Journal of Convex Analysis
Volume 15 (2008), No. 4, 869–890

A Turnpike Result for a Class of Problems of the Calculus of Variations with Extended-Valued Integrands

Alexander J. Zaslavski
Department of Mathematics, The Technion-Israel Institute of Technology, 32000 Haifa, Israel
ajzasl@tx.technion.ac.il

Received: October 3, 2006

In this work we study the structure of approximate solutions of an autonomous variational problem with a lower semicontinuous integrand $f : R^n \times R^n \to R^1 \cup \{\infty\}$, where R^n is the n-dimensional Euclidean space. We are interested in a turnpike property of the approximate solutions which are independent of the length of the interval, for all sufficiently large intervals.

Keywords: Good function, infinite horizon, overtaking optimal function, turnpike property

1991 Mathematics Subject Classification: 49J99

1. Introduction and main results

The study of the existence and the structure of solutions of optimal control problems defined on infinite intervals and on sufficiently large intervals has recently been a rapidly growing area of research. See, for example, [4, 5, 7, 9, 12, 14, 18, 19, 22-25, 29-33] and the references mentioned therein. These problems arise in engineering [1, 15], in models of economic growth [2, 6, 7, 10, 11, 13, 17, 20, 21, 25-27, 33], in infinite discrete models of solid-state physics related to dislocations in one-dimensional crystals [3, 28] and in the theory of thermodynamical equilibrium for materials [16, 18, 19]. In this paper we study the structure of solutions of a continuous-time optimal control system describing a general model of economic dynamics. More precisely, we consider the following variational problem

$$\int_0^T f(v(t), v'(t))dt \to \min, \tag{P}$$

$$v : [0, T] \to R^n \text{ is an absolutely continuous (a.c.) function such that } v(0) = x,$$

where $x \in R^n$. Here R^n is the n-dimensional Euclidean space with the Euclidean norm $|\cdot|$ and $f : R^n \times R^n \to R^1 \cup \{\infty\}$ is an extended-valued integrand.

We are interested in a turnpike property of the approximate solutions of (P) which is independent of the length of the interval T, for all sufficiently large intervals. To have this property means, roughly speaking, that the approximate solutions of the variational problems are determined mainly by the integrand f, and are essentially independent of T and x. Turnpike properties are well known in mathematical economics. The term was

ISSN 0944-6532 / \$ 2.50

first coined by Samuelson in 1948 (see [26]) where he showed that an efficient expanding economy would spend most of the time in the vicinity of a balanced equilibrium path (also called a von Neumann path). This property was further investigated for optimal trajectories of models of economic dynamics (see, for example, [2, 10, 11, 17, 20, 21, 25, 27, 33] and the references mentioned there). In the classical turnpike theory the function f has the turnpike property (TP) if there exists $\bar{x} \in R^n$ (a turnpike) which satisfies the following condition:

For each $M, \epsilon > 0$ there is a natural number L such that for each number $T \geq 2L$, each $x \in R^n$ satisfying $|x| \leq M$ and each solution $v : [0, T] \to R^n$ of the problem (P) the inequality $|v(t) - \bar{x}| \leq \epsilon$ holds for all $t \in [L, T - L]$.

Note that L depends neither on T nor on x.

In the classical turnpike theory [2, 11, 20, 21] the cost function f is strictly convex. Under this assumption the turnpike property can be established and the turnpike $\bar{x}$ is a unique solution of the minimization problem $f(x, 0) \to \min$, $x \in R^n$. In this situation it is shown that for each a.c. function $v : [0, \infty) \to R^n$ either the function

$$T \to \int_0^T f(v(t), v'(t))dt - Tf(\bar{x}, 0), \quad T \in (0, \infty)$$

is bounded (in this case the function v is called (f)-good) or it diverges to ∞ as $T \to \infty$. Moreover, it is also established that any (f)-good function converges to the turnpike $\bar{x}$. In the sequel this property is called as the asymptotic turnpike property.

Recently it was shown that the turnpike property is a general phenomenon which holds for large classes of variational and optimal control problems without convexity assumptions. (See, for example, [18, 29, 30, 32, 33] and the references mentioned therein). For these classes of problems a turnpike is not necessarily a singleton but may instead be an non-stationary trajectory (in the discrete time nonautonomous case) [32, 33] or an absolutely continuous function on the interval $[0, \infty)$ (in the continuous time nonautonomous case) [33] or a compact subset of the space X (in the autonomous case) [19, 29, 30, 33]. Note that all of these recent results were obtained for finite-valued integrands f (in other words, for unconstrained variational problems). In this paper we study the problems (P) with an extended-valued integrand $f : R^n \times R^n \to R^1 \cup \{\infty\}$ (in other words, constrained variational problems). Clearly, these constrained problems with extended-valued integrands are more difficult and less understood than their unconstrained prototypes in [29-33]. They are also more realistic from the point of view of mathematical economics. As we have mentioned before in general a turnpike is not necessarily a singleton. Nevertheless problems of the type (P) for which the turnpike is a singleton are of great importance because of the following reasons: there are many models of economic growth for which a turnpike is a singleton; if a turnpike is a singleton, then approximate solutions of (P) have very simple structure and this is very important for applications; if a turnpike is a singleton, then it can be easily calculated as a solution of the problem $f(x, 0) \to \min$, $x \in R^n$.

In this paper our goal is to understand when the turnpike property holds with the turnpike being a singleton. We will show that the turnpike property follows from the asymptotic turnpike property. More precisely, we assume that any (f)-good function converges to a unique solution $\bar{x}$ of the problem $f(x, 0) \to \min$, $x \in R^n$ and show that the turnpike property holds and $\bar{x}$ is the turnpike (see Theorem 1.3). Note that we do not use convexity assumptions. It should be mentioned that analogous results which show that turnpike properties follow from asymptotic turnpike properties for unconstrained variational problems with finite-valued integrands were obtained in [18, 33].

We denote by mes(E) the Lebesgue measure of a Lebesgue measurable set $E \subset R^1$, denote by $|\cdot|$ the Euclidean norm of the space R^n and by $\langle \cdot, \cdot \rangle$ the inner product of R^n.

Let $a > 0$, $\psi : [0, \infty) \to [0, \infty)$ be an increasing function such that

$$\lim_{t \to \infty} \psi(t) = \infty \tag{1}$$

and let $f : R^n \times R^n \to R^1 \cup \{\infty\}$ be a lower semicontinuous function such that the set

$$\mathrm{dom}(f) := \{(x, y) \in R^n \times R^n : \ f(x, y) < \infty\} \tag{2}$$

is nonempty, convex and closed and that

$$f(x, y) \geq \max\{\psi(|x|),\ \psi(|y|)|y|\} - a \quad \text{for each } x, y \in R^n. \tag{3}$$

For each $x \in R^n$ and each number $T > 0$ set

$$\begin{aligned} \sigma(f, T, x) = \ \inf\Bigg\{ \int_0^T f(v(t), v'(t))dt :\ & v : [0, T] \to R^n \\ & \text{is an absolutely continuous (a.c.) function satisfying } v(0) = x \Bigg\}, \end{aligned} \tag{4}$$

$$\sigma(f, T) = \inf\left\{ \int_0^T f(v(t), v'(t))dt :\ v : [0, T] \to R^n \text{ is an a.c. function} \right\}. \tag{5}$$

We suppose that there exists $\bar{x} \in R^n$ such that

$$f(\bar{x}, 0) \leq f(x, 0) \quad \text{for each } x \in R^n \tag{6}$$

and that the following assumptions hold:

(A1) $(\bar{x}, 0)$ is an interior point of the set dom(f) and f is continuous at $(\bar{x}, 0)$;

(A2) for each $M > 0$ there exists $c_M > 0$ such that

$$\sigma(f, T, x) \geq T f(\bar{x}, 0) - c_M$$

for each $x \in R^n$ satisfying $|x| \leq M$ and each $T > 0$;

(A3) for each $x \in R^n$ the function $f(x, \cdot) : R^n \to R^1 \cup \{\infty\}$ is convex.

Remark 1.1. By (A3) for each a.c. function $v : [0, \infty) \to R^n$ the function

$$T \to \int_0^T f(v(t), v'(t))dt - T f(\bar{x}, 0), \quad T \in (0, \infty)$$

is bounded from below.

Note that the relation (6) and the assumptions (A1)-(A3) are common in the literature and hold for many infinite horizon optimal control problems [7, 33]. In particular, we need

(6) and (A2) in the cases when the problem (P) possesses the turnpike property and $\bar{x}$ is its turnpike. The assumption (A2) means that the constant function $\bar{v}(t) = \bar{x}$, $t \in [0, \infty)$ is an approximate solution of the infinite horizon variational problem with the integrand f related to the problem (P).

We say that an a.c. function $v : [0, \infty) \to R^n$ is called (f)-good [11, 33] if

$$\sup \left\{ \left| \int_0^T f(v(t), v'(t))dt - Tf(\bar{x}, 0) \right| : \; T \in (0, \infty) \right\} < \infty.$$

The following result will be proved in Section 3.

Proposition 1.2. *Let $v : [0, \infty) \to R^n$ be an a.c. function. Then either v is (f)-good or*

$$\int_0^T f(v(t), v'(t))dt - Tf(\bar{x}, 0) \to \infty \quad \text{as } T \to \infty.$$

Moreover, if v is (f)-good, then $\sup\{|v(t)| : \; t \in [0, \infty)\} < \infty$.

For each $T_1 \in R^1$, $T_2 > T_1$ and each a.c. function $v : [T_1, T_2] \to R^n$ set

$$I^f(T_1, T_2, v) = \int_{T_1}^{T_2} f(v(t), v'(t))dt. \tag{7}$$

For each $M > 0$ denote by X_M the set of all $x \in R^n$ such that $|x| \le M$ and there exists an a.c. function $v : [0, \infty) \to R^n$ which satisfies

$$v(0) = x, \qquad I^f(0, T, v) - Tf(\bar{x}, 0) \le M \quad \text{for each } T \in (0, \infty). \tag{8}$$

In this paper we will establish the following turnpike result.

Theorem 1.3. *Suppose that the following assumption holds:*

(A4) *(the asymptotic turnpike property) for each (f)-good function $v : [0, \infty) \to R^n$,* $\lim_{t \to \infty} |v(t) - \bar{x}| = 0$.

Let $\epsilon, M > 0$. Then there exist a natural number L and a positive number δ such that for each real $T > 2L$ and each a.c. function $v : [0, T] \to R^n$ which satisfies

$$v(0) \in X_M \quad \text{and} \quad I^f(0, T, v) \le \sigma(f, T, v(0)) + \delta$$

there exist $\tau_1 \in [0, L]$ and $\tau_2 \in [T - L, T]$ such that

$$|v(t) - \bar{x}| \le \epsilon \quad \text{for all } t \in [\tau_1, \tau_2]$$

and if $|v(0) - \bar{x}| \le \delta$, then $\tau_1 = 0$.

Theorem 1.3 will be proved in Section 5.

In the sequel we use a notion of an overtaking optimal function introduced in [2, 11, 27].

An a.c. function $v : [0, \infty) \to R^n$ is called (f)-overtaking optimal if for each a.c. function $u : [0, \infty) \to R^n$ satisfying $u(0) = v(0)$

$$\limsup_{T \to \infty} \left[I^f(0, T, v) - I^f(0, T, u) \right] \le 0.$$

The following result establishes the existence of an overtaking optimal function.

Theorem 1.4. *Suppose that* (A4) *holds. Assume that* $x \in R^n$ *and there exists an* (f)*-good function* $v : [0, \infty) \to R^n$ *satisfying* $v(0) = x$*. Then there exists an* (f)*-overtaking optimal function* $u_* : [0, \infty) \to R^n$ *such that* $u_*(0) = x$*.*

Theorem 1.4 will be proved in Section 6. Examples of integrands f satisfying (A1)–(A4) are considered in Section 7.

2. Preliminaries

Proposition 2.1. *Let* M_0, M_1 *be positive numbers. Then there exists* $M_2 > 0$ *such that for each* $T > 0$ *and each a.c. function* $v : [0, T] \to R^n$ *which satisfies*

$$|v(0)| \le M_0, \qquad I^f(0, T, v) \le Tf(\bar{x}, 0) + M_1 \tag{9}$$

the following inequality holds:

$$|v(t)| \le M_2 \quad \textit{for all } t \in [0, T]. \tag{10}$$

Proof. By (1) there exists $\Gamma > M_0 + 1$ such that

$$\psi(\Gamma) > 2|f(\bar{x}, 0)| + 4 + a. \tag{11}$$

By (A2) there exists $c(\Gamma) > 0$ such that

$$\sigma(f, T, x) \ge Tf(\bar{x}, 0) - c(\Gamma) \quad \text{for each } T > 0 \text{ and each } x \in R^n \text{ satisfying } |x| \le \Gamma. \tag{12}$$

Choose a positive number M_2 such that

$$M_2 > 4\Gamma + 4 + (M_1 + 2c(\Gamma))(4\Gamma + 2a + 1 + |f(\bar{x}, 0)|). \tag{13}$$

Assume that $T > 0$ and that an a.c. function $v : [0, T] \to R^n$ satisfies (9). We will show that (10) holds. Let us assume the contrary. Then there exists $t_0 \in [0, T]$ such that

$$|v(t_0)| > M_2. \tag{14}$$

In view of (14), (9), (13) and the inequality $\Gamma > M_0 + 1$

$$t_0 \in (0, T]. \tag{15}$$

By (14), (15), (9), (13) and the inequality $\Gamma > M_0 + 1$ there exists $t_1 \in (0, t_0)$ such that

$$|v(t_1)| = \Gamma \quad \text{and} \quad |v(t)| > \Gamma \quad \text{for each } t \in (t_1, t_0). \tag{16}$$

There are two cases:

$$|v(t)| \ge \Gamma, \quad t \in [t_0, T]; \tag{17}$$

$$\inf\{|v(t)| : t \in [t_0, T]\} < \Gamma. \tag{18}$$

If (17) holds, then we set $t_2 = T$. If (18) is true, then there exists

$$t_2 \in (t_0, T) \tag{19}$$

for which

$$|v(t_2)| = \Gamma \quad \text{and} \quad |v(t)| > \Gamma \quad \text{for each } t \in (t_0, t_2). \tag{20}$$

By (4), the choice of t_2, (20) and (12),

$$I^f(t_2, T, v) \geq \sigma(f, T - t_2, v(t_2)) \geq (T - t_2) f(\bar{x}, 0) - c(\Gamma). \tag{21}$$

In view of (4), (16) and (12)

$$I^f(0, t_1, v) \geq \sigma(f, t_1, v(0)) \geq t_1 f(\bar{x}, 0) - c(\Gamma). \tag{22}$$

Relations (9) and (21) imply that

$$\begin{aligned} I^f(0, t_2, v) - t_2 f(\bar{x}, 0) &= I^f(0, T, v) - T f(\bar{x}, 0) - I^f(t_2, T, v) + (T - t_2) f(\bar{x}, 0) \\ &\leq M_1 - \left[I^f(t_2, T, v) - (T - t_2) f(\bar{x}, 0)\right] \leq M_1 + c(\Gamma). \end{aligned} \tag{23}$$

It follows from (23) and (22) that

$$\begin{aligned} &I^f(t_1, t_2, v) - (t_2 - t_1) f(\bar{x}, 0) \\ &= I^f(0, t_2, v) - t_2 f(\bar{x}, 0) - \left[I^f(0, t_1, v) - t_1 f(\bar{x}, 0)\right] \leq M_1 + 2c(\Gamma). \end{aligned} \tag{24}$$

In view of (16) and the choice of t_2 (see (17), (20))

$$|v(t)| \geq \Gamma \quad \text{for all } t \in [t_1, t_2]. \tag{25}$$

Together with (3) and (11) the inequality above implies that for $t \in [t_1, t_2]$ (a.e.)

$$f(v(t), v'(t)) \geq \psi(|v(t)|) - a \geq \psi(\Gamma) - a \geq 2|f(\bar{x}, 0)| + 4$$

and

$$I^f(t_1, t_2, v) - (t_2 - t_1) f(\bar{x}, 0) \geq 4(t_2 - t_1). \tag{26}$$

Combined with (24) this inequality implies that

$$t_2 - t_1 \leq M_1 + c(\Gamma). \tag{27}$$

Set

$$E_1 = \{t \in [t_1, t_0] : \ |v'(t)| \geq \Gamma\}, \qquad E_2 = [t_1, t_0] \setminus E_1. \tag{28}$$

Relations (16) and (28) imply that

$$\begin{aligned} M_2 - \Gamma \leq |v(t_0)| - |v(t_1)| \leq |v(t_0) - v(t_1)| \leq \int_{t_1}^{t_0} |v'(t)| t \\ = \int_{E_1} |v'(t)| dt + \int_{E_2} |v'(t)| dt \leq \int_{E_1} |v'(t)| dt + (t_0 - t_1)\Gamma. \end{aligned}$$

Together with (27) and the choice of t_2 this relation implies that

$$\int_{E_1} |v'(t)| dt \geq M_2 - \Gamma - \Gamma(M_1 + c(\Gamma)). \tag{29}$$

By (3), (28), (11), (27), the choice of t_2 and (29)

$$\begin{aligned}
&\int_{E_1} f(v(t), v'(t))dt \\
&\geq \int_{E_1} [\psi(|v'(t)|)|v'(t)| - a]\, dt \geq \int_{E_1} \psi(|v'(t)|)|v'(t)|dt - a(t_0 - t_1) \qquad (30)\\
&\geq 4\int_{E_1} |v'(t)|dt - a(M_1 + c(\Gamma)) \geq 4(M_2 - \Gamma) - 4\Gamma(M_1 + c(\Gamma)) - a(M_1 + c(\Gamma)).
\end{aligned}$$

It follows from the choice of t_2 (see (17), (19)), (28), (3), (13), (30) and (27) that

$$\begin{aligned}
&\int_{t_1}^{t_2} f(v(t), v'(t))dt \\
&= \int_{E_1} f(v(t), v'(t))dt + \int_{E_2} f(v(t), v'(t))dt + \int_{t_0}^{t_2} f(v(t), v'(t))dt \\
&\geq \int_{E_1} f(v(t), v'(t))dt - a(\mathrm{mes}(E_2)) - a(t_2 - t_0) \geq \int_{E_1} f(v(t), v'(t))dt - a(t_2 - t_1) \\
&\geq 2M_2 - (M_1 + c(\Gamma))(4\Gamma + a) - a(M_1 + c(\Gamma)).
\end{aligned}$$

Combined with (24) and (27) this inequality implies that

$$\begin{aligned}
2M_2 &\leq (M_1 + c(\Gamma))(4\Gamma + 2a) + I^f(t_1, t_2, v) \\
&\leq (M_1 + c(\Gamma))(4\Gamma + 2a) + M_1 + 2c(\Gamma) + (t_2 - t_1)f(\bar{x}, 0) \\
&\leq (M_1 + 2c(\Gamma))(4\Gamma + 2a + 1) + |f(\bar{x}, 0)|(M_1 + c(\Gamma)) \\
&\leq (M_1 + 2c(\Gamma))(4\Gamma + 2a + 1 + |f(\bar{x}, 0)|).
\end{aligned}$$

This inequality contradicts (13). The contradiction we have reached proves that (10) holds. Proposition 2.1 is proved. □

Proposition 2.2 (8, Chapter 10). *Let $T > 0$ and let $v_k : [0, T] \to R^n$, $k = 1, 2, \ldots$ be a sequence of a.c. functions such that the sequence $\{I^f(0, T, v_k)\}_{k=1}^{\infty}$ is bounded and that the sequence $\{v_k(0)\}_{k=1}^{\infty}$ is bounded. Then there exist a strictly increasing sequence of natural numbers $\{k_i\}_{i=1}^{\infty}$ and an a.c. function $v : [0, T] \to R^n$ such that*

$$v_{k_i}(t) \to v(t) \quad \text{as } i \to \infty \text{ uniformly on } [0, T],$$

$$I^f(0, T, v) \leq \liminf_{i\to\infty} I^f(0, T, v_{k_i}).$$

Proposition 2.3. *Let $\epsilon > 0$. Then there exists $\delta > 0$ such that if an a.c. function $v : [0, 1] \to R^n$ satisfies $|v(0) - \bar{x}|$, $|v(1) - \bar{x}| \leq \delta$, then*

$$I^f(0, 1, v) \geq f(\bar{x}, 0) - \epsilon.$$

Proof. In view of (A2) the following property holds:

(P1) $I^f(0, 1, u) \geq f(\bar{x}, 0)$ for each a.c. function $u : [0, 1] \to R^n$ satisfying $u(0) = u(1) = \bar{x}$.

Assume that the proposition is wrong. Then for each integer $i \geq 1$ there is an a.c. function $v_i : [0,1] \to R^n$ such that

$$|v_i(0) - \bar{x}|, \qquad |v_i(1) - \bar{x}| \leq 1/i, \qquad I^f(0,1,v_i) < f(\bar{x},0) - \epsilon. \tag{31}$$

By Proposition 2.2 extracting a subsequence and re-indexing if necessary we may assume that there is an a.c. function $v : [0,1] \to R^n$ such that

$$v_i(t) \to v(t) \quad \text{as } i \to \infty \text{ uniformly on } [0,1],$$

$$I^f(0,1,v) \leq \liminf_{i\to\infty} I^f(0,1,v_i) \leq f(\bar{x},0) - \epsilon.$$

Together with (31) this implies that

$$v(0) = \bar{x}, \qquad v(1) = \bar{x}, \qquad I^f(0,1,v) \leq f(\bar{x},0) - \epsilon.$$

These relations contradict (P1). The contradiction we have reached proves Proposition 2.3. □

3. Proof of Proposition 1.2

By Remark 1.1 there is $c_0 > 0$ such that

$$\int_0^T f(v(t), v'(t))dt - Tf(\bar{x},0) \geq -c_0 \quad \text{for each } T > 0. \tag{32}$$

Assume that there exists a strictly increasing sequence of positive numbers $\{T_k\}_{k=1}^\infty$ such that

$$T_k \geq k \quad \text{for each integer } k \geq 1, \tag{33}$$

$$\sup\{I^f(0,T_k,v) - T_k f(\bar{x},0) : \ k \text{ is a natural number}\} < \infty. \tag{34}$$

In order to prove the proposition it is sufficient to show that v is (f)-good and that $\sup\{|v(t)| : \ t \in [0,\infty)\} < \infty$. By (1) and (3) there is a number M_0 such that

$$M_0 > |v(0)| + 1, \tag{35}$$

$$f(y,z) \geq 2(|f(\bar{x},0)| + 1) \quad \text{for each } y,z \in R^n \text{ satisfying } |y| \geq 4^{-1}M_0.$$

We show that $\liminf_{t\to\infty} |v(t)| < M_0$. Let us assume the contrary. Then there exists $S_0 > 0$ such that

$$|v(t)| \geq 2^{-1}M_0 \quad \text{for each } t \geq S_0. \tag{36}$$

By (3), (36) and (35) for each natural number k such that $T_k > S_0$,

$$\begin{aligned}
&\int_0^{T_k} f(v_k(t), v_k'(t))dt - T_k f(\bar{x},0) \\
&= \int_0^{S_0} f(v(t), v'(t))dt - S_0 f(\bar{x},0) + \int_{S_0}^{T_k} f(v(t), v'(t))dt - (T_k - S_0)f(\bar{x},0) \\
&\geq S_0(-a - f(\bar{x},0)) + (T_k - S_0)[2(|f(\bar{x},0)| + 1) - f(\bar{x},0)] \to \infty \quad \text{as } k \to \infty.
\end{aligned}$$

This contradicts (34). The contradiction we have reached proves that

$$\liminf_{t\to\infty} |v(t)| < M_0. \tag{37}$$

By (A2) there is $c_1 > 0$ such that

$$\sigma(f, T, x) \geq Tf(\bar{x}, 0) - c_1 \quad \text{for each } T > 0 \text{ and each } x \in R^n \text{ satisfying } |x| \leq M_0. \tag{38}$$

In view of (34) there is $c_2 > 0$ such that

$$I^f(0, T_k, v) - T_k f(\bar{x}, 0) \leq c_2 \quad \text{for each natural number } k. \tag{39}$$

Let $T > 0$. By(37) there exists $\tau \geq T$ such that:

$$|v(\tau)| \leq M_0;$$

$$\text{if a number } t \text{ satisfies } T \leq t < \tau, \text{ then} \quad |v(t)| > M_0. \tag{40}$$

In view of (40)

$$\begin{aligned}
&I^f(0, T, v) - Tf(\bar{x}, 0) \\
&= I^f(0, \tau, v) - \tau f(\bar{x}, 0) - I^f(T, \tau, v) + (\tau - T)f(\bar{x}, 0) \\
&\leq I^f(0, \tau, v) - \tau f(\bar{x}, 0) - (\tau - T)[2(|f(\bar{x}, 0)| + 1) - f(\bar{x}, 0)] \\
&\leq I^f(0, \tau, v) - \tau f(\bar{x}, 0).
\end{aligned} \tag{41}$$

Choose a natural number k such that

$$T_k > \tau + 1. \tag{42}$$

Relations (40) and (38) imply that

$$I^f(\tau, T_k, v) \geq \sigma(f, T_k - \tau, v(\tau)) \geq (T_k - \tau)f(\bar{x}, 0) - c_1. \tag{43}$$

It follows from (41), (39) and (43) that

$$\begin{aligned}
&I^f(0, T, v) - Tf(\bar{x}, 0) \\
&\leq I^f(0, \tau, v) - \tau f(\bar{x}, 0) \\
&\leq I^f(0, T_k, v) - T_k f(\bar{x}, 0) - I^f(\tau, T_k, v) + (T_k - \tau)f(\bar{x}, 0) \\
&\leq c_2 - I^f(\tau, T_k, v) + (T_k - \tau)f(\bar{x}, 0) \\
&\leq c_2 - (T_k - \tau)f(\bar{x}, 0) + c_1 + (T_k - \tau)f(\bar{x}, 0) = c_2 + c_1.
\end{aligned}$$

Thus we have shown that for each $T > 0$

$$I^f(0, T, v) - Tf(\bar{x}, 0) \leq c_2 + c_1 \tag{44}$$

and v is (f)-good. It follows from (44) and Proposition 2.1 that $\sup\{|v(t)| : \; t \in [0, \infty)\} < \infty$. Proposition 1.2 is proved.

4. Auxiliary results

In this section we assume that (A4) holds. Namely, for each (f)-good function $v : [0, \infty) \to R^n$,

$$\lim_{t\to\infty} |v(t) - \bar{x}| = 0. \tag{45}$$

Lemma 4.1. *Let $M, \epsilon > 0$. Then there exists a number $T > 0$ such that for each a.c. function $v : [0, T] \to R^n$ which satisfies*

$$|v(0)| \le M, \qquad I^f(0, T, v) \le Tf(\bar{x}, 0) + M$$

the following inequality holds:

$$\min\{|v(t) - \bar{x}| : \ t \in [0, T]\} \le \epsilon.$$

Proof. Let us assume the contrary. Then for each integer $k \ge 1$ there exists an a.c. function $v_k : [0, k] \to R^n$ such that

$$|v_k(0)| \le M, \qquad I^f(0, k, v_k) \le kf(\bar{x}, 0) + M, \tag{46}$$

$$\min\{|v_k(t) - \bar{x}| : \ t \in [0, k]\} > \epsilon. \tag{47}$$

By Proposition 2.1 and (46) there is a number $M_1 > 0$ such that for each integer $k \ge 1$

$$|v_k(t)| \le M_1, \quad t \in [0, k]. \tag{48}$$

By (A2) there is $c_1 > 0$ such that

$$\sigma(f, T, x) \ge Tf(\bar{x}, 0) - c_1 \quad \text{for each } T > 0 \text{ and each } x \in R^n \text{ satisfying } |x| \le M_1. \tag{49}$$

Let $q \ge 1$ be an integer. It follows from (46), (48) and (49) that for each integer $k > q$

$$\begin{aligned}
&I^f(0, q, v_k) - qf(\bar{x}, 0) \\
&= I^f(0, k, v_k) - kf(\bar{x}, 0) - \big[I^f(q, k, v_k) - (k - q)f(\bar{x}, 0)\big] \\
&\le M - \big[I^f(q, k, v_k) - (k - q)f(\bar{x}, 0)\big] \\
&\le M - [\sigma(f, k - q, v_k(q)) - (k - q)f(\bar{x}, 0)] \le M + c_1
\end{aligned}$$

and

$$I^f(0, q, v_k) \le qf(\bar{x}, 0) + M + c_1 \quad \text{for each integer } k > q. \tag{50}$$

In view of (50), (48) and Proposition 2.2 there exist a subsequence $\{v_{k_i}\}_{i=1}^{\infty}$ and an a.c. function $v : [0, \infty) \to R^n$ such that for each natural number q

$$v_{k_i}(t) \to v(t) \quad \text{as } i \to \infty \text{ uniformly on } [0, q], \tag{51}$$

$$I^f(0, q, v) \le qf(\bar{x}, 0) + M + c_1. \tag{52}$$

Proposition 1.2 and (52) imply that v is an (f)-good function. In view of (A4) $\lim_{t\to\infty} v(t) = \bar{x}$. Therefore there is $\tau > 0$ such that $|v(\tau) - \bar{x}| < \epsilon/4$. Combined with (51) this implies that there is an integer $i \ge 1$ such that $k_i > \tau$ and

$$|v_{k_i}(\tau) - v(\tau)| < \epsilon/4.$$

Now we have

$$|v_{k_i}(\tau) - \bar{x}| \le |v_{k_i}(\tau) - v(\tau)| + |v(\tau) - \bar{x}| < \epsilon/2.$$

This inequality contradicts (47). The contradiction we have reached proves Lemma 4.1. □

Lemma 4.2. *Let $M, \epsilon > 0$. Then there exists $L_0 > 0$ such that for each number $T \geq L_0$, each a.c. function $v : [0, T] \to R^n$ satisfying*

$$|v(0)| \leq M, \qquad I^f(0, T, v) \leq Tf(\bar{x}, 0) + M \tag{53}$$

and each $s \in [0, T - L_0]$,

$$\min\{|v(t) - \bar{x}| : t \in [s, s + L_0]\} \leq \epsilon.$$

Proof. By Proposition 2.1 there exists $M_0 > M$ such that for each $T > 0$ and each a.c. function $v : [0, T] \to R^n$ which satisfies (53) the following inequality holds:

$$|v(t)| \leq M_0, \quad t \in [0, T]. \tag{54}$$

In view of (A2) there is $c_0 > 0$ such that

$$\sigma(f, T, x) \geq Tf(\bar{x}, 0) - c_0 \quad \text{for each } T > 0 \text{ and each } x \in R^n \text{ satisfying } |x| \leq M_0. \tag{55}$$

Lemma 4.1 implies that there is a number $L_0 > 0$ such that for each a.c. function $v : [0, L_0] \to R^n$ which satisfies

$$|v(0)| \leq M_0, \qquad I^f(0, L_0, v) \leq L_0 f(\bar{x}, 0) + M + 2c_0$$

the following inequality holds:

$$\min\{|v(t) - \bar{x}| : t \in [0, L_0]\} \leq \epsilon. \tag{56}$$

Assume that $T \geq L_0$, an a.c. function $v : [0, T] \to R^n$ satisfies (53) and that $S \in [0, T - L_0]$. By the choice of M_0

$$|v(S)| \leq M_0, \qquad |v(S + L_0)| \leq M_0. \tag{57}$$

In view of the choice of c_0, (55), (53) and (57)

$$I^f(0, S, v) \geq \sigma(f, S, v(0)) \geq Sf(\bar{x}, 0) - c_0, \tag{58}$$

$$I^f(S + L_0, T, v) \geq \sigma(f, T - (S + L_0), v(S + L_0)) \geq (T - (S + L_0))f(\bar{x}, 0) - c_0. \tag{59}$$

By (53), (58) and (59)

$$\begin{aligned} &I^f(S, S + L_0, v) \\ &= I^f(0, T, v) - I^f(0, S, v) - I^f(S + L_0, T, v) \\ &\leq Tf(\bar{x}, 0) + M - Sf(\bar{x}, 0) + c_0 - (T - S - L_0)f(\bar{x}, 0) + c_0 \\ &= L_0 f(\bar{x}, 0) + M + 2c_0. \end{aligned} \tag{60}$$

It follows from (60), (57) and the choice of L_0 (see (56)) that

$$\min\{|v(t) - \bar{x}| : t \in [S, L_0 + S]\} \leq \epsilon.$$

Lemma 4.2 is proved. □

For each $T > 0$ and each $x, y \in R^n$ define

$$\sigma(f,T,x,y) = \inf\bigl\{I^f(0,T,v) : \ v : [0,T] \to R^n \text{ is an a.c. function such that } v(0) = x, \ v(T) = y\bigr\}. \tag{61}$$

(We recall that infimum over empty set is ∞). By (A1) there exists $\bar{r} \in (0,1)$ such that:

$$\Omega_0 := \{(x,y) \in R^n \times R^n : \ |x - \bar{x}| \le \bar{r} \text{ and } |y| \le \bar{r}\} \subset \ \mathrm{dom}(f); \tag{62}$$

$$f \text{ is bounded on the set } \Omega_0. \tag{63}$$

It is not difficult to see that $\sigma(f,T,x,y)$ is finite for each $T \ge 1$ and each $x, y \in R^n$ such that $|x - \bar{x}|, \ |y - \bar{x}| \le \bar{r}/2$.

Lemma 4.3. *Let $\epsilon > 0$. Then there exists $\delta \in (0, \bar{r}/2)$ such that for each $T \ge 2$ and each a.c. function $v : [0,T] \to R^n$ which satisfies*

$$|v(0) - \bar{x}|, \qquad |v(T) - \bar{x}| \le \delta,$$

$$I^f(0,T,v) \le \sigma(f,T,v(0),v(T)) + \delta$$

the inequality $|v(t) - \bar{x}| \le \epsilon$ holds for all $t \in [0,T]$.

Proof. By (A1) for each natural number k there exists

$$\delta_k \in (0, 4^{-k}\bar{r}) \tag{64}$$

such that

$$|f(x,y) - f(\bar{x},0)| \le 4^{-k} \tag{65}$$

for each $x, y \in R^n$ satisfying

$$|x - \bar{x}|, \qquad |y| \le 2\delta_k. \tag{66}$$

We may assume without loss of generality that $\delta_{k+1} < \delta_k$ for all integers $k \ge 1$.

Assume that the lemma is wrong. Then for each natural number k there exist $T_k \ge 2$ and an a.c. function $v_k : [0,T_k] \to R^n$ such that

$$|v_k(0) - \bar{x}|, \qquad |v_k(T_k) - \bar{x}| \le \delta_k, \tag{67}$$

$$I^f(0,T_k,v_k) \le \sigma(f,T_k,v_k(0),v_k(T_k)) + \delta_k, \tag{68}$$

$$\max\{|v_k(t) - \bar{x}| : \ t \in [0,T_k]\} > \epsilon. \tag{69}$$

Let $k \ge 1$ be an integer. Define an a.c. function $u_k : [0,T_k] \to R^n$ as follows:

$$u_k(t) = v_k(0) + t(\bar{x} - v_k(0)), \quad t \in [0,1], \quad u_k(t) = \bar{x}, \quad t \in (1, T_k - 1],$$

$$u_k(t) = \bar{x} + (T_k - t - 1)(v_k(T_k) - \bar{x}), \quad t \in (T_k - 1, T_k]. \tag{70}$$

By (70) and (67) for each $t \in [0,1] \cup [T_k - 1, T_k]$

$$|u_k(t) - \bar{x}|, \qquad |u_k'(t)| \le \delta_k. \tag{71}$$

Together with (65) this implies that for $t \in [0,1] \cup [T_k - 1, T_k]$ a.e.

$$|f(u_k(t), u_k'(t)) - f(\bar{x}, 0)| \le 4^{-k}. \tag{72}$$

It follows from (68), (70), (72) and (54) that

$$\begin{aligned} I^f(0, T_k, v_k) &\le \sigma(f, T_k, v_k(0), v_k(T_k)) + \delta_k \le I^f(0, T_k, u_k) + \delta_k \\ &= I^f(0, 1, u_k) + I^f(T_k - 1, T_k, u_k) + (T_k - 2) f(\bar{x}, 0) + \delta_k \\ &\le T_k f(\bar{x}, 0) + 2 \cdot 4^{-k} + \delta_k \le T_k f(\bar{x}, 0) + 3 \cdot 4^{-k}. \end{aligned} \tag{73}$$

Set

$$\bar{v}_k(t) = v_k(t), \quad t \in [0, T_k], \quad \bar{v}_k(T_k + t) = v_k(T_k) + t(v_{k+1}(0) - v_k(T_k)), \quad t \in (0, 1]. \tag{74}$$

Clearly, $\bar{v}_k : [0, T_k + 1] \to R^n$ is an a.c. function,

$$\bar{v}_k(0) = v_k(0), \qquad \bar{v}_k(T_k + 1) = v_{k+1}(0). \tag{75}$$

By (74), (67) and the inequality $\delta_{k+1} < \delta_k$ for $t \in [T_k, T_k + 1]$

$$\begin{aligned} |\bar{v}_k(t) - \bar{x}| &= |(1 - t + T_k) v_k(T_k) + (t - T_k) v_{k+1}(0) - \bar{x}| \\ &\le (1 - t + T_k)|v_k(T_k) - \bar{x}| + (t - T_k)|v_{k+1}(0) - \bar{x}| \\ &\le (1 - t + T_k)\delta_k + (t - T_k)\delta_{k+1} \le \delta_k, \end{aligned} \tag{76}$$

$$|\bar{v}_k'(t)| = |v_{k+1}(0) - v_k(T_k)| \le |v_{k+1}(0) - \bar{x}| + |\bar{x} - v_k(T_k)| \le \delta_{k+1} + \delta_k \le 2\delta_k. \tag{77}$$

In view of (76), (77), (65) and (66) for $t \in [T_k, T_k + 1]$ a.e.

$$|f(\bar{v}_k(t), \bar{v}_k'(t)) - f(\bar{x}, 0)| \le 4^{-k}. \tag{78}$$

By (74), (73) and (78)

$$\begin{aligned} I^f(0, T_k + 1, \bar{v}_k) &= I^f(0, T_k, v_k) + I^f(T_k, T_k + 1, \bar{v}_k) \\ &\le T_k f(\bar{x}, 0) + 3 \cdot 4^{-k} + f(\bar{x}, 0) + 4^{-k} = (T_k + 1) f(\bar{x}, 0) + 4^{-k+1}. \end{aligned} \tag{79}$$

By (75) there exists an a.c. function $u : [0, \infty) \to R^n$ such that

$$u(t) = \bar{v}_1(t), \quad t \in [0, T_1 + 1] \tag{80}$$

and that for each integer $k \ge 1$

$$u\Big(\sum_{i=1}^{k} (T_i + 1) + t\Big) = \bar{v}_{k+1}(t), \quad t \in [0, T_{k+1} + 1]. \tag{81}$$

In view of (80), (81) and (79) for each integer $k \ge 1$

$$\begin{aligned} I^f\Big(0, \sum_{i=1}^{k+1} (T_i + 1), u\Big) &= \sum_{i=1}^{k} I^f\Big(\sum_{j=1}^{i} (T_j + 1), \sum_{j=1}^{i+1} (T_j + 1), u\Big) + I^f(0, T_1 + 1, u) \\ &= \sum_{i=1}^{k} I^f(0, 1 + T_{i+1}, \bar{v}_{i+1}) + I^f(0, T_1 + 1, \bar{v}_1) \\ &= \sum_{i=1}^{k+1} I^f(0, T_i + 1, \bar{v}_i) \\ &\le \sum_{i=1}^{k+1} [(T_i + 1) f(\bar{x}, 0) + 4^{-i+1}] \le \sum_{i=1}^{k+1} (T_i + 1) f(\bar{x}, 0) + 4. \end{aligned}$$

Since this relation holds for any integer $k \geq 1$ it follows from Proposition 1.2 that the function u is (f)-good. Together with (A4) this implies that

$$\lim_{t\to\infty} |u(t) - \bar{x}| = 0.$$

On the other hand by (80), (81), (74) and (69) $\limsup_{t\to\infty} |u(t)-\bar{x}| \geq \epsilon$. The contradiction we have reached proves Lemma 4.3. □

5. Completion of the proof of Theorem 1.3

Let $\bar{r} \in (0,1)$ satisfy (62) and (63). We may assume without loss of generality that $\epsilon < \bar{r}/2$. By Lemma 4.3 there exists a positive number $\delta < \epsilon/2$ such that the following property holds:

(P2) For each $T \geq 2$ and each a.c. function $v : [0,T] \to R^n$ which satisfies $|v(0) - \bar{x}|$, $|v(T) - \bar{x}| \leq \delta$ and

$$I^f(0,T,v) \leq \sigma(f,T,v(0),v(T)) + \delta$$

the inequality $|v(t) - \bar{x}| \leq \epsilon$ holds for all $t \in [0,T]$.

By Lemma 4.2 there exists $L_0 > 0$ such that the following property holds:

(P3) For each $T \geq L_0$, each a.c. function $v : [0,T] \to R^n$ satisfying

$$|v(0)| \leq M, \qquad I^f(0,T,v) \leq Tf(\bar{x},0) + M + 1$$

and each $S \in [0, T - L_0]$

$$\min\{|v(t) - \bar{x}| :\ t \in [S, S + L_0]\} \leq \delta.$$

Choose a natural number

$$L > 4L_0 + 4. \tag{82}$$

Assume that $T > 2L$ and an a.c. function $v : [0,T] \to R^n$ satisfies

$$v(0) \in X_M, \qquad I^f(0,T,v) \leq \sigma(f,T,v(0)) + \delta. \tag{83}$$

In view of (83)

$$|v(0)| \leq M \tag{84}$$

and there exists an ac. function $u : [0,\infty) \to R^n$ such that

$$u(0) = v(0), \qquad I^f(0,\tau,u) - \tau f(\bar{x},0) \leq M \quad \text{for each } \tau \in (0,\infty). \tag{85}$$

By (83) and (85)

$$I^f(0,T,v) \leq \delta + \sigma(f,T,v(0)) \leq 1 + I^f(0,T,u) \leq Tf(\bar{x},0) + M + 1. \tag{86}$$

It follows from (84), (86), (82) and the property (P3) that there exist

$$\tau_1 \in [0, L_0], \qquad \tau_2 \in [T - L_0, T] \tag{87}$$

such that

$$|v(\tau_i) - \bar{x}| \leq \delta, \quad i = 1, 2. \tag{88}$$

If $|v(0) - \bar{x}| \leq \delta$, then set $\tau_1 = 0$. Clearly, $\tau_2 - \tau_1 \geq T - 2L_0 > 4$. By (83)

$$I^f(\tau_1, \tau_2, v) \leq \sigma(f, \tau_2 - \tau_1, v(\tau_1), v(\tau_2)) + \delta. \tag{89}$$

It follows from (89), (88) and the relation $\tau_2 - \tau_1 > 4$ that $|v(t) - \bar{x}| \leq \epsilon$ for all $t \in [\tau_1, \tau_2]$. Theorem 1.3 is proved.

6. Proof of Theorem 1.4

Let $x \in R^n$ and let $v : [0, \infty) \to R^n$ be an (f)-good function satisfying $v(0) = x$. Let $\{T_k\}_{k=1}^{\infty}$ be a strictly increasing sequence of natural numbers. By definition there is $c_0 > 0$ such that

$$|I^f(0, S, v) - Sf(\bar{x}, 0)| \leq c_0 \quad \text{for each } T \in (0, \infty). \tag{90}$$

By Proposition 2.2 for each natural number T_k there exists an a.c. function $v_k : [0, T_k] \to R^n$ such that

$$v_k(0) = x, \qquad I^f(0, T_k, v_k) = \sigma(f, T_k, x). \tag{91}$$

In view of (90) and (91) for each integer $k \geq 1$

$$I^f(0, T_k, v_k) \leq I^f(0, T_k, v) \leq T_k f(\bar{x}, 0) + c_0. \tag{92}$$

It follows from (92), (91) and Proposition 2.1 that there exists $M_0 > 0$ such that for each integer $k \geq 1$

$$|v_k(t)| \leq M_0, \quad t \in [0, T_k]. \tag{93}$$

By (A2) there is $c_1 > 0$ such that

$$\sigma(f, S, z) \geq Sf(\bar{x}, 0) - c_1 \quad \text{for each } S > 0 \text{ and each } z \in R^n \text{ satisfying } |z| \leq M_0. \tag{94}$$

Relations (93) and (94) imply that for each integer $k \geq 1$ and each $S \in [0, T_k)$

$$I^f(S, T_k, v_k) \geq (T_k - S) f(\bar{x}, 0) - c_1. \tag{95}$$

Together with (92) this inequality implies that for each integer $k \geq 1$ and each $S \in (0, T_k)$

$$\begin{aligned} I^f(0, S, v_k) &= I^f(0, T_k, v_k) - I^f(S, T_k, v_k) \\ &\leq T_k f(\bar{x}, 0) + c_0 - (T_k - S) f(\bar{x}, 0) + c_1 = Sf(\bar{x}, 0) + c_0 + c_1. \end{aligned} \tag{96}$$

By (96) for each integer $m \geq 1$ the sequence $\{I^f(0, m, v_k)\}_{k=m}^{\infty}$ is bounded. Together with Proposition 2.2 this implies that there exist a strictly increasing sequence of natural numbers $\{k_i\}_{i=1}^{\infty}$ and an a.c. function $u : [0, \infty) \to R^n$ such that for each integer $m \geq 1$

$$v_{k_i}(t) \to u(t) \quad \text{as } i \to \infty \text{ uniformly on } [0, m], \tag{97}$$

$$I^f(0, m, u) \leq \liminf_{i \to \infty} I^f(0, m, v_{k_i}). \tag{98}$$

In view of (98) and (96) for each integer $m \geq 1$

$$I^f(0, m, u) \leq mf(\bar{x}, 0) + c_0 + c_1. \tag{99}$$

Therefore u is an (f)-good function and

$$\lim_{t \to \infty} |u(t) - \bar{x}| = 0. \tag{100}$$

We show that u is an (f)-overtaking optimal function. Let us assume the contrary. Then there exists an a.c. function $w : [0, \infty) \to R^n$ such that

$$w(0) = u(0), \qquad \limsup_{T \to \infty} [I^f(0, T, u) - I^f(0, T, w)] > \Delta, \tag{101}$$

where Δ is a positive constant. Since u is (f)-good it follows from (101) and Proposition 1.2 that w is an (f)-good function. Hence

$$\lim_{t\to\infty} |w(t) - \bar{x}| = 0. \tag{102}$$

By (A1) and Proposition 2.3 there is $\delta \in (0,1)$ such that:

$$\{(y,z) \in R^n \times R^n : \ |y - \bar{x}| \le 4\delta,\ |z| \le 4\delta\} \subset \ \mathrm{dom}(f), \tag{103}$$

$$|f(y,z) - f(\bar{x},0)| \le \Delta/16 \tag{104}$$

for each $y \in R^n$ satisfying $|y - \bar{x}| \le 4\delta$ and each $z \in R^n$ satisfying $|z| \le 4\delta$;

for each a.c. function $v : [0,1] \to R^n$ satisfying $|v(0) - \bar{x}|,\ |v(1) - \bar{x}| \le 4\delta$ we have

$$I^f(0,1,v) \ge f(\bar{x},0) - \Delta/16. \tag{105}$$

Relations (100) and (102) imply that there exists $\tau_0 \ge 4$ such that

$$|w(t) - \bar{x}|,\ |u(t) - \bar{x}| \le \delta/4 \quad \text{for all numbers } t \ge \tau_0. \tag{106}$$

By (101) there exists an integer $\tau_1 \ge 4(\tau_0 + 4)$ such that

$$I^f(0,\tau_1,u) - I^f(0\tau_1,w) > \Delta. \tag{107}$$

In view of (97) and (98) there exists a natural number q such that

$$T_q > 4(\tau_1 + 4), \tag{108}$$

$$|v_q(t) - u(t)| \le \delta/16, \quad t \in [0, 4\tau_1 + 4], \tag{109}$$

$$I^f(0,\tau_1,u) \le I^f(0,\tau_1,v_q) + \Delta/64. \tag{110}$$

Define an a.c. function $\tilde{v} : [0,T_q] \to R^n$ as follows:

$$\begin{aligned}
&\tilde{v}(t) = w(t), \quad t \in [0,\tau_1],\\
&\tilde{v}(t) = w(\tau_1) + (t - \tau_1)(v_q(\tau_1 + 1) - w(\tau_1)), \quad t \in (\tau_1, \tau_1 + 1],\\
&\tilde{v}(t) = v_q(t), \quad t \in (\tau_1 + 1, T_q].
\end{aligned} \tag{111}$$

By (111), (110) and (107)

$$\begin{aligned}
&I^f(0,T_q,\tilde{v}) - I^f(0,T_q,v_q)\\
&= I^f(0,\tau_1+1,\tilde{v}) - I^f(0,\tau_1+1,v_q)\\
&= I^f(0,\tau_1,w) - I^f(0,\tau_1,v_q) + I^f(\tau_1,\tau_1+1,\tilde{v}) - I^f(\tau_1,\tau_1+1,v_q)\\
&\le I^f(0,\tau_1,w) - I^f(0,\tau_1,u) + \Delta/64 + I^f(\tau_1,\tau_1+1,\tilde{v}) - I^f(\tau_1,\tau_1+1,v_q)\\
&\le \ -\Delta + \Delta/64 + I^f(\tau_1,\tau_1+1,\tilde{v}) - I^f(\tau_1,\tau_1+1,v_q).
\end{aligned} \tag{112}$$

In view of (109) and (106) for $s = \tau_1,\ \tau_1 + 1$

$$|v_q(s) - \bar{x}| \le |v_q(s) - u(s)| + |u(s) - \bar{x}| \le \delta/16 + \delta/4. \tag{113}$$

Combined with (105) this inequality implies that

$$I^f(\tau_1, \tau_1 + 1, v_q) \geq f(\bar{x}, 0) - \Delta/16. \tag{114}$$

Relations (111), (113) and (106) imply that for all $t \in (\tau_1, \tau_1 + 1)$

$$\begin{aligned} |\tilde{v}(t) - \bar{x}| &\leq (1 - t + \tau_1)|w(\tau_1) - \bar{x}| + (t - \tau_1)|v_q(\tau_1 + 1) - \bar{x}| \\ &\leq (1 - t + \tau_1)\delta/4 + (t - \tau_1)\delta/16 + (t - \tau_1)\delta/4 < \delta/2. \end{aligned} \tag{115}$$

It follows from (111), (109) and (106) that for all $t \in (\tau_1, \tau_1 + 1)$

$$\begin{aligned} |\tilde{v}'(t)| &= |v_q(\tau_1 + 1) - w(\tau_1)| \\ &\leq |v_q(\tau_1 + 1) - u(\tau_1 + 1)| + |u(\tau_1 + 1) - \bar{x}| \\ &\qquad + |\bar{x} - w(\tau_1)| \leq \delta/16 + \delta/4 + \delta/4 < (3/4)\delta. \end{aligned} \tag{116}$$

In view of (115), (116) and (104) for all $t \in (\tau_1, \tau_1 + 1)$, $f(\tilde{v}(t), \tilde{v}'(t)) \leq f(\bar{x}, 0) + \Delta/16$ and

$$I^f(\tau_1, \tau_1 + 1, \tilde{v}) \leq f(\tilde{x}, 0) + \Delta/16.$$

Combined with (112) and (114) this inequality implies that

$$\begin{aligned} &I^f(0, T_q, \tilde{v}) - I^(0, T_q, v_q) \\ &\leq -\Delta + \Delta/64 + f(\tilde{x}, 0) + \Delta/16 - f(\bar{x}, 0) + \Delta/16 < -\Delta/2. \end{aligned}$$

Since $\tilde{v}(0) = w(0) = u(0) = x = v_q(0)$ the inequality above contradicts (91). The contradiction we have reached shows that u is an (f)-overtaking optimal function. Theorem 1.4 is proved.

7. Examples

Example 7.1. Let $a_0 > 0$, $\psi_0 : [0, \infty) \to [0, \infty)$ be an increasing function satisfying

$$\lim_{t \to \infty} \psi_0(t) = \infty$$

and let $L : R^n \times R^n \to [0, \infty]$ be a lower semicontinuous function such that

$$\mathrm{dom}(L) := \{(x, y) \in R^n \times R^n : \ L(x, y) < \infty\} \tag{117}$$

is nonempty, convex, closed set and

$$L(x, y) \geq \max\{\psi_0(|x|), \ \psi_0(|y|)|y|\} - a_0 \quad \text{for each } x, y \in R^n. \tag{118}$$

Assume that for each $x \in R^n$ the function $L(x, \cdot) : R^n \to R^1 \cup \{\infty\}$ is convex and that there exists $\bar{x} \in R^n$ such that

$$L(x, y) = 0 \quad \text{if and only if } (x, y) = (\bar{x}, 0), \tag{119}$$

$(\bar{x}, 0)$ is an interior point of $\mathrm{dom}(L)$ and that L is continuous at $(\bar{x}, 0)$.

Let $\mu \in R^1$ and $l \in R^n$. Define

$$f(x, y) = L(x, y) + \mu + \langle l, y \rangle, \quad x, y \in R^n. \tag{120}$$

We will show that all the assumptions introduced in Section 1 hold for f.

First note that the function f is lower semicontinuous and that dom(f) =dom(L). Set $\psi(t) = (3/4)\psi_0(t),\ t \in [0, \infty)$. Clearly, there exists $K_0 > 1$ such that

$$\psi_0(K_0) > 4|l| + 4. \tag{121}$$

Set

$$a = a_0 + |\mu| + |l|K_0. \tag{122}$$

We show that (3) holds. Let $x, y \in R^n$. If $|y| \le K_0$, then by (120) and (118)

$$\begin{aligned} f(x,y) &= L(x,y) + \mu + \langle l, y\rangle \\ &\ge \max\{\psi_0(|x|),\ \psi_0(|y|)|y|\} - a_0 - |l||y| - |\mu| \\ &\ge \max\{\psi_0(|x|),\ \psi_0(|y|)|y|\} - a_0 - |\mu| - |l|K_0 \\ &\ge \max\{\psi(|x|), \psi(|y|)|y|\} - a. \end{aligned}$$

Thus (3) holds if $|y| \le K_0$. Assume that

$$|y| > K_0. \tag{123}$$

There are two cases:

$$\psi_0(|x|) \ge \psi_0(|y|)|y|; \tag{124}$$

$$\psi_0(|x|) < \psi_0(|y|)|y|. \tag{125}$$

Assume that (124) holds. By (123), (121) and (124)

$$|\langle l, y\rangle| \le |l||y| \le 4^{-1}\psi_0(K_0)|y| \le 4^{-1}\psi_0(|y|)|y| \le 4^{-1}\psi_0|(x)|.$$

Together with (120), (118) and (122) this implies that

$$\begin{aligned} f(x,y) &= L(x,y) + \mu + \langle l, y\rangle \\ &\ge \max\{\psi_0(|x|),\ \psi_0(|y|)|y|\} - a_0 - |\mu| - |\langle l, y\rangle| \\ &\ge \max\{\psi_0(|x|),\ \psi_0(|y|)|y|\} - a_0 - |\mu| - 4^{-1}\max\{\psi_0(|y|)|y|, \psi_0(|x|)\} \\ &= (3/4)\max\{\psi_0(|x|),\ \psi_0(|y|)|y|\} - a_0 - |\mu| \\ &\ge \max\{\psi(|x|),\ \psi(|y|)|y|\} - a. \end{aligned}$$

Thus (3) holds if (124) is valid.

Assume that (125) holds. Then by (121) and (123)

$$|\langle l, y\rangle| \le |l||y| \le 4^{-1}\psi_0(K_0)|y| \le 4^{-1}\psi_0(|y|)|y|.$$

Together with (120), (118) and (125) this inequality implies that

$$\begin{aligned} f(x,y) = L(x,y) + \mu + \langle l, y\rangle &\ge \psi_0(|y|)|y| - a_0 - |\mu| - 4^{-1}\psi_0(|y|)|y| \\ &\ge -a_0 - |\mu| + (3/4)\psi_0(|y|)|y| \\ &\ge -a_0 - |\mu| + (3/4)\psi_0(|x|). \end{aligned}$$

Together with (122) and the definition of ψ this implies that

$$f(x,y) \ge (3/4)\max\{\psi_0(|x|),\ \psi_0(|y|)|y|\} - a_0 - |\mu| \ge \max\{\psi(|x|),\ \psi(|y)|y|\} - a.$$

Thus (3) holds if (125) is valid and

$$f(x,y) \geq \max\{\psi(|x|),\ \psi(|y|)|y|\} - a \quad \text{for all } x, y \in R^n.$$

By (120) and (119),

$$\mu = f(\bar{x}, 0) \leq f(x, 0) \quad \text{for each } x \in R^n.$$

Clearly, (A1) and (A3) hold.

Proposition 7.2. (A2) *holds.*

Proof. Let $M > 0$. By definition of ψ there is $M_0 > M + 1$ such that

$$\psi(M_0) > |\mu| + 1 + a. \tag{126}$$

Set

$$c_M = (|l| + 1)(2M_0 + 1). \tag{127}$$

Let $T > 0$ and $x \in R^n$ satisfy $|x| \leq M$. We will show that

$$\sigma(f, T, x) \geq T f(\bar{x}, 0) - c_M = T\mu - c_M. \tag{128}$$

We may assume without loss of generality that $\sigma(f, T, x)$ is finite. There exists an a.c. function $v : [0, T] \to R^n$ such that

$$v(0) = x, \qquad \int_0^T f(v(t), v'(t))dt \leq \sigma(f, T, x) + 1 \tag{129}$$

By the relation $|x| \leq M$ and (129) there exists $T_0 \in (0, T]$ such that

$$|v(T_0)| \leq M_0, \qquad |v(t)| > M_0 \quad \text{if } t \text{ satisfies } T_0 < t \leq T. \tag{130}$$

By (130), (128), (126), (120) and (129)

$$\int_0^T f(v(t), v'(t))dt = \int_0^{T_0} f(v(t), v'(t))dt + \int_{T_0}^T f(v(t), v'(t))dt$$
$$\geq \int_0^{T_0} f(v(t), v'(t))dt + (T - T_0)(\psi(M_0) - a)$$
$$\geq \int_0^{T_0} f(v(t), v'(t))dt + (T - T_0)|\mu| \geq \int_0^{T_0} [\mu + \langle l, v'(t)\rangle]dt + (T - T_0)|\mu|$$
$$\geq T\mu + \langle l, v(T_0) - v(0)\rangle \geq T\mu - |l|2M_0,$$

$$\sigma(f, T, x) \geq T\mu - 2|l|M_0 - 1.$$

Proposition 7.2 is proved. □

The next result shows that (A4) holds for the integrand f.

Proposition 7.3. *Let $v : [0, \infty) \to R^n$ be an (f)-good function. Then*

$$\lim_{t \to \infty} |v(t) - \bar{x}| = 0.$$

Proof. By Proposition 1.2 and (130),

$$\sup\{|v(t)| : \ t \in [0,\infty)\} < \infty, \tag{131}$$

$$\lim_{T\to\infty} \int_0^T L(v(t), v'(t))dt < \infty. \tag{132}$$

For each integer $i \ge 0$ set

$$v_i(t) = v(t+i), \quad t \in [0,1]. \tag{133}$$

Assume that the assertion of the proposition does not hold. Then there exist $\epsilon > 0$ and a strictly increasing sequence of natural numbers $\{i_k\}_{k=1}^{\infty}$ such that for all integers $k \ge 1$

$$\sup\{|v_{i_k}(t) - \bar{x}| : \ t \in [0,1]\} \ge \epsilon. \tag{134}$$

By Proposition 2.2 and (131)–(133), extracting a sequence and re-indexing if necessary, we may assume without loss of generality that there exists an a.c. function $u : [0,1] \to R^n$ such that

$$v_{i_k}(t) \to u(t) \quad \text{as } k \to \infty \text{ uniformly on } [0,1], \tag{135}$$

$$I^f(0,1,u) \le \liminf_{k\to\infty} I^f(0,1,v_{i_k}). \tag{136}$$

Relations (134) and (135) imply that

$$\sup\{|u(t) - \bar{x}| : \ t \in [0,1]\} \ge \epsilon/4. \tag{137}$$

In view of (120), (136), (135), (133) and (132)

$$\begin{aligned}
\int_0^1 L(u(t), u'(t))dt &= \int_0^1 f(u(t), u'(t))dt - \mu - \int_0^1 \langle l, u'(t)\rangle dt \\
&\le \liminf_{k\to\infty} \int_0^1 f(v_{i_k}(t), v'_{i_k}(t))dt - \mu - \lim_{k\to\infty} \int_0^1 \langle l, v'_{i_k}(t)\rangle dt \\
&\le \liminf_{k\to\infty} \int_0^1 L(v_{i_k}(t), v'_{i_k}(t))dt = 0.
\end{aligned}$$

Therefore $L(u(t), u'(t)) = 0, \ t \in [0,1]$, (a.e.) and in view of (119) $u(t) = \bar{x}$ for all $t \in [0,1]$. This contradicts (137). The contradiction we have reached proves Proposition 7.3. □

Thus all the assumptions introduced in Section 1 hold for f.

Example 7.4. Let $a > 0$, $\psi : [0,\infty) \to [0,\infty)$ be an increasing function such that $\lim_{t\to\infty} \psi(t) = \infty$ and let $f : R^n \times R^n \to R^1 \cup \{\infty\}$ be a convex lower semicontinuous function such that the set $\mathrm{dom}(f)$ is nonempty, convex and closed and that

$$f(x,y) \ge \max\{\psi(|x|), \ \psi(|y|)|y|\} - a \quad \text{for each } x, y \in R^n.$$

We assume that there exists $\bar{x} \in R^n$ such that

$$f(\bar{x}, 0) \le f(x, 0) \quad \text{for each } x \in R^n$$

and that $(\bar{x}, 0)$ is an interior point of the set $\mathrm{dom}(f)$. It is known that f is continuous at $(\bar{x}, 0)$. It is well-known fact of convex analysis that there is $l \in R^n$ such that

$$f(x, y) \geq f(\bar{x}, 0) + \langle l, y \rangle \quad \text{for each } x, y \in R^n.$$

We assume that for each (x_1, y_1), $(x_2, y_2) \in \mathrm{dom}(f)$ satisfying $(x_1, y_1) \neq (x_2, y_2)$ and each $\alpha \in (0, 1)$

$$f(\alpha(x_1, y_1) + (1 - \alpha)(x_2, y_2)) < \alpha f(x_1, y_1) + (1 - \alpha) f(x_2, y_2).$$

Set

$$L(x, y) = f(x, y) - f(\bar{x}, 0) - \langle l, y \rangle \quad \text{for each } x, y \in R^n.$$

It is not difficult to see that there exist $a_0 > 0$ and an increasing function $\psi_0 : [0, \infty) \to [0, \infty)$ such that

$$L(x, y) \geq \max\{\psi_0(|x|),\ \psi_0(|y|)|y|\} - a_0 \quad \text{for all } x, y \in R^n.$$

It is also clear that L is a convex, lower semicontinuous functon and $L(x, y) = 0$ if and only if $(x, y) = (\bar{x}, 0)$. Now it is easy to see that our example is a particular case of Example 7.1 and all the assumptions introduced in Section 1 hold for f.

References

[1] B. D. O. Anderson, J. B. Moore: Linear Optimal Control, Prentice-Hall, Englewood Cliffs (1971).

[2] H. Atsumi: Neoclassical growth and the efficient program of capital accumulation, Rev. Econ. Stud. 32 (1965) 127–136.

[3] S. Aubry, P. Y. Le Daeron: The discrete Frenkel-Kontorova model and its extensions. I, Physica D 8 (1983) 381–422.

[4] J. Blot, P. Cartigny: Optimality in infinite-horizon variational problems under sign conditions, J. Optimization Theory Appl. 106 (2000) 411–419.

[5] J. Blot, P. Michel: The value-function of an infinite-horizon linear quadratic problem, Appl. Math. Lett. 16 (2003) 71–78.

[6] D. A. Carlson: The existence of catching-up optimal solutions for a class of infinite horizon optimal control problems with time delay, SIAM J. Control Optimization 28 (1990) 402–422.

[7] D. A. Carlson, A. Haurie, A. Leizarowitz: Infinite Horizon Optimal Control, Springer, Berlin (1991).

[8] L. Cesari: Optimization – Theory and Applications, Springer, New York (1983).

[9] A. V. Dmitruk, N. V. Kuz'kina: Existence theorem in the optimal control problem on an infinite time interval, Math. Notes 78 (2005) 466–480.

[10] Z. Dzalilov, A. M. Rubinov, P. E. Kloeden: Lyapunov sequences and a turnpike theorem without convexity, Set-Valued Anal. 6 (1998) 277–302.

[11] D. Gale: On optimal development in a multi-sector economy, Rev. Econ. Stud. 34 (1967) 1–18.

[12] V. Glizer, J. Shinar: On the structure of a class of time-optimal trajectories, Optim. Control Appl. Methods 14 (1993) 271–279.

[13] M. Ali Khan, T. Mitra: On choice of technique in the Robinson-Solow-Srinivasan model, Int. J. Econ. Theory 1 (2005) 83–110.

[14] A. Leizarowitz: Infinite horizon autonomous systems with unbounded cost, Appl. Math. Optimization 13 (1985) 19–43.

[15] A. Leizarowitz: Tracking nonperiodic trajectories with the overtaking criterion, Appl. Math. Optimization 14 (1986) 155–171.

[16] A. Leizarowitz, V. J. Mizel: One dimensional infinite horizon variational problems arising in continuum mechanics, Arch. Ration. Mech. Anal. 106 (1989) 161–194.

[17] V. L. Makarov, A. M. Rubinov: Mathematical Theory of Economic Dynamics and Equilibria, Springer, New York (1977).

[18] M. Marcus, A. J. Zaslavski: The structure of extremals of a class of second order variational problems, Ann. Inst. H. Poincaré, Anal. Non Linéaire 16 (1999) 593–629.

[19] M. Marcus, A. J. Zaslavski: The structure and limiting behavior of locally optimal minimizers, Ann. Inst. H. Poincaré, Anal. Non Linéaire 19 (2002) 343–370.

[20] L. W. McKenzie: Turnpike theory, Econometrica 44 (1976) 841–866.

[21] L. W. McKenzie: Classical General Equilibrium Theory, MIT Press, Cambridge (2002).

[22] B. Mordukhovich: Minimax design of a class of distributed parameter systems, Autom. Remote Control 50 (1990) 1333–1340.

[23] B. Mordukhovich, I. Shvartsman: Optimization and feedback control of constrained parabolic systems under uncertain perturbations, in: Optimal Control, Stabilization and Nonsmooth Analysis (Baton Rouge, 2003), M. de Queiroz et al. (ed.), Lect. Notes Control Inf. Sci., Springer, Berlin (2004) 121–132.

[24] S. Pickenhain, V. Lykina: Sufficiency conditions for infinite horizon optimal control problems, in: Recent Advances in Optimization (Avignon, 2004), A. Seeger (ed.), Lect. Notes Econ. Math. Syst. 563, Springer, Berlin (2006) 217–232.

[25] A. M. Rubinov: Superlinear Multivalued Mappings and their Applications to Problems of Mathematical Economics, Nauka, Leningrad (1980) (in Russian).

[26] P. A. Samuelson: A catenary turnpike theorem involving consumption and the golden rule, Am. Econ. Rev. 55 (1965) 486–496.

[27] C. C. von Weizsäcker: Existence of optimal programs of accumulation for an infinite horizon, Rev. Econ. Stud. 32 (1965) 85–104.

[28] A. J. Zaslavski: Ground states in Frenkel-Kontorova model, Math. USSR, Izv. 29 (1987) 323–354.

[29] A. J. Zaslavski: Optimal programs on infinite horizon. I, SIAM J. Control Optimization 33 (1995) 1643–1660.

[30] A. J. Zaslavski: Optimal programs on infinite horizon. II, SIAM J. Control Optimization 33 (1995) 1661–1686.

[31] A. J. Zaslavski: Turnpike theorem for convex infinite dimensional discrete-time control systems, J. Convex Analysis 5 (1998) 237–248.

[32] A. J. Zaslavski: Turnpike theorem for nonautonomous infinite dimensional discrete-time control systems, Optimization 48 (2000) 69–92.

[33] A. J. Zaslavski: Turnpike Properties in the Calculus of Variations and Optimal Control, Springer, New York (2006).