ for $\omega \in \Omega$, belongs to E. Assume that a Banach space X is an Asplund space. It is shown that a subset C of $E'(X^*)$ is relatively $\sigma(E'(X^*), E(X))$-compact iff the set $\{\widetilde{g} : g \in E'(X^*)\}$ in E' is relatively $\sigma(E', E)$-compact. We consider the topology $\overline{\tau(E, E')}$ on $E(X)$ associated with the Mackey topology $\tau(E, E')$ on E. It is shown that $\overline{\tau(E, E')}$ is strongly Mackey topology; hence $\overline{\tau(E, E')}$ coincides with the Mackey topology $\tau(E(X), E'(X^*))$. Moreover, $E'(X^*)$ is $\sigma(E'(X^*), E(X))$-sequentially complete whenever E' is perfect. We examine the space $\mathcal{L}_\tau(E(X), Y)$ of all $(\tau(E(X), E'(X^*)), \|\cdot\|_Y)$-continuous linear operators from $E(X)$ to a Banach space $(Y, \|\cdot\|_Y)$, equipped with the weak operator topology (briefly WOT) and the strong operator topology (briefly SOT). It is shown that if E is perfect, then $\mathcal{L}_\tau(E(X), Y)$ is WOT-sequentially complete, and every SOT-compact subset of $\mathcal{L}_\tau(E(X), Y)$ is $(\tau(E(X), E'(X^*)), \|\cdot\|_Y)$-equicontinuous. Moreover, a Vitali-Hahn-Saks type theorem for $\mathcal{L}_\tau(E(X), Y)$ is obtained.

Keywords: Vector-valued function spaces, Mackey topologies, strongly Mackey topologies, weak compactness, Radon-Nikodym property, Asplund spaces, sequential completeness, convex compactness property, weak operator topology, strong operator topology, linear operators

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1. Introduction and preliminaries

We denote by $\sigma(L, K)$ and $\tau(L, K)$ the weak-topology and the Mackey topology on L with respect to a dual pair $\langle L, K \rangle$. Given a Hausdorff locally convex space (L, ξ) by $(L, \xi)^*$ or L^*_ξ we will denote its topological dual. Recall that ξ is said to be a *strongly Mackey topology* if every relatively countably $\sigma(L^*_\xi, L)$-compact subset of L^*_ξ is ξ-equicontinuous (see [22, Definition 4.1]). Clearly, if ξ is a strongly Mackey topology, then ξ is a Mackey topology, i.e., $\xi = \tau(L, L^*_\xi)$.

First we establish terminology concerning Riesz spaces and function spaces (see [1], [10], [11], [26]). Let (Ω, Σ, μ) be a complete σ-finite measure space. Let L^0 denote the space of μ-equivalence classes of all Σ-measurable real-valued functions defined and finite a.e. on Ω. Let χ_A stand for the characteristic function of a set A, and let $\mathbb{N}$ and $\mathbb{R}$ denote the sets of all natural and real numbers. Let E be an ideal of L^0 with $\operatorname{supp} E = \Omega$, and let E' stand for the Köthe dual of E, i.e., $E' = \{v \in L^0 : \int_\Omega |u(\omega)v(\omega)| d\mu < \infty$ for all $u \in E\}$. Throughout the paper we assume that $\operatorname{supp} E' = \Omega$. Let $E^\sim$ and $E^\sim_n$

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stand for the order dual and the order continuous dual of E, resp. Then $E_n^\sim$ separates points of E and it can be identified with E' through the mapping: $E' \ni v \mapsto \varphi_v \in E_n^\sim$, where $\varphi_v(u) = \int_\Omega u(\omega)v(\omega)d\mu$ for all $u \in E$. E is said to be *perfect* whenever the natural embedding from E into $(E_n^\sim)_n^\sim$ is onto, i.e., $E'' = E$.

For terminology and basic concepts from the theory of vector-valued function spaces $E(X)$, we refer to the three main monographs: J. Diestel and J. J. Uhl's "Vector Measures" [8], P. Cembranos and J. Mendoza's "Banach spaces of Vector Valued Functions" [5] and Pei-Kee Lin's "Köthe-Bochner Function Spaces" [11].

Now we recall terminology and some basic results concerning the topological properties and the duality theory of vector-valued function spaces $E(X)$ as set out in [2], [3], [4], [5], [8], [9], [11], [14], [15], [16]. Let $(X, \|\cdot\|_X)$ be a real Banach space and let X^* stand for the Banach dual of X. Let S_X, B_X stand for the unit sphere and the closed unit ball in X. By $L^0(X)$ we denote the set of μ-equivalence classes of all strongly Σ-measurable functions $f : \Omega \longrightarrow X$. For $f \in L^0(X)$ let us set $\widetilde{f}(\omega) := \|f(\omega)\|_X$ for $\omega \in \Omega$. Let

$$E(X) = \{ f \in L^0(X) : \widetilde{f} \in E \}.$$

Recall that the algebraic tensor product $E \otimes X$ is the subspace of $E(X)$ spanned by the functions of the form $u \otimes x$, $(u \otimes x)(\omega) = u(\omega)x$, where $u \in E$, $x \in X$.

A subset H of $E(X)$ is said to be *solid* whenever $\widetilde{f_1} \leq \widetilde{f_2}$ and $f_1 \in E(X)$, $f_2 \in H$ imply $f_1 \in H$. A linear topology τ on $E(X)$ is said to be *locally solid* if it has a local base at zero consisting of solid sets. A linear topology τ on $E(X)$ that is as the same time locally solid and locally convex will be called a *locally convex-solid* topology on $E(X)$. A seminorm ϱ on $E(X)$ is called *solid* if $\varrho(f_1) \leq \varrho(f_2)$ whenever $f_1, f_2 \in E(X)$ and $\widetilde{f_1} \leq \widetilde{f_2}$. It is known that a locally convex topology τ on $E(X)$ is locally convex-solid if and only if it is generated by some family of solid seminorms defined on $E(X)$ (see [9]). A locally solid topology τ on $E(X)$ is said to be a *Lebesgue topology* whenever for a net (f_α) in $E(X)$, $\widetilde{f_\alpha} \xrightarrow{(0)} 0$ in E implies $f_\alpha \xrightarrow{\tau} 0$ (see [16, Definition 2.2]).

Let (E, ξ) be a Hausdorff locally convex-solid function space. Then one can topologize the space $E(X)$ as follows (see [9]). Let $\{p_\alpha : \alpha \in \mathcal{A}\}$ be a family of Riesz seminorms on E that generates ξ. By putting

$$\bar{p}_\alpha(f) := p_\alpha(\widetilde{f}) \ \text{ for } f \in E(X) \ (\alpha \in \mathcal{A}),$$

we obtain a family $\{\bar{p}_\alpha : \alpha \in \mathcal{A}\}$ of solid seminorms on $E(X)$ that defines a Hausdorff locally convex-solid topology $\bar{\xi}$ on $E(X)$ (called the *topology associated with* ξ). Then $\bar{\xi}$ is a Lebesgue topology whenever ξ is a Lebesgue topology (see [9]).

Conversely, let τ be a Hausdorff locally convex-solid topology on $E(X)$ and let $\{\varrho_\alpha : \alpha \in \mathcal{A}\}$ be a family of solid seminorms on $E(X)$ that generates τ. By putting for a fixed $x_0 \in S_X$

$$\widetilde{\varrho}_\alpha(u) := \varrho_\alpha(u \otimes x_0) \quad \text{for } u \in E \ (\alpha \in \mathcal{A}),$$

we obtain a family $\{\widetilde{\varrho}_\alpha : \alpha \in \mathcal{A}\}$ of Riesz seminorms on E that defines a Hausdorff locally convex-solid topology $\widetilde{\tau}$ on E.

One can note that $\widetilde{\bar{\xi}} = \xi$ and $\bar{\widetilde{\tau}} = \tau$ (see [9]). Thus every Hausdorff locally convex-solid topology τ on $E(X)$ can be represented as the topology associated with some Hausdorff locally convex-solid topology $\xi\,(=\widetilde{\tau}\,)$ on E. In particular, for a Banach function space $(E, \|\cdot\|_E)$ the space $E(X)$ provided with the norm $\|f\|_{E(X)} := \|\widetilde{f}\|_E$ is usually called a *Köthe-Bochner space*.

A linear functional F on $E(X)$ is said *order continuous* whenever for a net (f_α) in $E(X)$, $\widetilde{f}_\alpha \xrightarrow{(0)} 0$ in E implies $F(f_\alpha) \longrightarrow 0$. The set consisting of all order continuous linear functionals on $E(X)$ will be denoted by $E(X)^\sim_n$ and called the *order continuous dual* of $E(X)$ (see [2], [13]). Since we assume that $\operatorname{supp} E' = \Omega$, $E(X)^\sim_n$ separates points of $E(X)$. A Hausdorff locally convex-solid topology τ on $E(X)$ has the Lebesgue property if and only if $E(X)^*_\xi \subset E(X)^\sim_n$ (see [16, Theorem 2.4]).

To present the integral representation of $E(X)^\sim_n$ we now recall terminology concerning the spaces of weak*-measurable functions (see [5], [2], [4], [3]). For a given function $g : \Omega \to X^*$ and $x \in X$ we denote by g_x the real function on Ω defined by $g_x(\omega) = g(\omega)(x)$ for $\omega \in \Omega$. A function $g : \Omega \to X^*$ is said to be weak*-measurable if the functions g_x are measurable for each $x \in X$. We shall say the two weak*-measurable functions g_1, g_2 are weak*-equivalent whenever $g_1(\omega)(x) = g_2(\omega)(x)$ μ-a.e. for each $x \in X$. Let $L^0(X^*, X)$ be the set of weak*-equivalence classes of all weak*-measurable functions $g : \Omega \to X^*$. Following [2], [4] one can define the so-called *abstract norm* $\vartheta : L^0(X^*, X) \to L^0$ by $\vartheta(g) := \sup\{|g_x| : x \in B_X\}$. Then for $f \in L^0(X)$ and $g \in L^0(X^*, X)$ the function $\langle f, g\rangle :\ \Omega \longrightarrow \mathbb{R}$ defined by $\langle f, g\rangle(\omega) := \langle f(\omega), g(\omega)\rangle$ is measurable and $|\langle f, g\rangle| \le \widetilde{f}\,\vartheta(g)$. Moreover, $\vartheta(g) = \widetilde{g}$ for $g \in L^0(X^*)$. Let

$$E'(X^*, X) = \{\, g \in L^0(X^*, X) : \vartheta(g) \in E'\,\}.$$

Due to A. V. Bukhvalov (see [2, Theorem 4.1]) $E(X)^\sim_n$ can be identified with $E'(X^*, X)$ through the mapping: $E'(X^*, X) \ni g \mapsto F_g \in E(X)^\sim_n$, where

$$F_g(f) = \int_\Omega \langle f(\omega), g(\omega)\rangle \,\mathrm{d}\mu \quad \text{for all } f \in E(X)\,.$$

Clearly $E'(X^*) \subset E'(X^*, X)$. Moreover, the identities:

$$E'(X^*) = E'(X^*, X) \qquad \text{and} \qquad E(X)^\sim_n = \{F_g : g \in E'(X^*)\}$$

hold whenever the Banach space X^* has the Radon-Nikodym property (see [8, Chap. 3.1], [3, Theorem 3.5]). Recall that a Banach space X is called an *Asplund space* if every real-valued continuous convex function on an open convex domain in X is Fréchet differentiable on a dense G_δ subset of its domain (see [12], [18]). The following theorem provides a characterization of Banach spaces X whose duals have the Radon-Nikodym property (see [12], [18], [8, p. 213]).

Theorem 1.1. *For a Banach space X the following statements are equivalent:*

(*i*) *X^* has the Radon-Nikodym property.*

(*ii*) *X is an Asplund space.*

In the theory of function spaces the Mackey topology $\tau(E, E')(= \tau(E, E^\sim_n))$ on E is of importance (see [6]). It is well known that $\tau(E, E')$ is the finest Hausdorff locally convex-solid topology on E with the Lebesgue property.

In this paper we consider the topology $\overline{\tau(E,E')}$ on $E(X)$ associated with $\tau(E,E')$, i.e., $\overline{\tau(E,E')}$ is defined by solid seminorms:

$$\overline{p_M}(f) := p_M(\widetilde{f}) = \sup\Big\{ \int_\Omega \widetilde{f}(\omega)|v(\omega)|d\mu : v \in M\Big\} \text{ for } f \in E(X),$$

where M runs over the family of all absolutely convex and solid $\sigma(E',E)$-compact sets in E'. The basic properties of $\overline{\tau(E,E')}$ can be summarized in the following theorem (see [16, Theorem 2.5, Theorem 2.6]).

Theorem 1.2. *Let E be an ideal of L^0 and X be a Banach space. Then*

(*i*) $\overline{\tau(E,E')}$ *is the finest locally convex-solid topology on $E(X)$ with the Lebesgue property.*

(*ii*) $(E(X),\overline{\tau(E,E')}\,)^* = E(X)_n^\sim$, *and hence* $\overline{\tau(E,E')} \subset \tau(E,(X),E(X)_n^\sim)$.

(*iii*) $(E(X),\overline{\tau(E,E')})$ *is complete whenever E is perfect.*

We can state the following *Mackey problem*: under which conditions on X is $\overline{\tau(E,E')}$ a strongly Mackey topology on $E(X)$? Equivalently, when every relatively countably $\sigma(E(X)_n^\sim,E(X))$-compact sets in $E(X)_n^\sim$ are $\overline{\tau(E,E')}$-equicontinuous? We show that it holds whenever X is an Asplund space.

In Section 2 we obtain some results concerning duality of the space $E(X)$ that will be needed. In Section 3 we investigate the relationship between the $\overline{\tau(E,E')}$-equicontinuos subsets of $E(X)_n^\sim$ and relatively $\sigma(E',E)$-compact subsets of E'. Moreover, in case when X is an Asplund space, we obtain a characterization of relatively countably $\sigma(E'(X^*), E(X))$-compact subsets of $E'(X^*)$. As a consequence, in Section 4 in case X is an Asplund space, we obtain that $\overline{\tau(E,E')}$ is a strongly Mackey topology; hence $\overline{\tau(E,E')} = \tau(E(X),E'(X^*))$. Moreover, it is shown that the space $E'(X^*)$ is $\sigma(E'(X^*),E(X))$-sequentially complete. We characterize $\overline{\tau(L^\infty,L^1)}$ on $L^\infty(X)$ and $\overline{\tau(L^\Phi,L^{\Phi^*})}$ on the Orlicz-Bochner spaces $L^\Phi(X)$. In Section 5 we examine the space $\mathcal{L}_\tau(E(X),Y)$ of all $(\tau(E(X),E(X)_n^\sim),\|\cdot\|_Y)$-continuous linear operators from $E(X)$ to a Banach space $(Y,\|\cdot\|_Y)$, equipped with the weak operator topology (briefly WOT) and the strong operator topology (briefly SOT). It is shown that if E is perfect and X is an Asplund space, then the space $\mathcal{L}_\tau(E(X),Y)$ is WOT-sequentially complete, and every SOT-compact subset of $\mathcal{L}_\tau(E(X),Y)$ is $(\tau(E(X),E'(X^*)),\|\cdot\|_Y)$-equicontinuous. As an application, a Vitali-Hahn-Saks type theorem for $\mathcal{L}_\tau(E(X),Y)$ is obtained.

2. Duality of vector-valued function spaces

In this section we establish terminology and prove a general result concerning duality of vector-valued function spaces $E(X)$ (see [2], [3], [4], [14] for more details). For a linear functional F on $E(X)$ let us put

$$|F|(f) = \sup\{\,|F(h)| : h \in E(X), \widetilde{h} \le \widetilde{f}\,\} \text{ for } f \in E(X).$$

The set

$$E(X)^\sim = \{F \in E(X)^\# : |F|(f) < \infty \text{ for all } f \in E(X)\}$$

will be called the *order dual* of $E(X)$ (here $E(X)^\#$ denotes the algebraic dual of $E(X)$). For $F_1,F_2 \in E(X)^\sim$ we will write $|F_1| \le |F_2|$ whenever $|F_1|(f) \le |F_2|(f)$ for all

$f \in E(X)$. A subset A of $E(X)^\sim$ is said to be *solid* whenever $|F_1| \le |F_2|$ with $F_1 \in E(X)^\sim$ and $F_2 \in A$ imply $F_1 \in A$. A linear subspace I of $E(X)^\sim$ will be called an *ideal* of $E(X)^\sim$ whenever I is solid. It is known that if τ is a locally solid topology on $E(X)$, then $(E(X),\tau)^*$ is an ideal of $E(X)^\sim$ (see [14, Theorem 3.2]). Every subset A of $E(X)^\sim$ is contained in the smallest (with respect to inclusion) solid set called the solid hull of A and denoted by $S(A)$. One can note that $S(A) = \{F \in E(X)^\sim : |F| \le |G|$ for some $G \in A\}$. Let $F \in E(X)^\sim$ and $x_0 \in S_X$ be fixed. For $u \in E^+$ let us set:

$$\varphi_F(u) := |F|(u \otimes x_0) = \sup\{|F(h)| : \ h \in E(X),\ \widetilde{h} \le u\}.$$

Then $\varphi_F : E^+ \to \mathbb{R}^+$ is an additive mapping and φ_F has a unique positive extension to a linear mapping from E to $\mathbb{R}$ (denoted by φ_F again) and given by

$$\varphi_F(u) := \varphi_F(u^+) - \varphi_F(u^-) \ \text{ for all } u \in E.$$

(see [4, Lemma 7]). Observe that $\varphi_F \in E^\sim$.

Assume now that τ is a Hausdorff locally convex-solid topology on $E(X)$. Then $E(X)^*_\tau$ is an ideal of $E(X)^\sim$ (see [13, Theorem 3.2]), and the pair $\langle E(X), E(X)^*_\tau\rangle$, under its natural duality $\langle f, F\rangle = F(f)$ is a solid dual system (see [14, p. 206]). For a subset A of $E(X)$ and a subset B of $E(X)^*_\tau$ let us put

$$A^0 = \{F \in E(X)^*_\tau : |\langle f, F\rangle| \le 1 \ \text{ for all } f \in A\},$$

$$^0B = \{f \in E(X) : |\langle f, F\rangle| \le 1 \ \text{ for all } F \in B\}.$$

Now given a Hausdorff locally convex-solid function space (E,ξ) we characterize $\overline{\xi}$-equicontinuous subsets of $E(X)^*_{\overline{\xi}}$.

Proposition 2.1. *Let ξ be a Hausdorff locally convex-solid topology on E. Then for a subset Z of $E(X)^*_{\overline{\xi}}$ the following statements are equivalent:*

(*i*) *Z is $\overline{\xi}$-equicontinuous.*
(*ii*) *$\mathrm{conv}\,(S(Z))$ is $\overline{\xi}$-equicontinuous.*
(*iii*) *$S(Z)$ is $\overline{\xi}$-equicontinuous.*
(*iv*) *$\{\varphi_F : F \in Z\}$ in E^*_ξ is ξ-equicontinuous.*

Proof. (*i*) $\Longrightarrow$ (*ii*) Let Z be $\overline{\xi}$-equicontinuous. Then there exists a convex-solid $\overline{\xi}$-neighborhood V of 0 such that $Z \subset V^0$. Hence $\mathrm{conv}\,(S(Z)) \subset \mathrm{conv}\,(S(V^0)) = \mathrm{conv}\,(V^0) = V^0$ (see [14, Theorem 3.3]), and this means that $\mathrm{conv}\,(S(Z))$ is still $\overline{\xi}$-equicontinuous.

(*ii*) $\Longrightarrow$ (*iii*) It is obvious.

(*iii*) $\Longrightarrow$ (*iv*) Assume that the subset $S(Z)$ of $E(X)^*_{\overline{\xi}}$ is $\overline{\xi}$-equicontinuous. Let $\{p_\alpha : \alpha \in \mathcal{A}\}$ be a family of Riesz seminorms on E that generates ξ. Given $\varepsilon > 0$ there exist $\alpha_1,\dots,\alpha_n \in \mathcal{A}$ and $\eta > 0$ such that $\sup\{|F(f)| : F \in S(Z)\} \le \varepsilon$ whenever $\overline{p}_{\alpha_i}(f) = p_{\alpha_i}(\widetilde{f}) \le \varepsilon$, for $i = 1,2,\dots,n$. To show that $\{\varphi_F : F \in Z\}$ is ξ-equicontinuous, it is enough to show that $\sup\{|\varphi_F(u)| : F \in Z\} \le \varepsilon$ whenever $p_{\alpha_i}(u) \le \eta$ for $i = 1,2,\dots,n$. Indeed, let $u \in E$, $x_0 \in S_X$ and $p_{\alpha_i}(u) = p_{\alpha_i}(|u|) = \overline{p}_{\alpha_i}(u \otimes x_0) \le \eta$ for $i = 1,2,\dots,n$, so $\sup\{|F(u \otimes x_0)| : F \in S(Z)\} \le \varepsilon$. Hence in view of [14, Lemma 2.1] we obtain that $\sup\{|F(u \otimes x_0)| : F \in S(Z)\} = \sup\{\varphi_F(|u|) : F \in Z\}$, and since $|\varphi_F(u)| \le \varphi_F(|u|)$, the proof is complete.

$(iv) \Longrightarrow (i)$ Assume that the set $\{\varphi_F : F \in Z\}$ in E^*_ξ is ξ-equicontinuous. Let $\{p_\alpha : \alpha \in \mathcal{A}\}$ be a family of Riesz seminorms on E that generates ξ. Given $\varepsilon > 0$ there exist $\alpha_1, \dots, \alpha_n \in \mathcal{A}$ and $\eta > 0$ such that $\{|\varphi_F(u)| : F \in Z\} \leq \varepsilon$ whenever $u \in E$ and $p_{\alpha_i}(u) \leq \eta$ for $i = 1, 2, \dots, n$. Hence, for $f \in E(X)$ with $\overline{p}_{\alpha_i}(f) = p_{\alpha_i}(\widetilde{f}) \leq \eta$ for $i = 1, 2, \dots, n$ we get $\sup\{|\varphi_F(\widetilde{f})| : F \in Z\} \leq \varepsilon$. But $|F(f)| \leq |F|(f) = \varphi_F(\widetilde{f})$, so $\sup\{|F(f)| : F \in Z\} \leq \varepsilon$, and this means that Z in $\overline{\xi}$-equicontinuous. □

3. Weak compactness in the order continuous dual of vector-valued function spaces

In this section we examine the relationship between the $\overline{\tau(E, E')}$-equicontinuous subsets of $E(X)^\sim_n$, $\tau(E, E')$-equicontinuous subsets of $E^\sim_n$ and relatively $\sigma(E, E')$-compact subsets of E'. As an application, in case X is an Asplund space, we obtain a characterization of $\sigma(E'(X^*), E(X))$-compact subset of $E'(X^*)$.

We start by recalling a characterization of $\sigma(E', E)$-compact subsets of E'. Assume that M is a $\sigma(E', E)$-bounded subset of E'. Then M is also $|\sigma|(E', E)$-bounded, so one can define a Riesz seminorm p_M on E by

$$p_M(u) = \sup_{v \in M} \int_\Omega |u(\omega)v(\omega)| d\mu.$$

Proposition 3.1. *Let E be an ideal of L^0. Then for a $\sigma(E', E)$-bounded subset M of E' the following statements are equivalent:*

- (i) *For every $u \in E$ the set $\{uv : v \in M\}$ in L^1 is uniformly integrable.*
- (ii) *p_M is absolutely continuous, i.e., $p_M(\chi_{A_n} u) \longrightarrow 0$ for every $u \in E$, as $A_n \searrow_\mu \emptyset$ (i.e., $A_n \downarrow$ and $\mu(\bigcap_{n=1}^\infty A_n) = 0$).*
- (iii) *p_M is σ-order continuous.*
- (iv) *$S(M)$ is relatively $\sigma(E', E)$-compact.*
- (v) *M is relatively $\sigma(E', E)$-compact.*
- (vi) *$\operatorname{conv}(S(M))$ is relatively $\sigma(E', E)$-compact.*

Proof. $(i) \Longleftrightarrow (ii)$ It follows from the definition of uniform integrability in L^1.

$(ii) \Longleftrightarrow (iii)$ It is obvious.

$(iii) \Longleftrightarrow (iv)$ See [1, Theorem 20.3].

$(iv) \Longleftrightarrow (v)$ It is obvious.

$(v) \Longleftrightarrow (vi)$ See [1, Corollary 20.10]. □

Now we recall some terminology and prove some technical results concerning the space $E'(X^*, X)$. In view of [2, Theorem 4.1] for $g \in E'(X^*, X)$ we have:

$$|F_g|(f) = \int_\Omega \widetilde{f}(\omega)\vartheta(g)(\omega) d\mu = \varphi_{\vartheta(g)}(\widetilde{f}) \text{ for all } f \in E(X),$$

and

$$\varphi_{F_g}(u) = |F_g|(u \otimes x_0) = \varphi_{\vartheta(g)}(u) \text{ for } u \in E^+.$$

Lemma 3.2. *For $g_1, g_2 \in E'(X^*, X)$ the following statements are equivalent.*

(*i*) $\vartheta(g_1) \leq \vartheta(g_2)$.
(*ii*) $|F_{g_1}|(f) \leq |F_{g_2}|(f)$ *for all* $f \in E(X)$.
(*iii*) $\varphi_{\vartheta(g_1)}(u) \leq \varphi_{\vartheta(g_2)}(u)$ *for all* $u \in E^+$.

Proof. (*i*) $\Longleftrightarrow$ (*ii*) See [14, Corollary 2.5].

(*ii*) $\Longrightarrow$ (*iii*) For $u \in E^+$ we have

$$\varphi_{\vartheta(g_1)}(u) = \varphi_{F_{g_1}}(u) = |F_{g_1}|(u \otimes x_0) \leq |F_{g_2}|(u \otimes x_0) = \varphi_{F_{g_2}}(u) = \varphi_{\vartheta(g_2)}(u).$$

(*iii*) $\Longrightarrow$ (*ii*) For $f \in E(X)$ we have

$$|F_{g_1}|(f) = \varphi_{\vartheta(g_1)}(\widetilde{f}) \leq \varphi_{\vartheta(g_2)}(\widetilde{f}) = |F_{g_2}|(f).$$

□

A subset C of $E'(X^*, X)$ is said to be *solid* if $\vartheta(g_1) \leq \vartheta(g_2)$ with $g_1 \in E'(X^*, X)$ and $g_2 \in C$ imply $g_1 \in C$. Every subset A of $E'(X^*, X)$ is contained in the smallest (with respect to inclusion) solid set in $E'(X^*, X)$ called the *solid hull* of A and denoted by $S(A)$. One can note that

$$S(A) = \{g \in E'(X^*, X) : \vartheta(g) \leq \vartheta(h) \text{ for some } h \in A\}.$$

Lemma 3.3. *Let* C *be a subset of* $E'(X^*, X)$ *and* $F_C = \{F_g : g \in C\}$. *Then*

$$\operatorname{conv}(S(F_C)) = F_{\operatorname{conv}(S(C))}.$$

Proof. Assume that $F \in \operatorname{conv}(S(F_C))$. Then $F = \Sigma_{i=1}^n \alpha_i F_{g_i} = F_{\Sigma_{i=1}^n \alpha_i g_i}$, where $g_1 \in E'(X^*, X)$ and $\alpha_i \geq 0$ for $i = 1, 2, \ldots, n$ with $\Sigma_{i=1}^n \alpha_i = 1$, and $|F_{g_i}| \leq |F_{g_i'}|$ for some $g_i' \in C$ and $i = 1, 2, \ldots, n$. In view of Lemma 3.2 $\vartheta(g_i) \leq \vartheta(g_i')$, i.e., $g_i \in S(C)$ for $i = 1, 2, \ldots, n$ and $\Sigma_{i=1}^n \alpha_i g_i \in \operatorname{conv}(S(C))$. This means that $F \in F_{\operatorname{conv}(S(C))}$.

Assume that $F \in F_{\operatorname{conv}(S(C))}$. Then $F = F_{\Sigma_{i=1}^n \alpha_i g_i} = \Sigma_{i=1}^n \alpha_i F_{g_i}$, where $g_i \in E'(X^*, X)$ and $\alpha_i \geq 0$ for $i = 1, 2, \ldots, n$ with $\Sigma_{i=1}^n \alpha_i = 1$, and $\vartheta(g_i) \leq \vartheta(g_i')$ for some $g_i' \in C$ and $i = 1, 2, \ldots, n$. By Lemma 3.2, $|F_{g_i}|(f) \leq |F_{g_i'}|(f)$ for all $f \in E(X)$ and $i = 1, 2, \ldots, n$, so $F \in \operatorname{conv}(S(F_C))$. □

Now we ready to characterize $\overline{\tau(E, E')}$-equicontinuous subsets of $E(X)_n^\sim$.

Proposition 3.4. *Let* E *be an ideal of* L^0 *and let* X *be a Banach space. Then for a subset* C *of* $E'(X^*, X)$ *and a subset* $\vartheta(C)$ $(= \{\vartheta(g) : g \in C\})$ *of* E' *the following statements are equivalent:*

(*i*) $\{F_g : g \in C\}$ *in a* $\overline{\tau(E, E')}$ *-equicontinuous subset of* $E(X)_n^\sim$.
(*ii*) $\{F_g : g \in \operatorname{conv} S(C)\}$ *is a* $\overline{\tau(E, E')}$ *-equicontinuous subset of* $E(X)_n^\sim$.
(*iii*) $\{F_g : g \in S(C)\}$ *is a* $\overline{\tau(E, E')}$ *-equicontinuous subset of* $E(X)_n^\sim$.
(*iv*) $\{\varphi_{\vartheta(g)} : g \in C\}$ *is a* $\overline{\tau(E, E')}$ *-equicontinuous subset of* $E_n^\sim$.
(*v*) $\{\varphi_{\vartheta(g)} : g \in C\}$ *is a relatively* $\sigma(E_n^\sim, E)$*-compact subset of* $E_n^\sim$.
(*vi*) $\vartheta(C)$ *is a relatively* $\sigma(E', E)$*-compact subset of* E'.

Proof. $(i) \iff (ii) \iff (iii) \iff (iv)$ It follows from Proposition 2.1 and Lemma 3.3.

$(iv) \iff (v) \iff (vi)$ It is obvious (see [1, p. 142]).

$(vi) \iff (iv)$ Assume that $\vartheta(C)$ is a relatively $\sigma(E', E)$-compact subset of E'. Hence by Proposition 3.1, $\overline{\operatorname{abs\,conv}\vartheta(C)}$ (the closure taken for $\sigma(E', E)$) is absolutely convex $\sigma(E', E)$-compact. It follows that the set $\{\varphi_v : v \in \overline{\operatorname{abs\,conv}\vartheta(C)}\}$ is $\tau(E, E_n^\sim)$-equicontinuous, so $\{\varphi_{\vartheta(g)} : g \in C\}$ is also $\tau(E, E')$-equicontinuous. □

As an application of Proposition 3.4 we get:

Corollary 3.5. *Let E be an ideal of L^0 and let X be a Banach space. Let C be a subset of $E'(X^*, X)$ such that the set $\vartheta(C)$ in E' is relatively $\sigma(E', E)$-compact. Then the set* $\operatorname{conv}(S(C))$ *in $E'(X^*, X)$ is relatively $\sigma(E'(X^*, X), E(X))$-compact.*

Proof. In view of Proposition 3.4, $\{F_g : g \in \operatorname{conv}(S(C))\}$ is a $\overline{\tau(E, E')}$-equicontinuous subset of $E(X)_n^\sim$, so it is relatively $\sigma(E(X)_n^\sim, E(X))$-compact. This means that $\operatorname{conv}(S(C))$ is a relatively $\sigma(E'(X^*, X), E(X))$-compact subset of $E'(X^*, X)$. □

Now we are in position to state a characterization of $\sigma(E'(X^*), E(X))$-compact subsets of $E'(X^*)$ whenever X is an Asplund space.

Proposition 3.6. *Let E be an ideal of L^0 and X be an Asplund space. Then for a subset C of $E'(X^*)$ the following statements are equivalent:*

- (i) *C is relatively $\sigma(E'(X^*), E(X))$-compact.*
- (ii) *C is relatively countably $\sigma(E'(X^*), E(X))$-compact.*
- (iii) *$\widetilde{C}$ is relatively $\sigma(E', E)$-compact.*
- (iv) $\operatorname{conv}(S(C))$ *is relatively $\sigma(E'(X^*), E(X))$-compact.*
- (v) $\operatorname{abs\,conv}(C)$ *is relatively $\sigma(E'(X^*), E(X))$-compact.*

Proof. $(i) \Longrightarrow (ii)$ It is obvious.

$(ii) \Longrightarrow (iii)$ Assume that C is relatively countably $\sigma(E'(X^*), E(X))$-compact. In view of Proposition 3.1 it is enough to show that the set $\widetilde{C}$ is $\sigma(E', E)$-bounded and for every $u \in E$ the set $\{u\widetilde{g} : g \in C\}$ in L^1 is uniformly integrable. Clearly C is $\sigma(E'(X^*), E(X))$-bounded (see [23, Problem 6-4-106, p. 86]). Hence $\{F_g : g \in C\}$ is a $\sigma(E'(X^*), E(X))$-bounded subset of $E(X)_n^\sim$. We shall show that $\widetilde{C}$ is a $|\sigma|(E', E)$-bounded subset of E'. Indeed, let $u_0 \in E$ and $f_0 = u_0 \otimes x_0$ for some fixed $x_0 \in S_X$. Hence by [15, Theorem 2.1] we have

$$\begin{aligned} \sup\Big\{ \int_\Omega |u_0(\omega)|\, \widetilde{g}(\omega) d\mu : g \in C \Big\} &= \sup\Big\{ \int_\Omega \widetilde{f_0}(\omega)\widetilde{g}(\omega) d\mu : g \in C \Big\} \\ &= \sup\{|F_g|(f_0) : g \in C\} < \infty. \end{aligned}$$

Now we shall show that for every $u \in E$ the set $\{u\,\widetilde{g} : g \in C\}$ in $L^1(\mu)$ is uniformly integrable. Assume on the contrary that there exists $u_0 \in E^+$ such that the set $\{u_0\,\widetilde{g} : g \in C\}$ is not uniformly integrable. For each $g \in C$ let us put

$$\nu_g(A) = \int_A u_0(\omega)\widetilde{g}(\omega) d\mu \quad \text{for all } A \in \Sigma.$$

Then for every $g \in C$, $\nu_g : \Sigma \to [0,\infty)$ is μ-continuous countably additive measure but the family $\{\nu_g : g \in C\}$ is not uniformly μ-continuous. In view of [17, Proposition 2.2] there exist a pairwise disjoint sequence (B_n) in Σ, a sequence (g_n) in C and number $\varepsilon_0 > 0$ such that for every $n \in \mathbb{N}$:

$$\nu_{g_n}(B_n) = \int_{B_n} u_0(\omega)\widetilde{g}_n(\omega)d\mu > \varepsilon_0. \tag{1}$$

Since $\chi_{B_n} u_0 g_n \in L^1(X^*) \subset L^1(X^*, X)$, in view of [2, Theorem 1.1(3), p. 24] we have

$$\begin{aligned}\nu_{g_n}(B_n) &= \|\chi_{B_n} u_0 g_n\|_{L^1(X^*,X)} \\ &= \sup\Big\{\Big|\int_{B_n} \langle f(\omega), u_0(\omega)g_n(\omega)\rangle \, d\mu\Big| : f \in L^\infty(X), \|f\|_{L^\infty(X)} \le 1\Big\}.\end{aligned}$$

Hence by (1) one can choose a sequence (f_n) in $L^\infty(X)$ with $\|f_n\|_{L^\infty(X)} \le 1$, $\chi_{\Omega\setminus B_n} f_n = 0$ and such that

$$\Big|\int_{B_n} \langle f_n(\omega), u_0(\omega)g_n(\omega)\rangle \, d\mu\Big| > \varepsilon_0 \quad \text{for all } n \in \mathbb{N}. \tag{2}$$

Let

$$f_0(\omega) = \begin{cases} f_n(\omega) & \text{for } \omega \in B_n, \; n = 1, 2, \ldots, \\ 0 & \text{for } \omega \in \Omega \setminus \bigcup_{n=1}^\infty B_n. \end{cases}$$

Then $f_0 \in L^\infty(X)$, $\chi_{B_n} f_0 = f_n$ for $n \in \mathbb{N}$ and $u_0 f_0 \in E(X)$. Note that for every $g \in E'(X^*)$ we have

$$\begin{aligned}&\sum_{n=1}^\infty \Big|\int_\Omega \langle u_0(\omega)f_n(\omega), g(\omega)\rangle d\mu\Big| \\ \le\ &\sum_{n=1}^\infty \int_{B_n} |\langle u_0(\omega)f_n(\omega), g(\omega)\rangle| d\mu \;\le\; \sum_{n=1}^\infty \int_{B_n} u_0(\omega)\widetilde{f}_n(\omega)\widetilde{g}(\omega)d\mu \\ =\ &\sum_{n=1}^\infty \int_{B_n} u_0(\omega)\widetilde{f}_0(\omega)\widetilde{g}(\omega)d\mu \;=\; \int_\Omega u_0(\omega)\widetilde{f}_0(\omega)\widetilde{g}(\omega)d\mu < \infty.\end{aligned}$$

Hence we can define a linear operator

$$T_{f_0} : E'(X^*) \ni g \mapsto \Big(\int_\Omega \langle u_0(\omega)f_n(\omega), g(\omega)\rangle d\mu\Big)_{n=1}^\infty \in l^1.$$

Given $(s_n) \in l^\infty$ we define

$$h(\omega) = \begin{cases} s_n u_0(\omega)f_n(\omega) & \text{if } \omega \in B_n, \; n \in \mathbb{N}, \\ 0 & \text{if } \omega \in \Omega \setminus \bigcup_{n=1}^\infty B_n. \end{cases}$$

Clearly $h \in E(X)$ and

$$\begin{aligned}\Big|\sum_{n=1}^\infty \Big(s_n \int_\Omega \langle u_0(\omega)f_n(\omega), g(\omega)\rangle d\mu\Big)\Big| &= \Big|\sum_{n=1}^\infty \int_{B_n} \langle u_0(\omega)s_n f_n(\omega), g(\omega)\rangle d\mu)\Big| \\ &= \Big|\int_\Omega \langle h(\omega), g(\omega)\rangle d\mu\Big|.\end{aligned}$$

It follows that the operator T_{f_0} is $(\sigma(E'(X^*), E(X)), \sigma(l^1, l^\infty))$-continuous. Hence the set $\{T_{f_0}(g) : g \in C\}$ is relatively countably $\sigma(l^1, l^\infty)$-compact in l^1, and by the Eberlein theorem it is also relatively sequentially $\sigma(l^1, l^\infty)$-compact. On the other hand, in view of Schur's theorem the set $\{T_{f_0}(g) : g \in C\}$ is relatively compact in the Banach space l^1, and it follows that for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\sup\Big\{\sum_{n=n_\varepsilon}^{\infty}\Big|\int_\Omega \langle u_0(\omega) f_n(\omega), g(\omega)\rangle d\mu\Big| : g \in C\Big\} \le \varepsilon.$$

In particular, it follows that for $n \ge n_{\varepsilon_0}$ we get

$$\Big|\int_\Omega \langle u_0(\omega) f_n(\omega), g_n(\omega)\rangle d\mu\Big| \le \varepsilon_0$$

which contradicts (2).

$(iii) \Longrightarrow (iv)$ It follows from Corollary 3.3.

$(iv) \Longrightarrow (v)$ It is obvious because $\operatorname{abs\,conv}(C) \subset \operatorname{conv}(S(C))$.

$(v) \Longrightarrow (i)$ It is obvious. □

Recall that a locally convex space (L, ξ) is said to have the *convex compactness property* if the closed absolutely convex hull of every ξ-compact subset of L is still ξ-compact (see [22, p. 156], [23, Definition 9-2-8]).

As a consequence of Proposition 3.6 we have:

Corollary 3.7. *Let E be an ideal of L^0 and X be an Asplund space. Then the space $(E'(X^*), \sigma(E'(X^*), E(X)))$ has the convex compactness property.*

4. Strongly Mackey topologies on vector-valued function spaces

Now we are in position to state our main result:

Theorem 4.1. *Let E be an ideal of L^0 and let X be an Asplund space. Then $\overline{\tau(E, E')}$ is a strongly Mackey topology on $E(X)$; hence we have*

$$\overline{\tau(E, E')} = \tau(E(X), E'(X^*)).$$

Proof. We have $E(X)_n^\sim = \{F_g : g \in E'(X^*)\}$ because X is an Asplund space. Assume that for $C \subset E'(X^*)$ the set $\{F_g : g \in C\}$ is a relatively countably $\sigma(E(X)_n^\sim, E(X))$-compact subset of $E(X)_n^\sim$, i.e., C is a relatively countably $\sigma(E'(X^*), E(X))$-compact subset of $E'(X^*)$. Then by Proposition 3.6 $\{\widetilde{g} : g \in C\}$ is a relatively $\sigma(E', E)$-compact subset of E'. Hence, in view of Proposition 3.4, $\{F_g : g \in C\}$ is a $\overline{\tau(E, E')}$-equicontinuous subset of $E(X)_n^\sim$. This means that $\overline{\tau(E, E')}$ is a strongly Mackey topology on $E(X)$. □

As a consequence of Theorem 4.1 we get:

Corollary 4.2. *Let E be a perfect ideal of L^0 and X be an Asplund space. Then the space $E'(X^*)$ is $\sigma(E'(X^*), E(X))$-sequentially complete.*

Proof. In view of Theorem 4.1, $\tau(E(X), E(X)_n^\sim) = \tau(E(X), E'(X^*)) = \overline{\tau(E, E')}$, and it follows that $\tau(E(X), E(X)_n^\sim)$ is a locally solid topology on $E(X)$. Hence by [17, Theorem 3.6] the space $E(X)_n^\sim$ is $\sigma(E(X)_n^\sim, E(X))$-sequentially complete. This means that $E'(X^*)$ is $\sigma(E'(X^*), E(X))$-sequentially complete. □

Now we consider the topology $\overline{\tau(L^\infty, L^1)}$ on $L^\infty(X)$ associated with the Mackey topology $\tau(L^\infty, L^1)$ on L^∞. It is known that $\overline{\tau(L^\infty, L^1)}$ coincides with the natural *mixed topology* $\gamma[\mathcal{T}_\infty(X), \mathcal{T}_0(X)_{|L^\infty(X)}]$ (briefly $\gamma_{L^\infty(X)}$) (see [16, Theorem 4.2]). Here $\mathcal{T}_\infty(X)$ stands for the $\|\cdot\|_\infty$-norm topology on $L^\infty(X)$, and $\mathcal{T}_0(X)$ denotes the topology of the F-norm $\|\cdot\|_0$ on $L^0(X)$ that generates convergence in measure on sets of finite measure. Then $\gamma_{L^\infty(X)}$ is the finest locally convex-topology on $L^\infty(X)$ which agrees with $\mathcal{T}_0(X)$ on $\|\cdot\|_\infty$-bounded sets in $L^\infty(X)$ (see [24, 2.2.2]). This means that $(L^\infty(X), \gamma_{L^\infty(X)})$ is a *generalized DF-space* (see [20]). In particular, by Theorem 4.1 and Corollary 4.2 we get:

Corollary 4.3. *Assume that X is an Asplund space. Then the mixed topology $\gamma_{L^\infty(X)}$ is a strongly Mackey topology; hence we have:*

$$\gamma_{L^\infty(X)} = \overline{\tau(L^\infty, L^1)} = \tau(L^\infty(X), L^1(X^*)).$$

Moreover, the space $L^1(X^*)$ is $\sigma(L^1(X^*), L^\infty(X))$-sequentially complete.

Now we consider the topology $\overline{\tau(L^\Phi, L^{\Phi^*})}$ on the Orlicz-Bochner space $L^\Phi(X)$ associated with the Mackey topology $\tau(L^\Phi, L^{\Phi^*})$ on the Orlicz space L^Φ. For this purpose we first recall some terminology (see [19]). By a Young function we mean here a continuous convex mapping $\Phi : [0,\infty) \to [0,\infty)$ that vanishes only at 0 and $\Phi(t)/t \to 0$ as $t \to 0$, $\Phi(t)/t \to \infty$ as $t \to \infty$. The Orlicz space $L^\Phi = \{u \in L^0 : \int_\Omega \Phi(\lambda|u(\omega)|)d\mu < \infty$ for some $\lambda > 0\}$ can be equipped with a complete topology $\mathcal{T}_\Phi$ of the Riesz norm $\|u\|_\Phi := \inf\{\lambda > 0 : \int_\Omega \Phi(|u(\omega)|/\lambda)d\mu \leq 1\}$. It is known that $(L^\Phi)' = L^{\Phi^*}$, where Φ^* stands for the Young function complementary to Φ in the sense of Young. The Orlicz-Bochner space $L^\Phi(X) = \{f \in L^0(X) : \widetilde{f} \in L^\Phi\}$ can be equipped with the complete topology $\mathcal{T}_\Phi(X)$ of the norm $\|f\|_{L^\Phi(X)} := \|\widetilde{f}\|_\Phi$ for $f \in L^\Phi(X)$, i.e., $\mathcal{T}_\Phi(X) = \overline{\mathcal{T}_\Phi}$. Note that $L^\Phi(X)_n^\sim = \{F_g : g \in L^{\Phi^*}(X^*, X)\}$.

For $\varepsilon > 0$ let $U_\Phi(\varepsilon) = \{f \in L^\Phi(X) : \int_\Omega \Phi(\widetilde{f}(\omega))d\mu \leq \varepsilon\}$. Then the family of all sets of the form: $\bigcup_{n=1}^\infty(\Sigma_{i=1}^n U_\Phi(\varepsilon_i))$, where (ε_n) is a sequence of positive numbers, forms a local base at 0 (consisting of solid and convex subsets of $L^\Phi(X)$) for a locally convex topology $\mathcal{T}_\Phi^\wedge(X)$, and called the *modular topology* (see [9]). In particular, we will write $\mathcal{T}_\Phi^\wedge$ instead of $\mathcal{T}_\Phi^\wedge(\mathbb{R})$. Then $\mathcal{T}_\Phi^\wedge = \tau(L^\Phi, L^{\Phi^*})$ (see [13, Theorem 1.3]). The basic properties of the modular topology $\mathcal{T}_\Phi^\wedge(X)$ are included in the following theorem (see [9, Theorem 6.3, Theorem 6.5]).

Corollary 4.4. *Let Φ be a Young function and let X be a Banach space. Then*

(*i*) *$\mathcal{T}_\Phi^\wedge(X)$ is the finest locally convex-solid topology on $L^\Phi(X)$ with the Lebesgue property.*

(*ii*) $\mathcal{T}_\Phi^\wedge(X) = \overline{\mathcal{T}_\Phi^\wedge} = \overline{\tau(L^\Phi, L^{\Phi^*})}$.

(*iii*) *$\mathcal{T}_\Phi^\wedge(X)$ is generated by the family of norms $\|\cdot\|_{L^\Psi(X)}$, where Ψ runs over the*

family of all Young functions such that $\Psi \lhd \Phi$ (i.e., for every $c > 1$ there exists $d > 1$ such that $\Psi(ct) \leq d\Phi(t)$ for all $t \geq 0$).

Making use of Theorem 4.1, Corollary 4.2 and Theorem 4.4 we get:

Theorem 4.5. *Let Φ be a Young function and X be an Asplund space. Then the modular topology $\mathcal{T}_\Phi^\wedge(X)$ on $L^\Phi(X)$ is a strongly Mackey topology; hence we have:*

$$\mathcal{T}_\Phi^\wedge(X) = \overline{\tau(L^\Phi, L^{\Phi^*})} = \tau(L^\Phi(X), L^{\Phi^*}(X^*)).$$

Moreover, the space $L^{\Phi^*}(X^*)$ is $\sigma(L^{\Phi^*}(X^*), L^\Phi(X))$-sequentially complete.

5. Linear operators on vector-valued function spaces with Mackey topologies

From now on we assume that E is an ideal of L^0 and $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are real Banach spaces. For a linear operator $T : E(X) \to Y$ let $T^* : Y^\# \to E(X)^\#$ stand for the conjugate of T defined via the duality identity $\langle f, T^*(y^*)\rangle = \langle T(f), y^*\rangle$ for $f \in E(X)$, $y^* \in Y^\#$ (here $Y^\#$ denotes the algebraic dual of Y).

We start with the following well-known characterization.

Proposition 5.1. *For a linear operator $T : E(X) \to Y$ the following statements are equivalent:*

(*i*) T *is* $(\tau(E(X), E(X)_n^\sim), \|\cdot\|_Y)$*-continuous.*
(*ii*) T *is* $(\sigma(E(X), E(X)_n^\sim), \sigma(Y, Y^*))$*-continuous.*
(*iii*) $T^*(Y^*) \subset E(X)_n^\sim$.

Proof. (*i*) $\Longleftrightarrow$ (*ii*) See [23, Corollary 11-1-3, Corollary 11-2-6]; (*ii*) $\Longleftrightarrow$ (*iii*) See [23, Lemma 11-1-1]. □

Let $\mathcal{L}_\tau(E(X), Y)$ stand for the space of all $(\tau(E(X), E(X)_n^\sim), \|\cdot\|_Y)$-continuous linear operators from $E(X)$ to Y. The *strong operator topology* (briefly SOT) is a locally convex topology on $\mathcal{L}_\tau(E(X), Y)$ defined by the family of seminorms $\{p_f : f \in E(X)\}$, where $p_f(T) = \|T(f)\|_Y$ for all $T \in \mathcal{L}_\tau(E(X), Y)$. The *weak operator topology* (briefly WOT) is a locally convex topology on $\mathcal{L}_\tau(E(X), Y)$ defined by the family of seminorms $\{p_{f,y^*} : f \in E(X), y^* \in Y^*\}$, where $p_{f,y^*}(T) = |\langle T(f), y^*\rangle|$ for all $T \in \mathcal{L}_\tau(E(X), Y)$. Note that for $Y = \mathbb{R}$, both SOT and WOT coincide on $E(X)_n^\sim$ with $\sigma(E(X)_n^\sim, E(X))$.

Proposition 5.2. *Assume that $E(X)_n^\sim$ is $\sigma(E(X)_n^\sim, E(X))$-sequentially complete. Then the space $\mathcal{L}_\tau(E(X), Y)$ is* WOT*-sequentially complete.*

Proof. Let (T_n) be a WOT-Cauchy sequence in $\mathcal{L}_\tau(E(X), Y)$. Then one can define a linear operator $T : E(X) \to Y$ such that for every $f \in E(X)$, $y^* \in Y^*$,

$$\langle T(f), y^*\rangle := \lim_n \langle T_n(f), y^*\rangle. \tag{3}$$

It is enough to show that T is $(\tau(E(X), E(X)_n^\sim), \|\cdot\|_Y)$-continuous. In fact, in view of Proposition 5.1, $T_n^*(Y^*) \subset E(X)_n^\sim$ for every $n \in \mathbb{N}$. Moreover, one can easily observe that $T_n^* : Y^* \to E(X)_n^\sim$ is $(\sigma(Y^*, Y), \sigma(E(X)_n^\sim, E(X)))$-continuous for $n \in \mathbb{N}$. Let $y_0^* \in Y^*$ be given. Then for every $f \in E(X)$ by (3) we have:

$$T^*(y_0^*)(f) = \langle T(f), y_0^*\rangle = \lim_n \langle T_n(f), y_0^*\rangle = \lim_n T_n^*(y_0^*)(f). \tag{4}$$

It follows that $(T_n^*(y_0^*))$ is a $\sigma(E(X)_n^\sim, E(X))$-Cauchy sequence in $E(X)_n^\sim$, so there exists $F_0 \in E(X)_n^\sim$ such that $F_0(f) = \lim_n T_n^*(y_0^*)(f)$ for every $f \in E(X)$. By (4) $T^*(y_0^*) = F_0 \in E(X)_n^\sim$; hence $T^*(Y^*) \subset E(X)_n^\sim$, and this means that $T \in \mathcal{L}_\tau(E(X), Y)$ (see Proposition 5.1). □

As an application of Proposition 5.2 and Corollary 4.2 we get:

Corollary 5.3. *Assume that E is a perfect ideal of L^0 and X is an Asplund space. Then the space $\mathcal{L}_\tau(E(X),Y)$ is* WOT*-sequentially complete.*

The following general result will be usuful (see [21, Theorem 2]).

Proposition 5.4. *Let $\mathcal{K}$ be a* SOT*-compact subset of $\mathcal{L}_\tau(E(X),Y)$, and let A be a $\sigma(Y^*,Y)$-closed, equicontinuous subset of Y^*. Then the set $\bigcup\{T^*(A) : T \in \mathcal{K}\}$ is a $\sigma(E(X)_n^\sim, E(X))$-compact subset of $E(X)_n^\sim$.*

Now we are in position to state our main result.

Theorem 5.5. *Assume that X is an Asplund space and Y is a Banach space. Let $\mathcal{K}$ be a* SOT*-compact subset of $\mathcal{L}_\tau(E(X),Y)$. Then $\mathcal{K}$ is $(\tau(E(X),E'(X^*)), \|\cdot\|_Y)$-equicontinuous.*

Proof. In view of the Alaoglu theorem the unit ball in Y^* is $\sigma(Y^*,Y)$-closed and equicontinuous. Hence by Proposition 5.4 $Z = \bigcup\{T^*(B_{Y^*}) : T \in \mathcal{K}\}$ is $\sigma(E(X)_n^\sim, E(X))$-compact subset of $E(X)_n^\sim = \{F_g : g \in E'(X^*)\}$. Let $C_Z = \{g \in E'(X^*) : F_g \in Z\}$. Then by Corollary 3.7 the set $\overline{\operatorname{abs\,conv}(C_Z)}$ (the closure in $\sigma(E'(X^*),E(X))$) is still a $\sigma(E'(X^*),E(X))$-compact subset of $E'(X^*)$. Putting

$$p_0(f) = \sup\{|F_g(f)| : g \in \overline{\operatorname{abs\,conv}(C_Z)}\} \text{ for } f \in E(X)$$

we get

$$\begin{aligned} p_0(f) &\geq \sup\{|F(f)| : F \in Z\} = \sup_{T\in\mathcal{K}}\{|T^*(y^*)| : y^* \in B_{Y^*}\} \\ &= \sup_{T\in\mathcal{K}}\{|\langle T(f), y^*\rangle| : y^* \in B_{Y^*}\} = \sup_{T\in\mathcal{K}} \|T(f)\|_Y. \end{aligned}$$

Now, let $\varepsilon > 0$ be given. Then $W_\varepsilon = \{f \in E(X) : p_0(f) \leq \varepsilon\}$ is a neighborhood of 0 for $\tau(E(X),E'(X^*))$. It follows that $\|T(f)\|_Y \leq \varepsilon$ for every $T \in \mathcal{K}$ and all $f \in W_\varepsilon$, and this means that $\mathcal{K}$ is $(\tau(E(X),E'(X^*)), \|\cdot\|_Y)$-equicontinuous. □

As an application of Theorem 5.5 we get the following Vitali-Hahn-Saks type theorem.

Corollary 5.6. *Let E be a perfect ideal of L^0. Assume that X is an Asplund space, and Y a Banach space. Let (T_n) be a sequence in $\mathcal{L}_\tau(E(X),Y)$ and assume that $T(f) := \lim_n T_n(f)$ exists in $(Y, \|\cdot\|_Y)$ for every $f \in E(X)$. Then $T \in \mathcal{L}_\tau(E(X),Y)$ and the set $\{T_n : n \in \mathbb{N}\}$ in $\mathcal{L}_\tau(E(X),Y)$ is $(\tau(E(X),E'(X^*)), \|\cdot\|_Y)$-equicontinuous.*

Proof. Note that $\langle T(f), y^*\rangle = \lim_n \langle T_n(f), y\rangle$ for every $f \in E(X)$, $y^* \in Y^*$. Since $E(X)_n^\sim$ is $\sigma(E(X)_n^\sim, E(X))$ is sequentially complete (see Corollary 4.2), in view of the proof of Proposition 5.2, $T \in \mathcal{L}_\tau(E(X),Y)$ and $T_n \to T$ in $\mathcal{L}_\tau(E(X),Y)$ for SOT, so the set $\{T_n : n \in \mathbb{N}\}$ is relatively compact for SOT. Hence by Theorem 5.5 it is $(\tau(E(X),E'(X^*)), \|\cdot\|_Y)$-equicontinuous. □

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