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A Lower Semicontinuity Result in SBD

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A lower semicontinuity result is proved in the space of special functions of bounded deformation for a fracture energetic model according to Barenblatt's theory, i.e.

$$\int_{J_u} \varphi([u]\cdot\nu_u)d\mathcal{H}^{N-1},\ \ [u]\cdot\nu_u \geq 0\ \mathcal{H}^{N-1}\text{-a.e. on } J_u$$

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1. Introduction

We consider a surface energy, appearing in the context of Barenblatt's model of fracture mechanics (see [7]), which takes into account the fractures appearing under the regime of linearized elasticity. It depends on the normal component of the opening of the fracture and includes a constraint which prevents infinitesimal interpenetration (cf. the energy (1) below).

More precisely, the aim of this paper consists of finding sufficient conditions for the semicontinuity of the functional below, by seeking suitable properties on the energy surface density φ,

$$\mathcal{F}(u) := \begin{cases} \int_{J_u}\varphi([u]\cdot\nu_u)d\mathcal{H}^{N-1} & \text{if } [u]\cdot\nu_u \geq 0\ \mathcal{H}^{N-1}\text{-a.e. on } J_u, \\ +\infty & \text{otherwise,} \end{cases} \tag{1}$$

where the deformation u belongs to $SBD(\Omega)$, Ω being a bounded open set in $\mathbb{R}^N$, $[u]$ denotes the jump of u, J_u the jump set, and ν_u is the normal to the jump set.

The surface integral in (1) represents the energy dissipated in the fracture process and the constraint $[u]\cdot\nu_u \geq 0$ $\mathcal{H}^{N-1}$-a.e. on J_u takes into account infinitesimal noninterpenetration.

By virtue of the by now classical compactness result quoted in Theorem 2.1 below (see Theorem 1.1 in [8]), a natural requirement is to investigate the lower semicontinuity of the energy (1) with respect to the convergences (11) ÷ (13) below.

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To this end we introduce the family

$$\Phi := \{\varphi : [0,+\infty[\to [0,+\infty[, \varphi \text{ convex, subadditive and nondecreasing}\} \tag{2}$$

Indeed our main result is the following theorem.

Theorem 1.1. *Let Ω be a bounded open subset of $\mathbb{R}^N$, and let $\theta : [0,+\infty[\to [0,+\infty[$ be a non-decreasing function verifying* (9). *Let φ be in Φ in* (2). *Let $\{u_h\}$ be a sequence in $SBD(\Omega)$, such that $[u_h]\cdot\nu_{u_h} \geq 0$ $\mathcal{H}^{N-1}$-a.e. on J_{u_h} for every h, converging to u in $L^1(\Omega;\mathbb{R}^N)$ satisfying* (10). *Then*

$$[u]\cdot\nu_u \geq 0 \ \ \mathcal{H}^{N-1}\text{-a.e. on } J_u \tag{3}$$

and

$$\int_{J_u} \varphi([u]\cdot\nu_u)\,d\mathcal{H}^{N-1} \leq \liminf_{h\to+\infty} \int_{J_{u_h}} \varphi([u_h]\cdot\nu_{u_h})\,d\mathcal{H}^{N-1} \tag{4}$$

We remark that our conclusions cannot be obtained by mere extension to SBD of the results already avalaible in the framework of SBV spaces.

Furthermore we stress that our result (see Theorem 1.1) can be considered as a first step towards an extension to linearized elasticity of well known semicontinuity results in SBV due to Ambrosio (see [1], [3] and [4]) (see also [6] for more extensive and detailed results) and we do not regard it as optimal.

We also provide a characterization, in terms of Convex Analysis, of the surface energy densities φ in (1) which ensure lower semicontinuity.

To this aim, given two families $\{a_\alpha\}_{\alpha\in\mathcal{A}}$ and $\{b_\alpha\}_{\alpha\in\mathcal{A}}$ in $[0,+\infty[$, we consider the function φ defined by

$$\varphi(s) := \sup_{\alpha\in\mathcal{A}} (a_\alpha s + b_\alpha) \tag{5}$$

for every $s \in [0,+\infty[$ and we assume that $\varphi(s) < +\infty$ for every $s \in [0,+\infty[$.

It is worthwhile to observe that since $\mathbb{R}$ is second countable and by virtue of Lindelöff Theorem, one can assume without loss of generality that $\mathcal{A}$ in (5) is countable.

The following result will be proved.

Theorem 1.2. *Let $\varphi : [0,+\infty[\to [0+\infty[$. The following conditions are equivalent:*

(1) *φ belongs to Φ.*

(2) *φ is defined by* (5) *for some families $\{a_\alpha\}_{\alpha\in\mathcal{A}}$, and $\{b_\alpha\}_{\alpha\in\mathcal{A}}$ in $[0,\infty[$.*

(3) *φ is convex and nondecreasing, and $s \mapsto \frac{\varphi(s)}{s}$ is nonincreasing.*

(4) *φ is convex and nondecreasing, and*

$$\frac{\varphi(s)}{s} \geq \varphi'_+(s) \text{ for every } s > 0. \tag{6}$$

(5) *φ is convex and nondecreasing, and the polar function φ^* (extending φ to $+\infty$ in $]-\infty,0[$) is nonpositive on its effective domain.*

An example of functions in Φ is given by

$$\varphi : s \in \mathbb{R}^+ \mapsto (1 + s^p)^{\frac{1}{p}}.$$

We remark that the class of functions described by (3) in Theorem 1.2 above, is known in the literature in the context of ODE's as the class of functions of 'softening characteristic', see for instance [15].

The paper is organized as follows: in Section 2 we recall the definition and main properties of functions of bounded deformation and some results of Measure Theory that will be exploited in the proof of Theorem 1.1 and Theorem 1.2 respectively. Section 3 is essentially devoted to the proof of Theorem 1.1; at the same time several applications to minimum problems are presented in the case where volume energy terms are added to the surface energy in (1). In Section 4 several characterizations of the classes of surface integrands that provide lower semicontinuity to the functional (1) are given.

2. Notations and Preliminaries

We start by recalling the fundamental properties of functions of bounded deformation.

Here and in the sequel, let Ω be a bounded open subset of $\mathbb{R}^N$. The space $BD(\Omega)$ of *vector fields with bounded deformation* in a bounded open set Ω of $\mathbb{R}^N$ is defined as the set of vector fields $u = (u^1, \dots, u^N) \in L^1(\Omega; \mathbb{R}^N)$ whose distributional gradient $Du = \{D_i u^j\}$ has the symmetric part

$$Eu = \{E_{ij}u\}, E_{ij}u = (D_i u^j + D_j u^i)/2$$

which belongs to $\mathcal{M}_b(\Omega; M^{N\times N}_{sym})$, the space of bounded Radon measures in Ω with values in $M^{N\times N}_{sym}$, the space of symmetric $N\times N$ matrices.

For $u \in BD(\Omega)$, the *jump set* J_u is defined as the set of points $x \in \Omega$ where u has two different *one sided Lebesgue limits* $u^+(x)$ and $u^-(x)$, with respect to a suitable direction $\nu_u(x) \in S^{N-1} = \{\xi \in \mathbb{R}^N : |\xi| = 1\}$, i.e.

$$\lim_{\varrho\to 0^+} \frac{1}{\varrho^N} \int_{B^{\pm}_{\varrho}(x,\nu_u(x))} |u(y) - u^{\pm}(x)| dy = 0, \tag{7}$$

where $B^{\pm}_{\varrho}(x, \nu_u(x)) = \{y \in \mathbb{R}^N : |y - x| < \varrho, (y - x, \pm\nu_u(x)) > 0\}$.

In [5] it has been proved that for every $u \in BD(\Omega)$ the jump set J_u is Borel measurable and countably $(\mathcal{H}^{N-1}, N-1)$ rectifiable and $\nu_u(x)$ is normal to the approximate tangent space to J_u at x for $\mathcal{H}^{N-1}$-a.e. $x \in J_u$, where $\mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure (see [6] and [13]).

Let $u \in BD(\Omega)$, the Lebesgue decomposition of Eu is written as

$$Eu = E^a u + E^s u$$

with $E^a u$ the absolutely continuous part and $E^s u$ the singular part with respect to the Lebesgue measure $\mathcal{L}^N$.

The density of $E^a u$ with respect to $\mathcal{L}^N$ is denoted by $\mathcal{E}u$, i.e. $E^a u = \mathcal{E}u\mathcal{L}^N$.

We recall that $E^s u$ can be further decomposed as

$$E^s u = E^j u + E^c u$$

with $E^j u$, the *jump part* of Eu, i.e. the restriction of $E^s u$ to J_u and $E^c u$ the *Cantor part* of Eu, i.e. the restriction of $E^s u$ to $\Omega \setminus J_u$.

In [5] it has been proved that

$$E^j u = (u^+ - u^-) \odot \nu_u \mathcal{H}^{N-1} \lfloor J_u \tag{8}$$

where $\odot$ denotes the symmetric tensor product, defined by $a \odot b := (a \otimes b + b \otimes a)/2$ for every $a, b \in \mathbb{R}^N$, and $\mathcal{H}^{N-1} \lfloor J_u$ denotes the restriction of $\mathcal{H}^{N-1}$ to J_u, i.e. $(\mathcal{H}^{N-1} \lfloor J_u)(B) = \mathcal{H}^{N-1}(B \cap J_u)$ for every Borel set $B \subseteq \Omega$. Moreover in [5] it has been also proved that $|E^c u|(B) = 0$ for every Borel set $B \subseteq \Omega$ such that $\mathcal{H}^{N-1}(B) < +\infty$, where $|\cdot|$ stands for the total variation.

The space $SBD(\Omega)$ of *special vector fields with bounded deformation* is defined as the set of all $u \in BD(\Omega)$ such that $E^c u = 0$, or, in other words

$$Eu = \mathcal{E}u\mathcal{L}^N + (u^+ - u^-) \odot \nu_u \mathcal{H}^{N-1} \lfloor J_u \, .$$

In the sequel, for every $u \in L^1_{loc}(\Omega; \mathbb{R}^N)$ we denote by $[u]$ the vector $u^+ - u^-$, according to the notations introduced in (7).

We recall the following compactness result, proved in [8], (cf. Theorem 1.1 and Remark 2.3 therein) that will be exploited in the sequel.

Theorem 2.1. *Let* $\theta : [0, +\infty[\to [0, +\infty[$ *be a non-decreasing function such that*

$$\lim_{t \to +\infty} \frac{\theta(t)}{t} = +\infty. \tag{9}$$

Let $\{u_h\}$ *be a sequence in* $SBD(\Omega)$ *such that*

$$\|u_h\|_{L^\infty(\Omega;\mathbb{R}^N)} + \int_\Omega \theta(|\mathcal{E}u_h|)dx + \mathcal{H}^{N-1}(J_{u_h}) \leq K \tag{10}$$

for some constant K *independent of* h*. Then there exists a subsequence, still denoted by* $\{u_h\}$*, and a function* $u \in SBD(\Omega)$ *such that*

$$u_h \to u \text{ strongly in } L^1_{loc}(\Omega; \mathbb{R}^N), \tag{11}$$

$$\mathcal{E}u_h \rightharpoonup \mathcal{E}u \text{ weakly in } L^1(\Omega; M^{N\times N}_{sym}), \tag{12}$$

$$E^j u_h \rightharpoonup E^j u \text{ weakly* in } \mathcal{M}_b(\Omega; M^{N\times N}_{sym}), \tag{13}$$

$$\mathcal{H}^{N-1}(J_u) \leq \liminf_{h\to+\infty} \mathcal{H}^{N-1}(J_{u_h}) \tag{14}$$

We also recall the following result from Measure Theory that will be exploited in the sequel, (cf. [6] and [9]).

Theorem 2.2. *Let λ be a positive σ-finite Borel measure in Ω and let $\varphi_i : \Omega \to [0, \infty]$, $i \in \mathbb{N}$, be Borel functions. Then*

$$\int_\Omega \sup_i \varphi_i d\lambda = \sup \left\{ \sum_{i \in I} \int_{A_i} \varphi_i d\lambda \right\}$$

where the supremum ranges over all finite sets $I \subset \mathbb{N}$ and all families $\{A_i\}_{i \in I}$ of pairwise disjoint open sets with compact closure in Ω.

3. Lower Semicontinuity Results

This section is devoted to the proof of the lower semicontinuity Theorem 1.1. The proof of Theorem 1.1 relies on the two Lemmas below.

Lemma 3.1. *Let $\{u_h\}$ be a sequence in $SBD(\Omega)$ converging to $u \in SBD(\Omega)$ in $L^1(\Omega; \mathbb{R}^N)$, such that $[u_h] \cdot \nu_{u_h} \geq 0$ $\mathcal{H}^{N-1}$-a.e. on J_{u_h} for every h, and verifying* (13)*, then* (3) *holds.*

Proof. We observe that for every $u \in SBD(\Omega)$ the measure $[u] \cdot \nu_u \mathcal{H}^{N-1} \lfloor J_u$ can be written also as $\mathrm{tr}(E^j u)$, where $\mathrm{tr}(E^j u)$ denotes the trace of the matrix of measures $E^j u$.

By virtue of this fact, the convergence (13) stated in Theorem 2.1 and the linearity of the trace tr entail that

$$\mathrm{tr}(E^j u_h) \rightharpoonup \mathrm{tr}(E^j u) \text{ weakly* in } \mathcal{M}_b(\Omega; \mathbb{R}), \tag{15}$$

consequently, since $[u_h] \cdot \nu_{u_h} \geq 0$ $\mathcal{H}^{N-1}$-a.e. on J_{u_h}, and hence $\mathrm{tr}(E^j u_h)$ is a nonnegative measure, we deduce that $\mathrm{tr}(E^j u)$ is a nonnegative measure which gives (3). □

The next result has to be considered just a step towards the proof of the main semicontinuity theorem.

Lemma 3.2. *Let $a \geq 0$ and $b \geq 0$, and let $\{u_h\}$ be a sequence in $SBD(\Omega)$, converging to $u \in SBD(\Omega)$ in $L^1(\Omega; \mathbb{R}^N)$, satisfying* (13) *and* (14)*. Suppose that $[u_h] \cdot \nu_{u_h} \geq 0$ $\mathcal{H}^{N-1}$-a.e. on J_{u_h} for every h. Then*

$$\int_{J_u} (a[u] \cdot \nu_u + b) d\mathcal{H}^{N-1} \leq \liminf_{h \to +\infty} \int_{J_{u_h}} (a[u_h] \cdot \nu_{u_h} + b) d\mathcal{H}^{N-1}. \tag{16}$$

Proof. Since $[u_h] \cdot \nu_{u_h} \mathcal{H}^{N-1} \lfloor J_{u_h} = \mathrm{tr}(E^j u_h) \geq 0$ and, by virtue of Lemma 3.1, (3) holds, from (13) we obtain

$$\int_{J_u} [u] \cdot \nu_u d\mathcal{H}^{N-1} = (\mathrm{tr}(E^j u))(\Omega) \leq \liminf_{h \to \infty} (\mathrm{tr}(E^j u_h))(\Omega) = \liminf_{h \to \infty} \int_{J_{u_h}} [u_h] \cdot \nu_{u_h} d\mathcal{H}^{N-1}.$$

Since $a \geq 0$ and $b \geq 0$, (16) follows from this inequality and from (14) □

Proof of Theorem 1.1. The constraint is preserved in the limit as proven in Lemma 3.1. It remains to prove the lower semicontinuity of the surface term. To this aim we observe that the proof below relies on Theorem 1.2, i.e. we adopt the representation (5) for the energy density φ.

As observed after (5) we may assume $\mathcal{A}$ countable. Clearly if $\mathcal{A}$ is a singleton, the statement follows from Lemma 3.2. The general case can be proved essentially by applying Theorem 2.2. Indeed, if φ admits the representation (5), by superadditivity of liminf:

$$\liminf_h \int_{J_{u_h}} \varphi([u_h]\cdot\nu_{u_h})d\mathcal{H}^{N-1} \geq \sum_\alpha \liminf_h \int_{J_{u_h}\cap A_\alpha} (a_\alpha[u_h]\cdot\nu_{u_h}+b_\alpha)d\mathcal{H}^{N-1}$$

for any finite family of pairwise disjoint open sets $A_\alpha \subset \Omega$.

By Lemma 3.2 we have

$$\liminf_{h\to+\infty} \int_{J_{u_h}} (a_\alpha[u_h]\cdot\nu_{u_h}+b_\alpha)d\mathcal{H}^{N-1} \geq \int_{J_u\cap A_\alpha} (a_\alpha[u]\cdot\nu_u+b_\alpha)d\mathcal{H}^{N-1}.$$

Therefore

$$\liminf_{h\to+\infty} \int_{J_{u_h}} \varphi([u_h]\cdot\nu_{u_h})d\mathcal{H}^{N-1} \geq \sum_\alpha \int_{J_u\cap A_\alpha} (a_\alpha[u]\cdot\nu_u+b_\alpha)d\mathcal{H}^{N-1}$$

for any finite family of pairwise disjoint open sets $A_\alpha \subset \Omega$.

By Theorem 2.2 we can interchange integration and supremum over all such families, thus getting

$$\liminf_{h\to+\infty} \int_{J_{u_h}} \varphi([u_h]\cdot\nu_{u_h})d\mathcal{H}^{N-1} \geq \int_{J_u} \varphi([u]\cdot\nu_u)d\mathcal{H}^{N-1},$$

whence (4) follows.

□

3.1. Applications to Minimum Problems

The lower semicontinuity results proven above, together with some known lower semicontinuity results for volume energies of the kind $\int_\Omega f(x,\mathcal{E}u_h)dx$, (under suitable sets of assumptions on the function f) can be used to prove the existence of solutions to the following minimum problems dealing with the equilibrium of cracked elastic bodies, subject to volume forces, with Neumann boundary conditions.

Theorem 3.3. *Let $\theta : [0,+\infty[\to [0,+\infty[$ be a non-decreasing function verifying (9) and let $f : \Omega\times M^{N\times N}_{\mathrm{sym}} \to [0,+\infty]$ be such that*

i) *$f(x,\cdot)$ is convex and lower semicontinuous on $\mathbb{M}^{N\times N}_{\mathrm{sym}}$, for a.e. $x\in\Omega$,*
ii) *f is $\mathcal{L}(\Omega)\otimes\mathcal{B}(\mathbb{M}^{N\times N}_{\mathrm{sym}})$-measurable,*
iii) *$f(x,A)\geq\theta(|A|)$ for every $x\in\Omega$ and $A\in\mathbb{M}^{N\times N}_{\mathrm{sym}}$,*

where $\mathcal{L}(\Omega)$ and $\mathcal{B}(\mathbb{M}^{N\times N}_{\mathrm{sym}})$ represent respectively the Lebesgue σ-algebra on Ω and the Borel σ-algebra on $\mathbb{M}^{N\times N}_{\mathrm{sym}}$. Let $h\in L^1(\Omega;\mathbb{R}^N)$ and let $\{H(x)\}_{x\in\Omega}$ be a uniformly bounded family of closed subsets of $\mathbb{R}^N$. Let φ be as in Theorem 1.1 and assume that $\varphi(0)>0$. Then the constrained minimum problem

$$\min_{\substack{u\in SBD(\Omega)\\ u(x)\in H(x) \text{ a.e. in } \Omega\\ (u^+-u^-)\cdot\nu_u\geq 0\,\mathcal{H}^{N-1} \text{ a.e. on } J_u}} \left\{\int_\Omega f(x,\mathcal{E}u)dx + \int_{J_u}\varphi([u]\cdot\nu_u)d\mathcal{H}^{N-1} + \int_\Omega h\cdot u\,dx\right\} \qquad (17)$$

has a solution.

Proof. The assumptions on $H(x)$ provide an L^∞ bound on every minimizing sequence $\{u_h\}$ for (17). Consequently, since *iii)* holds and $\varphi(0) > 0$, Theorem 2.1 and Theorem 1.1 apply. Moreover, since $\mathcal{E}u_h \rightharpoonup \mathcal{E}u$ in L^1, the functional $w \in \mathcal{L}^1(\Omega; \mathbb{M}^{N\times N}_{Sym}) \to \int_\Omega f(x, w(x))dx$ is lower semicontinuous with respect to the weak $\mathcal{L}^1$ convergence under the above set of assumption on f, and the term describing the volume forces in (17) is linear, the Direct Methods of the Calculus of Variations ensure the existence of a solution. □

We observe that the hypotheses made on f in Theorem 3.3 are crucial for the lower semicontinuity of the volume term. On the other hand a lack of convexity on the function f requires more refined tecniques and other type of assumptions which guarantee lower semicontinuity. To this end we exploit the results contained in [11] (see Theorem 1.2 and Example 4.2 therein).

Theorem 3.4. *Let $p > 1$ and $f : \Omega \times \mathbb{M}^{N\times N}_{Sym} \to [0, \infty[$ be a Carathéodory function satisfying the following assumptions:*

(i) *for a.e. $x \in \Omega$, for every $\xi \in \mathbb{M}^{N\times N}_{Sym}$,*

$$\frac{1}{C}|\xi|^p \leq f(x,\xi) \leq \Psi(x) + C(1 + |\xi|^p)$$

for some constant $C > 0$ and $\Psi \in L^1(\Omega)$,

(ii) *for a.e. $x_0 \in \Omega$ $f(x_0, \cdot)$ is symmetric quasiconvex, i.e.*

$$f(x_0,\xi) \leq \frac{1}{|D|}\int_D f(x_0, \xi + \mathcal{E}\psi(x))dx$$

for every bounded open set D of $\mathbb{R}^N$ and every $\psi \in W^{1,\infty}_0(D;\mathbb{R}^N)$ and $\xi \in \mathbb{M}^{N\times N}_{Sym}$.

Let K be a non empty, not necessarily closed subset of Ω, such that $0 < \mathcal{H}^{N-1}(K) < +\infty$, $\{H(x)\}_{x\in\Omega}$ and φ as in Theorem 3.3, then the following minimum problem admits a solution

$$\min_{\substack{u\in SBD(\Omega)\\ J_u\subset K\\ u(x)\in H(x) \text{ a.e. in } \Omega\\ (u^+-u^-)\cdot\nu_u\geq 0 \mathcal{H}^{N-1} \text{ a.e. on } J_u}} \left\{\int_\Omega f(x,\mathcal{E}u)dx + \int_{J_u}\varphi([u]\cdot\nu_u)d\mathcal{H}^{N-1}\right\}. \tag{18}$$

Proof. We preliminarily recall that, as emphasized in [11], the assumption that K is not necessarily closed has been made to avoid trivial cases that can be solved with simpler methods and tools.

We observe that the set where the minimum is undertaken in (18) and the fact that $\varphi(0) > 0$ guarantee that Theorem 2.1 applies. Consequently both the assumptions of Theorem 1.2 in [11] and of Theorem 1.1 are fulfilled and the lower semicontinuity follows both for the bulk and for the surface energies in (18). The Direct Methods of Calculus of Variations and Example 4.2 in [11] lead us to the existence of a solution for (18). □

4. The class of surface integrands which ensure lower semicontinuity

This section is devoted to charachterize the class (2), by proving Theorem 1.2.

Proof of Theorem 1.2. We preliminarily observe that all the functions in Φ are continuous. For the sake of convenience we arrange the proof as (2) $\Rightarrow$ (1) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1) Since $a_\alpha \geq 0$ and $b_\alpha \geq 0$, the function $s \to a_\alpha s + b_\alpha$ is nonnegative, convex, subadditive, and nondecreasing, by virtue of (5), the same properties hold for φ since these properties are closed under suprema.

(1) $\Rightarrow$ (3) Let $0 < s \leq t$ and consider the following convex combination

$$t = \frac{s}{t} \cdot s + \left(1 - \frac{s}{t}\right)(s + t).$$

The convexity of φ entails that

$$\varphi(t) \leq \frac{s}{t}\varphi(s) + \left(1 - \frac{s}{t}\right)\varphi(s + t).$$

By the subadditivity of φ, it follows

$$\varphi(t) \leq \frac{s}{t}\varphi(s) + \left(1 - \frac{s}{t}\right)(\varphi(s) + \varphi(t))$$

i.e.

$$\frac{\varphi(t)}{t} \leq \frac{\varphi(s)}{s},$$

which proves that $\frac{\varphi(s)}{s}$ is nonincreasing.

(3) $\Rightarrow$ (4) We recall that convexity entails the existence of right derivatives for every $s > 0$. Hence taking the right derivative of $\frac{\varphi(s)}{s}$ we soon get $\frac{\varphi(s)}{s} \geq \varphi'_+(s)$.

(4) $\Rightarrow$ (5) We preliminarily observe that φ^* is nondecreasing, since

$$\varphi^*(r) = \sup_{s \geq 0} \{rs - \varphi(s)\}. \tag{19}$$

is the supremum of nondecreasing functions.

Furthermore we observe that (6), the monotonicity and the convexity of φ entail also that

$$\lim_{s \to 0^+} (\varphi(s) - s\varphi'_+(s)) \geq 0.$$

If $r = \varphi'_+(s)$ for some $s \in [0, +\infty[$, then, well know facts of Convex Analysis (see for instance Theorem 23.5 in [16]) ensure that $\varphi'_+(s)$ belongs to the subdifferential of φ at s and

$$\varphi^*(\varphi'_+(s)) = s\varphi'_+(s) - \varphi(s) \leq 0. \tag{20}$$

Since φ^* is nondecreasing, this means that if $r \leq \varphi'_+(s)$ for some $s \geq 0$, then $\varphi^*(r) \leq 0$. Since φ^* is lower semicontinuous, this holds for $r \leq \sup_{s \geq 0} \varphi'_+(s) = \beta$.

On the other hand, if $r > \beta$, the right derivative of $s \mapsto rs - \varphi(s)$ is larger than a positive constant, and this clearly implies

$$\varphi^*(r) = \sup_{s \geq 0}\{rs - \varphi(s)\} = \lim_{s \to +\infty}(rs - \varphi(s)) = +\infty$$

(5) $\Rightarrow$ (2) Well known properties of convex functions, allow us to write φ as

$$\varphi(s) = \sup_{r \in \mathrm{dom}(\varphi^*)}\{rs - \varphi^*(r)\} \tag{21}$$

Since $\varphi^*(r) = \varphi^*(0) = -\varphi(0)$ for $r \leq 0$ by (19) and by the monotonicity of φ , the right hand side of (21) (before taking the supremum) is $rs - \varphi^*(0)$ for every $r \leq 0$ which is less than or equal to $-\varphi^*(0)$. Hence the supremum is unaltered if we take $r \geq 0$ and this proves the claim. □

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