Journal of Convex Analysis
Volume 15 (2008), No. 4, 905–917

Nonlinear Contractive Model Predictive Control via Polytopic Robust Controllable Sets

Jian Wan
Institut d'Informàtica i Aplicacions, Universitat de Girona,
Campus Montilivi, 17071 Girona, Spain
jwan@eia.udg.es

Josep Vehí
Institut d'Informàtica i Aplicacions, Universitat de Girona,
Campus Montilivi, 17071 Girona, Spain
vehi@eia.udg.es

Ningsu Luo
Institut d'Informàtica i Aplicacions, Universitat de Girona,
Campus Montilivi, 17071 Girona, Spain
ningsu@eia.udg.es

Received: January 8, 2007
Revised manuscript received: October 15, 2007

A general framework for computing polytopic robust controllable sets of constrained nonlinear uncertain discrete-time systems as well as controlling such complex systems based on the computed polytopic robust controllable sets is introduced in this paper. The resulting one-step control approach turns out to be a robust model predictive control scheme with feasible unit control horizon and contractive constraint. The solvers of set inversion and constrained minimax optimization via interval analysis are applied to compute robust controllable sets and one-step control inputs in a reliable way. The computed robust controllable sets are unions of boxes and polytope geometry is applied to approximate a union of boxes innerly by one polytope.

Keywords: Nonlinear discrete-time systems, model predictive control, robust controllable sets, interval analysis, polytope geometry

1991 Mathematics Subject Classification: 65G40, 93B05, 93B51

1. Introduction

A fundamental control problem is to determine the subset of the state space that can be steered to a given terminal set via an admissible control sequence, while guaranteeing that the state constraint be satisfied for all uncertain cases. Such a subset is referred to as a robust controllable set and it is a more general interpretation of the classical controllability problem of unconstrained deterministic linear systems [6]. The analytical determination of robust controllable sets as well as the controllability of constrained discrete-time systems is extremely hard. A numerical approach is to compute robust controllable sets geometrically and any initial state within the computed maximal robust controllable set is robustly controllable to the selected terminal set in finite steps. In [12], robust controllable sets were computed geometrically via a set of set differences and projections for

ISSN 0944-6532 / $ 2.50

constrained discrete-time systems with additive uncertainty. Such an approach is quite effective for discrete-time linear or piecewise-affine systems with polytopes as their terminal sets, where the set computations can be performed efficiently through polyhedral algebra, linear programming and computational geometry software. However, the computation of robust controllable sets for general constrained nonlinear discrete-time systems is not straightforward and efficient. In [2, 3], a branch-and-bound algorithm based on interval arithmetic was introduced to compute the inner approximations of control invariant sets for constrained nonlinear systems with a given bound of error tolerance. This algorithm was extended in [9] to compute robust controllable sets of constrained nonlinear systems with additive uncertainty, where the computed robust controllable sets were utilized as a contractive sequence of sets to formulate a robust model predictive control scheme. However, intervals encountered here have two modalities: there exists a value in $[a, b]$ that possesses a property or some given properties, for example, there exists a control input $u \in [a, b]$ that can drive the system state x to the terminal set; all values in $[c, d]$ possess a property or some given properties, for example, for all system states $x \in [c, d]$, they can be driven to the terminal set by an admissible control input u. The branch and bound of the state space with universal modality and the control space with existential modality were mixed in their approaches, where extra treatments of the subboxes of the admissible state space are needed after each branch and bound of the admissible state space.

This paper deals with the computation of polytopic robust controllable sets for general constrained nonlinear uncertain discrete-time systems in a semantic way, where the branch and bound of the admissible state space with universal modality and the control space with existential modality are separated by two nested loops. The computed polytopic robust controllable sets are also utilized as a contractive sequence of sets to formulate a one-step model predictive control scheme for simplifying traditional multi-step minimax optimizations underlying robust model predictive control. The simplified one-step minimax optimizations are fulfilled by an interval-based solver of constrained minimax optimization. The paper is organized as follows: the problem statement is illustrated in Section 2; the basics of interval analysis and polytope geometry are introduced in Section 3; the computation of polytopic robust controllable sets is addressed in Section 4; nonlinear contractive model predictive control via the computed polytopic robust controllable sets is described in Section 5; an illustrative example is demonstrated in Section 6; and finally, some conclusions are drawn in Section 7.

2. Problem Statement

The system to be considered is described by the following constrained nonlinear uncertain discrete-time state-space model:

$$\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k), \mathbf{w}(k), \mathbf{u}(k)), \quad k = 0, 1, \ldots, \tag{1}$$

where $\mathbf{x}(k) \in \mathbf{X} \subset \mathbb{R}^n$ is a vector of n state variables and $\mathbf{X}$ is a compact set containing the origin; $\mathbf{w}(k) \in \mathbf{W} \subset \mathbb{R}^l$ is a vector of l uncertain parameters and (or) additive disturbances; $\mathbf{u}(k) \in \mathbf{U} \subset \mathbb{R}^m$ is a vector of m control inputs and $\mathbf{U}$ is a compact set containing the origin. The domains of $\mathbf{X}, \mathbf{W}$ and $\mathbf{U}$ are assumed to be described by boxes of their proper dimensions, i.e., every component of the vectors is an interval. Such a model represents a general class of physical systems with constrained state and control as well as uncertainty. The control target is to drive the system from the initial state $\mathbf{x}_0$

to a sufficiently small region around the origin asymptotically. The dual-mode approach of model predictive control is adopted here: at first, the one-step control deriving from contractive model predictive control drives the system state into a designed terminal robust control invariant set $\mathbb{T}$; and then the related local stabilizing feedback control law is applied instead to drive the system state to a sufficiently small region around the origin asymptotically.

The one-step robust controllable set $Pre(\mathbb{T})$ is the set of states in $\mathbf{X}$ within which an admissible control input $\mathbf{u}(k) \in \mathbf{U}$ exists that guarantees to drive the system to the terminal set $\mathbb{T}$ in one step for every allowable uncertainty $\mathbf{w}(k) \in \mathbf{W}$, i.e.,

$$Pre(\mathbb{T}) = \{\mathbf{x}(k) \in \mathbf{X} \mid \exists \mathbf{u}(k) \in \mathbf{U},\ \forall \mathbf{w}(k) \in \mathbf{W} : \mathbf{x}(k+1) \in \mathbb{T}\}. \tag{2}$$

The terminal set $\mathbb{T}$ is a robust control invariant set if and only if $\mathbb{T} \subseteq Pre(\mathbb{T})$ [6]. If the terminal set $\mathbb{T}$ is selected to be a robust control invariant set, then the one-step robust controllable set $Pre(\mathbb{T})$ is an enlarged robust control invariant set and it can also be referred to as the one-step robust stabilisable set [6, 11]. The ith-step robust controllable set $Pre_i(\mathbb{T})$ can be obtained by computing one-step robust controllable sets recursively, i.e.,

$$Pre_i(\mathbb{T}) = Pre(Pre_{i-1}(\mathbb{T})), \tag{3}$$

where $Pre_0(\mathbb{T}) = \mathbb{T}$ and the maximal robust controllable set $Pre_\infty(\mathbb{T})$ within the constrained state space $\mathbf{X}$ is reached when $Pre_{N+1}(\mathbb{T}) = Pre_N(\mathbb{T})$ for certain N.

Once the maximal robust controllable set $Pre_N(\mathbb{T})$ is obtained, the controllability of the nonlinear uncertain discrete-time system (1) is obvious: the system is robustly controllable to the terminal set $\mathbb{T}$ in finite steps if the initial state is within the maximal robust controllable set. The control inputs can be obtained through the strategy of robust model predictive control with feasible unit control horizon and contractive constraint when the system state is outside the terminal set $\mathbb{T}$, i.e., the one-step control inputs are obtained by solving the following constrained minimax optimization iteratively:

$$\mathbf{u}^*(\mathbf{x}(k)) = \arg \min_{\mathbf{u}(\mathbf{x}(k)) \in \mathbf{U}} \max_{\mathbf{w}(k) \in \mathbf{W}} [\mathbf{x}^T(k+1) Q \mathbf{x}(k+1) + \mathbf{u}^T(\mathbf{x}(k)) R \mathbf{u}(\mathbf{x}(k))] \tag{4}$$

subject to

$$\mathbf{x}(k+1) \in Pre_{i-1}(\mathbb{T}), \tag{5}$$

where $\mathbf{x}(k) \in Pre_i(\mathbb{T})$, but $\mathbf{x}(k)$ does not belong to $Pre_{i-1}(\mathbb{T})$; Q and R are weighted positive definite matrices; and $\mathbf{u}^*(\mathbf{x}(k))$ is the optimal one-step control input. This one-step control scheme differs from traditional minimax configurations of robust model predictive control, in which control horizons are usually selected to be large enough for satisfying the imposed terminal constraints and the resulting multi-step minimax optimizations are usually time-consuming. The feasibility and stability of the closed-loop system with the unit control horizon can be guaranteed since the system state is driven contractively along the computed robust controllable sets to the designed terminal set [9]. Once the system state enters the terminal set $\mathbb{T}$, the related local stabilizing feedback control law is applied instead to drive the system state to a sufficiently small region around the origin asymptotically [4].

3. Interval Analysis and Polytope Geometry

This section gives a brief introduction of interval analysis and polytope geometry, especially those concepts and manipulations used in the following sections.

3.1. Interval Analysis

The initial idea of interval analysis is to enclose real numbers in intervals and real vectors in boxes as a method of considering the imprecision of representing real numbers by finite digits in numerical computers. Interval analysis has become a fundamental numerical tool for representing uncertainties or errors, proving properties of sets, solving sets of equations or inequalities and optimizing globally via interval arithmetic. The key concepts of interval analysis are interval arithmetic, inclusion function and subpaving, whose definitions are as follows [5]:

Interval Arithmetic. Interval arithmetic is a special case of computation on convex sets, which includes real compact intervals $[a, b] = \{a \leq x \leq b,\ a \leq b,\ x, a, b \in \mathbb{R}\}$, real compact interval vectors and real compact interval matrices. The four elementary arithmetic operations $(+, -, \times, \div)$ are extended to intervals. Concretely, for any such binary operator, denoted by $\diamond$, performing the operation associated with $\diamond$ on the intervals $[a, b]$ and $[c, d]$ means computing $X \diamond Y = convh(\{x \diamond y \in \mathbb{R} \mid x \in [a, b],\ y \in [c, d]\})$, where $convh(\cdot)$ denotes the convex hull of $\{x \diamond y \in \mathbb{R} \mid x \in [a, b],\ y \in [c, d]\}$. Correspondingly, the set of all interval vectors in $\mathbb{R}^n$ is denoted to be $\mathbb{I}(\mathbb{R}^n)$.

Inclusion Function. Consider a function $\mathbf{f}$ from $\mathbb{R}^n$ to $\mathbb{R}^m$, the interval function $\mathbf{F}$ from $\mathbb{I}(\mathbb{R}^n)$ to $\mathbb{I}(\mathbb{R}^m)$ is an inclusion function for $\mathbf{f}$ if $\forall \mathbf{X} \in \mathbb{I}(\mathbb{R}^n),\ \mathbf{f}(\mathbf{X}) \subseteq \mathbf{F}(\mathbf{X})$. The natural inclusion function of $\mathbf{f}(\mathbf{X})$ can be obtained by replacing each occurrence of every variable with the corresponding interval variable, by executing all operations according to interval arithmetic, and by computing ranges of the standard functions.

Subpaving. A subpaving of a box $\mathbf{X} \in \mathbb{I}(\mathbb{R}^n)$ is a union of non-overlapping subboxes with non-zero width, where every subbox is a subset of the box $\mathbf{X}$. A subpaving of $\mathbf{X}$ is regular if each of its subboxes can be obtained from $\mathbf{X}$ by a finite succession of branches.

The fundamental concepts of interval analysis can be integrated to set up various algorithms for solving set inversion, global optimization and minimax optimization problems in a guaranteed numerical way [5]. A basic operation within these solvers is to bisect or branch an interval vector into two sub-interval vectors. Taking the interval vector $\mathbf{X} = [a_1, b_1] \times \ldots \times [a_n, b_n]$ as an example, its width is denoted to be:

$$\texttt{Width}(\mathbf{X}) = \max_{i=1,\ldots,n} |a_i - b_i|, \tag{6}$$

and the index j is denoted to be:

$$j = \min_{i=1,\ldots,n} \{i \mid (|a_i - b_i|) = \texttt{Width}(\mathbf{X})\}, \tag{7}$$

then the bisection $\texttt{Bisect}(\mathbf{X})$ returns two sub-interval vectors $\texttt{L}\mathbf{X}$ and $\texttt{R}\mathbf{X}$:

$$\begin{cases} \texttt{L}\mathbf{X} := [a_1, b_1] \times \ldots \times [a_j, \frac{(a_j+b_j)}{2}] \times \ldots \times [a_n, b_n] \\ \texttt{R}\mathbf{X} := [a_1, b_1] \times \ldots \times [\frac{(a_j+b_j)}{2}, b_j] \times \ldots \times [a_n, b_n]. \end{cases} \tag{8}$$

3.2. Polytope Geometry

Polytope is a bounded polyhedron $\mathcal{P} \subset \mathbb{R}^n$, which can be described as:

$$\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^n \mid P^x\mathbf{x} \leq P^b\}, \tag{9}$$

where P^x is a matrix of dimension $m \times n$ and P^b is a column vector of dimension m. Basic polytope manipulations are to compute the complement of a polytope, the intersection of two polytopes, the set difference of two polytopes and the convex hull of a union of polytopes, whose definitions are as follows [8, 1]:

Complement. The complement of a polytope $\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^n \mid P^x\mathbf{x} \leq P^b\}$ relative to $\mathbf{X} \subset \mathbb{R}^n$ is a union of polytopes $\mathcal{P}^C := \cup_{i=1}^m \{\mathbf{x} \in \mathbf{X} \mid P_i^x\mathbf{x} > P_i^b\}$, where P_i^x and P_i^b are the ith row of P^x and P^b, respectively.

Intersection. The intersection of two polytopes $\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^n \mid P^x\mathbf{x} \leq P^b\}$ and $\mathcal{Q} = \{\mathbf{x} \in \mathbb{R}^n \mid Q^x\mathbf{x} \leq Q^b\}$ is a polytope $\mathcal{P} \cap \mathcal{Q} := \{\mathbf{x} \in \mathbb{R}^n \mid P^x\mathbf{x} \leq P^b,\ Q^x\mathbf{x} \leq Q^b\}$.

Set Difference. The set difference of two polytopes $\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^n \mid P^x\mathbf{x} \leq P^b\}$ and $\mathcal{Q} = \{\mathbf{x} \in \mathbb{R}^n \mid Q^x\mathbf{x} \leq Q^b\}$ is a union of polytopes $\mathcal{P} \setminus \mathcal{Q} := \mathcal{P} \cap \mathcal{Q}^C$.

Convex Hull. The convex hull of a union of polytopes $\mathcal{P}_i \subset \mathbb{R}^n (i = 1, \ldots, p)$ is a polytope $convh(\cup_{i=1}^p \mathcal{P}_i) := \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = \sum_{i=1}^p \alpha_i\mathbf{x}_i, \mathbf{x}_i \in \mathcal{P}_i, 0 \leq \alpha_i \leq 1,\ \sum_{i=1}^p \alpha_i = 1\}$.

4. The Computation of Polytopic Robust Controllable Sets

Interval analysis and polytope geometry are applied to compute polytopic robust controllable sets of constrained nonlinear uncertain discrete-time systems in this section.

4.1. The First-step Robust Controllable Set Approximation Algorithm

Assume that the terminal set $\mathbb{T}$ is designed to be a robust control invariant polytope for the system (1), then an inner approximation of the first-step robust controllable set can be computed on the basis of the solver of set inversion via interval analysis [5]. The detailed first-step robust controllable set approximation algorithm is listed in Algorithm I, where ε is the bound of error tolerance and $\Sigma_\mathbf{X}$ is to store the computed first-step robust controllable set.

Algorithm I. The First-step Robust Controllable Set Approximation

In: $\mathbf{X}, \mathbf{W}, \mathbf{U}, \mathbb{T}, \varepsilon$; Out: $\Sigma_\mathbf{x}$

1. Initialize **Stack** $1 = \mathbf{X}$, $\Sigma_\mathbf{x} = \emptyset$;
2. while **Stack** $1 \neq \emptyset$
3. Pop out a $\mathbf{X}_i$ from **Stack** 1;
4. Compute $\mathbf{f}(\mathbf{X}_i, \mathbf{w}_{Local}, \mathbf{U})$;
5. if $\mathbf{f}(\mathbf{X}_i, \mathbf{w}_{Local}, \mathbf{U}) \cap \mathbb{T} = \emptyset$
6. Discard $\mathbf{X}_i$ and return to 2;
7. endif
8. Initialize **Stack** $2 = \mathbf{U}$;
9. while **Stack** $2 \neq \emptyset$
10. Pop out a $\mathbf{U}_j$ from **Stack** 2;
11. Compute $\mathbf{f}(\mathbf{X}_i, \mathbf{w}_{Local}, \mathbf{U}_j)$;
12. if $\mathbf{f}(\mathbf{X}_i, \mathbf{w}_{Local}, \mathbf{U}_j) \cap \mathbb{T} = \emptyset$

13. Discard $\mathbf{U}_j$ and return to 9;
14. elseif $\mathbf{f}(\mathbf{X}_i, \mathbf{W}, (\mathbf{u}_j)_{Local}) \subseteq \mathbb{T}$
15. $\Sigma_{\mathbf{x}} \leftarrow \mathbf{X}_i \cup \Sigma_{\mathbf{x}}$ and return to 2;
16. elseif Width($\mathbf{U}_j$) $\leq \varepsilon$, discard $\mathbf{U}_j$ and return to 9;
17. else
18. Bisect $\mathbf{U}_j$, push them on **Stack** 2 and return to 9;
19. endif
20. endwhile
21. if Width($\mathbf{X}_i$) $\leq \varepsilon$, then discard $\mathbf{X}_i$ and return to 2;
22. else
23. Bisect $\mathbf{X}_i$, push them on **Stack** 1 and return to 2;
24. endif
25. endwhile

As shown in Algorithm I, $\mathbf{w}_{Local}$ in Step 4 and 11 relates to a local search of a concrete value $\mathbf{w}_{Local} \in \mathbf{W}$: if there exists such a value $\mathbf{w}_{Local}$ that renders $\mathbf{f}(\mathbf{X}_i, \mathbf{w}_{Local}, \mathbf{U}) \cap \mathbb{T} = \emptyset$, then for all $\mathbf{u} \in \mathbf{U}$, it is impossible to drive the state subbox $\mathbf{X}_i$ to the terminal set $\mathbb{T}$ in case of $\mathbf{w}_{Local}$ at the next time instance, so $\mathbf{X}_i$ does not belong to the first-step robust controllable set and it is discarded in Step 6; however, only a part of $\mathbf{U}$ is tested in Step 12, i.e., $\mathbf{f}(\mathbf{X}_i, \mathbf{w}_{Local}, \mathbf{U}_j) \cap \mathbb{T} = \emptyset$, so for the whole $\mathbf{U}_j$, it is impossible to drive the state subbox $\mathbf{X}_i$ to the terminal set $\mathbb{T}$ in case of $\mathbf{w}_{Local}$ at the next time instance, then $\mathbf{U}_j$ is discarded instead in Step 13 and other parts of $\mathbf{U}$ are to be tested further. On the contrary, if there exists a concrete control input $(\mathbf{u}_j)_{Local} \in \mathbf{U}_j$ that renders $\mathbf{f}(\mathbf{X}_i, \mathbf{W}, (\mathbf{u}_j)_{Local}) \subseteq \mathbb{T}$, then for all uncertain cases $\mathbf{w}(k) \in \mathbf{W}$, the state subbox $\mathbf{X}_i$ can be driven to the terminal set $\mathbb{T}$ via the admissible control input $(\mathbf{u}_j)_{Local}$ at the next time instance, which signifies that $\mathbf{X}_i$ belongs to the first-step robust controllable set and it is stored in $\Sigma_{\mathbf{X}}$. If no semantic judgement can be made for $\mathbf{X}_i$ or $\mathbf{U}_j$ and the widths of them are smaller than the selected bound of error tolerance ε, just as in Step 16 and 21, they are to be discarded as well; otherwise, $\mathbf{X}_i$ or $\mathbf{U}_j$ is to be bisected further for a finer test, just as shown in Step 18 and 23. Obviously, the computed first-step robust controllable set $\Sigma_{\mathbf{x}}$ is an inner approximation of $Pre(\mathbb{T})$ because of the bound of error tolerance ε, i.e., $\Sigma_{\mathbf{x}} \in Pre(\mathbb{T})$. In order to simplify the computation, $\mathbf{w}_{Local}$ and $(\mathbf{u}_j)_{Local}$ are usually selected to be the center of the subbox $\mathbf{W}$ and the center of the subbox $\mathbf{U}_j$, respectively. It can be seen that the bisection and selection of the state space with universal modality and the control space with existential modality are performed separately by two nested loops, where a clear semantic interpretation exists. Thus the computation of controllable sets in Algorithm I is different from the corresponding algorithm in [2, 3], where the admissible domain $\mathbb{L} = (\mathbf{X}, \mathbf{U})$ was the combination of the admissible state space $\mathbf{X}$ and the admissible control space $\mathbf{U}$ and extra treatments of the subboxes of the admissible domain were needed after each bisection and selection of the admissible state space: if the action for $\mathbf{Z}_i = (\mathbf{X}_i, \mathbf{U}_i)$ was to insert to the solution set, then the algorithm in [2, 3] needed to erase from the set $\mathbb{L}$ all subboxes $\mathbf{Z}_j = (\mathbf{X}_j, \mathbf{U}_j)$ with $\mathbf{X}_j = \mathbf{X}_i$; if the action for $\mathbf{Z}_i = (\mathbf{X}_i, \mathbf{U}_i)$ was to bisect $\mathbf{X}_i$, then the algorithm in [2, 3] also needed to bisect all subboxes $\mathbf{Z}_j = (\mathbf{X}_j, \mathbf{U}_j)$ with $\mathbf{X}_j = \mathbf{X}_i$ in the set $\mathbb{L}$. The extra treatments of the subboxes of the admissible domain were time-consuming since the members of the set $\mathbb{L}$ were numerous and the action of bisection and selection was frequent. Furthermore, there was no semantic interpretation for discarding the subbox of the admissible state space: if the action for $\mathbf{Z}_i = (\mathbf{X}_i, \mathbf{U}_i)$ was to discard, this did not mean that $\mathbf{X}_i$ could not be a part of the one-step controllable set because $\mathbf{X}_i$ was likely to be controllable to the selected terminal set by other parts of the admissible control space $\mathbf{U}$.

4.2. The Revised Polytopic Approximation Algorithm

According to the first-step robust controllable set approximation algorithm in `Algorithm I`, the computed inner approximation of the first-step robust controllable set $\Sigma_{\mathbf{X}}$ is a union of boxes, which is also a union of polytopes as boxes are a specific class of polytopes. The union of polytopes can be further approximated innerly by one polytope according to the one-step set polytopic approximation algorithm proposed in [2, 3]. The benefits of representing a robust controllable set by one polytope rather than by a union of boxes range from less memory resources to easier synthesis of real-time constrained control. The convex hull of the union of polytopes is used in the following revised polytopic approximation algorithm for obtaining a possibly reduced number of complementary sets and separating a complementary set and the contracted convex hull instead of separating a complementary set and the contracted union of boxes.

Assume that the convex hull of the union of polytopes $\mathcal{H} = \mathtt{convh}(\Sigma_{\mathbf{X}})$ as well as its vertices has been obtained via vertex enumeration [8], i.e., $\mathcal{H} = \{\mathbf{x} \in \mathbb{R}^n \mid H^x\mathbf{x} \leq K^b\}$ and its vertices are $\{\mathbf{v}_k^{\mathcal{H}}\}_{k=1}^{n_h}$, where n_h is the number of vertices on $\mathcal{H}$. Then the complementary sets $\mathcal{C}$ of $\Sigma_{\mathbf{X}}$ relative to its convex hull $\mathcal{H}$ are a union of polytopes and they can be obtained by the set difference $\mathcal{C} = \mathcal{H} \setminus \Sigma_{\mathbf{X}} = \cup_{m=1}^{n_c} \mathcal{C}_m$, where n_c is the number of polytopes in $\mathcal{C}$. The vertices of each polytope $\mathcal{C}_m$ in $\mathcal{C}$ can be obtained as well and they are assumed to be $\{\mathbf{v}_j^{\mathcal{C}_m}\}_{j=1}^{n_{c_m}}$, where n_{c_m} is the number of vertices on $\mathcal{C}_m$. The $\alpha-$support hyperplane for $\mathcal{C}_m$ is a hyperplane $\mathbf{c}_m^T\mathbf{x} = 1$ such that [2, 3]: $\mathbf{c}_m^T\mathbf{x} > 1$ for every $\mathbf{x} \in \mathcal{C}_m$ and $\alpha \cdot \mathbf{c}_m^T\mathbf{x} \leq 1$ for every $\mathbf{x} \in \mathcal{H}$, where $\alpha \in [0, 1]$. The computation of the $\alpha-$support hyperplane for each $\mathcal{C}_m$ can be transformed to be a linear programming problem [2, 3]:

$$\min_{\{\mathbf{c}_m, \gamma\}} \gamma \tag{10}$$

subject to

$$\begin{cases} \mathbf{c}_m^T\mathbf{v}_j^{\mathcal{C}_m} > 1, & j = 1, \ldots, n_{c_m} \\ \mathbf{c}_m^T\mathbf{v}_k^{\mathcal{H}} \leq \gamma, & k = 1, \ldots, n_h \\ \gamma \geq 1. \end{cases} \tag{11}$$

Once the α-support hyperplane for $\mathcal{C}_m$ is obtained, those $\{\mathcal{C}_r | \mathbf{c}_m^T\mathbf{v}_j^{\mathcal{C}_r} > 1,\ j = \{1, \ldots, n_{c_r}\},\ r \in \{m+1, \ldots, n_c\}\}$ should be discarded to avoid redundant separations from their corresponding contracted convex hull. So the resulting polytopic approximation $\mathcal{P}_a$ for $\Sigma_{\mathbf{X}}$ is to be:

$$\mathcal{P}_a = \cap_{m=1}^{n_{c_f}} \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{c}_m^T\mathbf{x} \leq 1\}, \tag{12}$$

where n_{c_f} is the number of all processed complementary sets in $\mathcal{C}$. The detail of the revised polytopic approximation algorithm is shown in `Algorithm II`. It is worthy to note that the polytopic approximation algorithm is guaranteed to result in a non-empty $\mathcal{P}_a$ with the assumption of $0 \in \Sigma_{\mathbf{X}}$ since the constraints in (11) are then guaranteed to be feasible.

```
Algorithm II. The Revised Polytopic Approximation
In: Σx; Out: P^a
```

```
H = convh(Σx);
P^a = H;
C = H \ Σx = ∪_{m=1}^{n_c} C_m;
for m = 1 : 1 : n_c
    if C_m ∩ P^a ≠ ∅
        c_m = arg min_{c_m,γ} γ subject to (11);
        P^a = P^a ∩ {x ∈ R^n | c_m^T x ≤ 1};
    end
end
```

4.3. The Following-step Robust Controllable Set Approximation Algorithm

Once the inner approximation of the first-step robust controllable set $\Sigma_{\mathbf{X}}$ has been approximated innerly by one polytope $\mathcal{P}_a$ via the revised polytopic approximation algorithm, the following-step robust controllable sets can be computed recursively by renewing the terminal set $\mathbb{T}$ with $\mathcal{P}_i^a$ in `Algorithm I`, where $\mathcal{P}_i^a$ is the polytopic approximation of the ith-step robust controllable set and $\mathcal{P}_1^a = \mathcal{P}_a$. If the terminal set $\mathbb{T}$ is designed to be robust control invariant, then theoretically the computed first-step robust controllable set should contain it. However, the limitation of the bound of error tolerance ε of the interval-based algorithm in `Algorithm I` and the conservativeness of the revised polytopic approximation algorithm in `Algorithm II` might lead to:

$$\mathbb{T} \nsubseteqq \mathcal{P}_1^a, \tag{13}$$

which signifies that the obtained first-step polytopic robust controllable set $\mathcal{P}_1^a$ is not robust control invariant. A remedy for this problem is to replace $\mathcal{P}_1^a$ by the union $\mathcal{P}_1^a \cup \mathbb{T}$ in the computation of the second-step robust controllable set and obviously $\mathcal{P}_1^a \cup \mathbb{T}$ is a robust control invariant set for the system (1). Obtaining a union of polytopes as a robust controllable set also happens in piecewise-affine and hybrid systems [6]. Generally, the terminal set for computing $Pre_i(\mathbb{T})$ recursively is renewed to be $\cup_{j=0}^{i-1}\mathcal{P}_j^a$ at each step, where $\mathcal{P}_0^a = \mathbb{T}$. Corresponding inclusion test and exclusion test between a box and a union of polytopes are concerned instead in `Step 5`, `12` and `14` of `Algorithm I`, respectively. To judge whether a box is included in a union of polytopes can be transformed to judge whether their set difference is empty, which can be further transformed to be a linear programming problem [6], i.e., a polytope $\mathcal{H} = \{\mathbf{x} \in \mathbb{R}^n \mid H^x\mathbf{x} \leq H^b\}$ is empty if and only if $\zeta > 0$ where $\zeta = \arg\min \zeta$ subject to $H^x\mathbf{x} \leq H^b + \zeta \cdot 1$. To judge whether a box is excluded in a union of polytopes can be transformed to judge whether their intersections are empty, which is a similar linear programming problem. The polytopic approximation of the maximal robust controllable set $Pre_\infty(\mathbb{T})$ within the constrained state space is reached when $\mathcal{P}_N^a = \mathcal{P}_{N+1}^a$ for certain N. Nevertheless, the polytopic robust controllable set approximation algorithm can be simplified if $\mathcal{P}_{i-1}^a \subseteqq \mathcal{P}_i^a$ for all $i = 1, \ldots, N$, where corresponding inclusion tests and exclusion tests are fulfilled just between a box and a polytope.

5. Nonlinear Contractive Model Predictive Control via Polytopic Robust Controllable Sets

Once all the polytopic robust controllable sets $\mathcal{P}_j^a (j = 1, \ldots, N)$ have been obtained, robust controllability of any initial state can be judged accordingly. Assume that $\mathbf{x}_0 \in$

$\cup_{j=0}^{N}\mathcal{P}_j^a$, i.e., the initial state is robustly controllable to the selected terminal set $\mathbb{T}$ in finite steps, then the dual-mode approach of the nonlinear contractive model predictive control via polytopic robust controllable sets is illustrated in `Algorithm III`.

`Algorithm III. Nonlinear MPC via Polytopic Robust Controllable Sets`

`In:` $\mathbf{x}(0)$, $\mathcal{P}_j^a (j = 0, \ldots, N)$; `Out:` $\mathbf{u}^*(\mathbf{x}(k)), \mathbf{x}$

1. `Get the current state` $\mathbf{x}(k)$;
2. if $\mathbf{x}(k) \in \mathbb{T}$
3.   `Switch to a local stabilizing feedback control law;`
4. else
5.   `Find the` i: $i = \min_{j=1,\ldots,N}\{\mathbf{x}(k) \in \mathcal{P}_j^a\}$;
6.   `Compute` $\mathbf{u}^*(\mathbf{x}(k))$ `with the contractive constraint (5);`
7.   `Apply` $\mathbf{u}^*(\mathbf{x}(k))$ `to the system;`
8. end
9. `Return to 1 and repeat.`

According to `Algorithm III`, the control algorithm measures the current state in `Step 1` and then judges whether the system state has arrived into the terminal set $\mathbb{T}$ in `Step 2`. The related local stabilizing feedback control law is applied if the state has arrived into the terminal set $\mathbb{T}$; otherwise, the algorithm finds the smallest polytopic robust controllable set to which the current state belongs in `Step 5`; the one-step control scheme is formulated according to the strategy of robust model predictive control with feasible unit control horizon and contractive constraint in `Step 6`, where $Pre_i(\mathbb{T})$ is denoted to $\cup_{j=0}^{i}\mathcal{P}_j^a$ according to the addressed algorithm for computing polytopic robust controllable sets. It can be seen that any feasible solution that satisfies the imposed contractive constraint is an effective control input because such a control input is sufficient to guarantee the feasibility and stability of the closed-loop system [9]. A feasible control input can be obtained as well via an interval-based solver of constrained minimax optimization in a guaranteed numerical way [5].

6. An Illustrative Example

The addressed nonlinear contractive model predictive control via polytopic robust controllable sets is applied to control a highly nonlinear model of a Continuous Stirred-Tank Reactor (CSTR) [9, 10]. Assuming constant liquid volume, the CSTR for an exothermic, irreversible reaction, $\mathbf{A} \to \mathbf{B}$, is described by the following dynamic model based on a component balance for the reactant $\mathbf{A}$ and an energy balance:

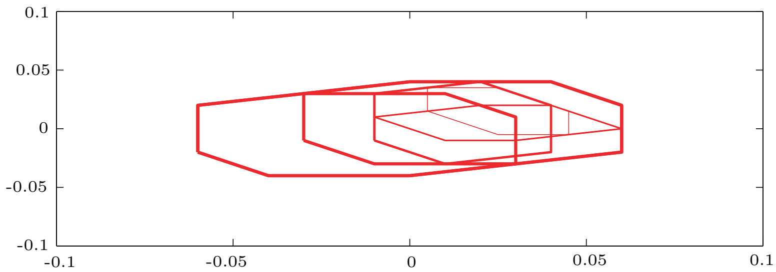
$$\begin{cases} \dot{C}_A = \frac{q}{V}(C_{Af} - C_A) - k_0 \exp(-\frac{E}{RT})C_A + w_1, \\ \dot{T} = \frac{q}{V}(T_f - T) + \frac{(-\triangle H)}{\rho C_p}k_0 \exp(-\frac{E}{RT})C_A + \frac{UA}{V\rho C_p}(T_c - T) + w_2, \end{cases} \tag{14}$$

where C_A is the concentration of $\mathbf{A}$ in the reactor, T is the reactor temperature, T_c is the temperature of the coolant stream, and $w_1 \in [-0.02, 0.02]$, $w_2 \in [-2, 2]$ are assumed to be additive disturbances. The constraints are $280 \text{ K} \le T_c \le 370 \text{ K}$, $280 \text{ K} \le T \le 370 \text{ K}$ and $0 \le C_A \le 1$ mol/l. The objective is to regulate C_A and T by manipulating T_c. The nominal operating conditions, which correspond to an unstable equilibrium $C_A^{eq} = 0.5$ mol/l, $T^{eq} = 350$ K, $T_c^{eq} = 300$ K are: $q = 100$ l/min, $C_{Af} = 1$ mol/l, $T_f = 350$ K, $V = 100$ l, $\rho = 1000$ g/l, $C_p = 0.239$ J/g K, $\triangle H = -5 \times 10^4$ J/mol, $E/R = 8750$ K, $k_0 = 7.2 \times 10^{10}$ min^{-1}, $UA = 5 \times 10^4$ J/min K. The nonlinear discrete-time state-space model is obtained by defining the state vector $\mathbf{x} = [C_A - C_A^{eq}\ (T - T^{eq})/100]^T$ as well as the manipulated input $u = (T_c - T_c^{eq})/100$ and by discretizing the ODE with a sampling time $\triangle\, t = 0.03$ min using the Euler method.

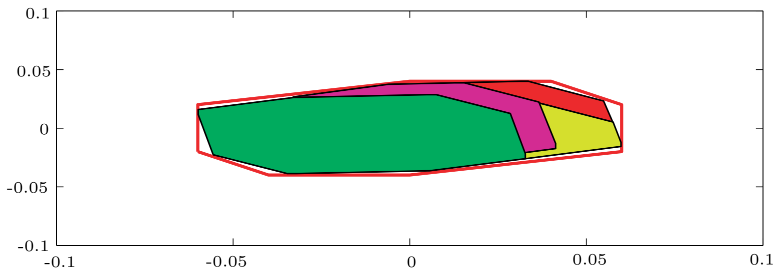
A local feedback control law $u = [-0.0690\ \ -4.3387]\mathbf{x}$ is designed in advance according to the linearized model and the LQ method [10]. With the designed local feedback control law, the terminal set is selected to be the following polytope:

$$\begin{bmatrix} 0.31623 & -0.94868 \\ -0.31623 & 0.94868 \\ -0.70711 & -0.70711 \\ 0.70711 & 0.70711 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \mathbf{x} \leq \begin{bmatrix} 0.037947 \\ 0.037947 \\ 0.056569 \\ 0.056569 \\ 0.06 \\ 0.06 \\ 0.04 \\ 0.04 \end{bmatrix}. \tag{15}$$

The selected polytope can be demonstrated geometrically to be robust control invariant for the discretized system with the related local feedback control law. The geometric demonstration of robust control invariance for the selected polytope is shown in Fig. 1, where the polytope is actually a zonotope [7]. The zonotope is bisected into sub-zonotopes and the evolutions of all sub-zonotopes under the related local feedback control law are within the original zonotope.



(a) The bisections of the selected terminal set



(b) The evolution of every sub-zonotope of the bisected terminal set

Figure 6.1: The geometric demonstration of robust control invariance

The first-step robust controllable set to the selected robust control invariant polytope can be computed via `Algorithm I` and the obtained first-step robust controllable set can be approximated innerly by one polytope via `Algorithm II`. The computed first-step robust controllable set and its polytopic approximation are shown in Fig. 2, where the bound of error tolerance is $\varepsilon = 0.002$. It is worthy to note that the computation time for the first-step robust controllable set is 2.14 hours and the computation time for the following polytopic approximation is 13 seconds using the addressed algorithms on a Pentium Centrino 1.4GHz Notebook while the corresponding computation times for the simplified system without consideration of uncertainty are respectively 19.79 hours and 15 seconds using the published algorithms in [3].

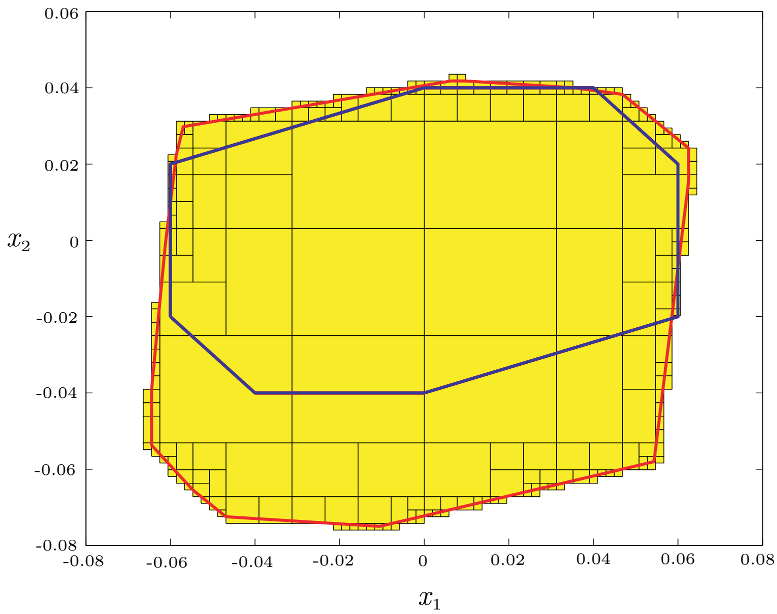


Figure 6.2: The first-step robust controllable set and its polytopic approximation

The following-step robust controllable sets can be computed accordingly by renewing the terminal set. The computed polytopic robust controllable sets of the discretized system are shown in Fig. 3. The resulting robust control process of the dual-mode approach of nonlinear contractive model predictive control via polytopic robust controllable sets for the discretized system with the initial state (0.4 mol/l,326 K) is shown in Fig. 4, where $Q = R = 1$ and the coordinates are transformed to be the original values of the controlled system. The underlying minimax optimizations for the one-step control inputs are performed by the solver of constrained minimax optimization via interval analysis [5].

7. Conclusion

This paper discusses the application of interval analysis and polytope geometry to compute polytopic robust controllable sets offline and compute one-step control inputs online for constrained nonlinear uncertain discrete-time systems. The union of the obtained polytopic robust controllable sets is guaranteed to be robust control invariant at each step and such a scenario is more general in practice, especially for piecewise-affine and hybrid systems. The proposed approach provides a general framework for computing polytopic robust controllable sets of constrained nonlinear uncertain systems with an initial robust control invariant polytope as well as controlling such complex systems with guaranteed feasibility and stability. However, the burden of computing polytopic robust controllable sets via the interval-based algorithm grows exponentially with the total dimension of the state space and the control space and thus structures of various constrained nonlinear discrete-time systems are needed to be explored further to improve the efficiency.

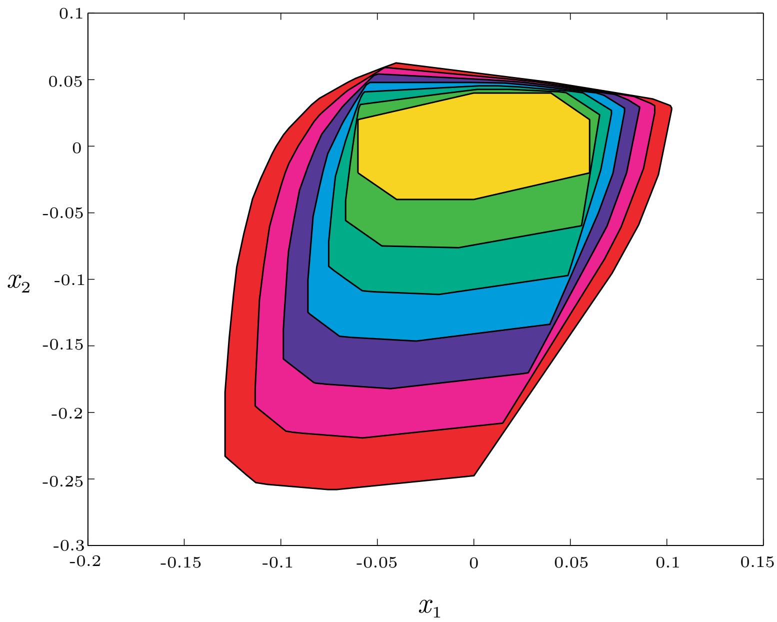


Figure 6.3: The computed polytopic robust controllable sets of the CSTR

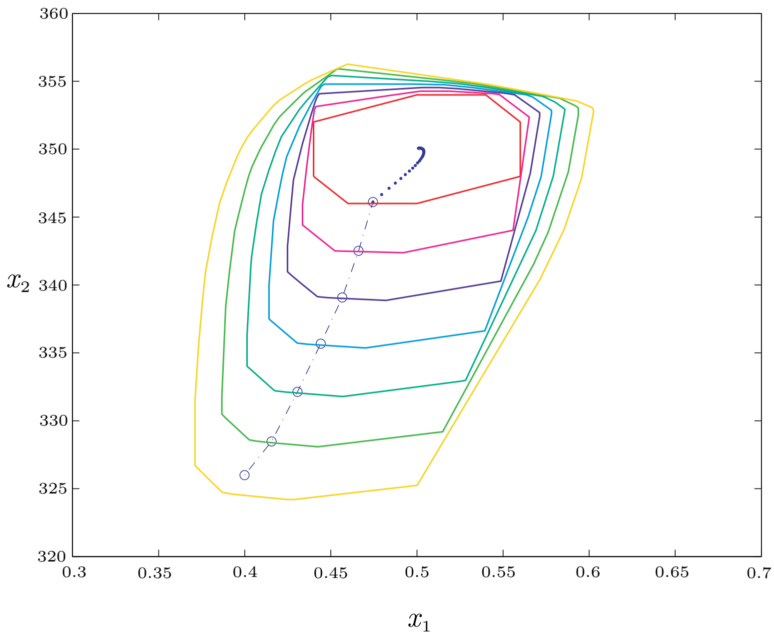


Figure 6.4: The dual-mode robust control process of the CSTR

Acknowledgements. This work was partially funded by the European Union (European Regional Development Fund) and the Spanish government (Plan Naciónal de Investigación Científica, Desarrollo e Innovación Tecnológica, Ministerio de Ciencia y Tecnología) through the coordinated research projects DPI2004-07167-C02-02, DPI2005-08668-C03-02 and DPI2007-66728-C02-02 and by the government of Catalonia through SGR00296. The uncommercial MATLAB Toolboxs of INTerval LABoratory (INTLAB) by Dr. Siegfried M. Rump, Invariant Set Toolbox by Dr. Eric C. Kerrigan and Multi-Parametric Toolbox (MPT) by M. Kvasnica, P. Grieder and M. Baotic have been used in the simulations and the authors are grateful for their implicit contributions. The authors are also very thankful to the anonymous reviewer for his/her pertinent comments and valuable suggestions, which are significant for correcting and improving the paper further.

References

[1] S. Boyd, L. Vandenberghe: Convex Optimization, Cambridge University Press, Cambridge (2004).

[2] J. M. Bravo, D. Limon, T. Alamo, E. F. Camacho: On the computation of invariant sets for constrained nonlinear systems: an interval arithmetic approach, in: Proc. European Control Conf. (Cambridge, 2003).

[3] J. M. Bravo, D. Limon, T. Alamo, E. F. Camacho: On the computation of invariant sets for constrained nonlinear systems: an interval arithmetic approach, Automatica 41(9) (2005) 1583–1589.

[4] M. Cannon, V. Deshmukh, B. Kouvaritakis: Nonlinear model predictive control with polytopic invariant sets, Automatica 39(8) (2003) 1487–1494.

[5] L. Jaulin, M. Kieffer, O. Didrit, E. Walter: Applied Interval Analysis, Springer, London (2001).

[6] E. C. Kerrigan: Robust Constraint Satisfaction: Invariant Sets and Predictive Control, Ph.D. Thesis, University of Cambridge, Cambridge (2000).

[7] W. Kühn: Rigorously computed orbits of dynamical systems without the wrapping effect, Computing 61(1) (1998) 47–67.

[8] M. Kvasnica, P. Grieder, M. Baotic: Multi-Parametric Toolbox (MPT), http://control.ee.ethz.ch/∼mpt/ (2005).

[9] D. Limon, T. Alamo, E. F. Camacho: Robust MPC control based on a contractive sequence of sets, in: IEEE Conf. on Decision and Control (Maui, 2003) 3706–3711.

[10] L. Magni, G. De Nicolao, L. Magnani, R. Scattolini: A stabilizing model-based predictive control algorithm for nonlinear systems, Automatica 37(9) (2001) 1351–1362.

[11] D. Q. Mayne, W. R. Schroeder: Robust time-optimal control of constrained linear systems, Automatica 33(12) (1997) 2103–2118.

[12] S. V. Rakovic, E. C. Kerrigan, D. Q. Mayne, J. Lygeros: Reachability analysis of discrete-time systems with disturbances, IEEE Trans. Autom. Control 51(4) (2006) 546–561.