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A Regularity Result for Asymptotically Convex Problems with Lower Order Terms

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We establish a local Lipschitz regularity result for minimizers of variational integrals with lower order terms, under the assumption that the integrand becomes appropriately convex at infinity.

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1. Introduction

Let us consider the integral functional

$$\mathcal{I}(u) = \int_\Omega f(Du(x))dx \tag{1}$$

where Ω is a bounded open subset of $I\!R^n$ and $u : \Omega \to I\!R^N$, $N \geq 1$, is a function in the Sobolev class $W^{1,p}(\Omega, I\!R^N)$, $p > 1$.

The regularity of local minimizers of $\mathcal{I}(u)$ has been widely investigated in case the integrand $f \in C^2$ is assumed to be convex or quasiconvex and to behave like $|\xi|^p$. In 1977, K. Uhlenbeck (see [20]), proved everywhere $C^{1,\alpha}$ regularity for local minimizers of the model case $f(\xi) = |\xi|^p$, with $p \geq 2$. This result was generalized first allowing dependence of the integrand on (x, u) (see [12], [13]), and next considering the case $1 < p < 2$ (see [1], [6], [15]).

Recently, many papers have been devoted to the study of the regularity of minimizers of non convex integrands satisfying suitable asymptotic growth assumptions. Remark that in the vectorial case it is known that minimizers may have singularities, and, as recently proved by Sverak and Yan [19], can be even unbounded. Anyway, prescribing a more regular behavior at infinity allows to have more regular minimizers. As far as we know, Lipschitz regularity results are available when f behaves asymptotically, in a C^2 sense, like $|\xi|^p$ (see [5] for the case $p = 2$, [13] for $p \geq 2$ and [17] for $1 < p < 2$).

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Moreover higher integrability results are available in case f behaves asymptotically in a C^1 or C^0 sense for every $1 < p < \infty$ ([7], [16]).

The aim of this paper is to establish the local boundedness of the gradient of local minimizers of variational integrals of the type

$$I(u) = \int_\Omega f(Du) + g(x,u)dx, \tag{2}$$

where the integrand f is asymptotically, in a C^2 sense, subquadratic at infinity, thus extending a previous result by the authors ([17]) to the case in which the functional depends also on (x,u) (see Theorem 2.1). The proof of our result is achieved comparing the minimizer of $I(u)$ with the minimizers of two more regular integrals and then using a standard iteration procedure.

We want to point out that, in case $p \geq 2$, Lipschitz regularity of local minimizers of (2) has been proved in [18].

With respect to [18], in order to obtain the same growth condition on the lower order terms, we have to face new difficulties due to subquadratic growth . Namely, the implementation of the result in the case $1 < p < 2$ requires subtle subquadratic estimates and a delicate technical effort. Here, we have to prove first that the gradient of a minimizer of (2) belongs to a suitable Morrey space (see Section 3). Then, we use this information to improve the regularity of Du via a comparison argument. We remark that in the scalar case our result allows the lower order term $g(x,u)$ to grow as any power $|u|^r$ with $0 < r < p^*$.

2. Statement and Technical Lemmas

Let Ω be a bounded open set of $\mathbb{R}^n$. Consider the integral

$$I(u) = \int_\Omega f(Du) + g(x,u)dx, \tag{3}$$

where the functions $f : \mathbb{R}^{nN} \to \mathbb{R}$, $g : \Omega \times \mathbb{R}^N \to \mathbb{R}$, $n \geq 2$, $N \geq 1$, satisfy the following assumptions, for $1 < p < 2$ and $\mu > 0$:

$$f \in C^2(\mathbb{R}^{nN}) \tag{4}$$

$$|D^2 f(\xi)| \leq L(\mu^2 + |\xi|^2)^{\frac{p-2}{2}}, \quad \text{for all } \xi \in \mathbb{R}^{nN} \tag{5}$$

$$\lim_{|\xi|\to\infty} |D^2 f(\xi) - D^2 H(\xi)|(\mu^2 + |\xi|^2)^{\frac{2-p}{2}} = 0, \tag{6}$$

where

$$H(\xi) := (\mu^2 + |\xi|^2)^{\frac{p}{2}} \tag{7}$$

$$g(x,\eta) \geq c_0 + c_1|\eta|^{q+1}, \tag{8}$$

$$|D_\eta g(x,\eta)| \leq c_2(|\eta|^q + 1) \tag{9}$$

$$|g(x,\eta) - g(x,\eta')| \leq c_2(|\eta|^2 + |\eta'|^2 + 1)^{\frac{q}{2}}|\eta - \eta'| \tag{10}$$

with $c_0, c_1, c_2 > 0$, and

$$0 < q + 1 < \frac{p^*}{p} \qquad \text{if } N > 1$$
$$0 < q + 1 < p^* \qquad \text{if } N = 1. \tag{11}$$

A function $u \in W^{1,p}(\Omega, I\!R^N)$ is a local minimizer of $I(u)$ if the following inequality

$$I(u) \leq I(u + \varphi),$$

holds for every test function $\varphi \in W_0^{1,p}(\Omega, I\!R^N)$ with compact support in Ω.

Our main result is the following theorem.

Theorem 2.1. *Let $f : I\!R^{nN} \to I\!R$ satisfy assumptions* (4), (5) *and* (6). *Moreover let $g : \Omega \times I\!R^N \to I\!R$ be a Carathéodory function satisfying the assumptions* (8), (9), (10) *and* (11). *If $u \in W^{1,p}(\Omega, I\!R^N)$ is a local minimizer of*

$$I(u) = \int_\Omega f(Du) + g(x, u)dx,$$

then Du is locally bounded in Ω. Moreover, for almost every $x_0 \in \Omega$, we have

$$|Du(x_0)| \leq C\left(1 + \left(⨍_{B_{R_0}(x_0)} |Du|^p dx\right)^{\frac{1}{p}}\right), \tag{12}$$

where $R_0 = \operatorname{dist}(x_0, \partial\Omega)$ and C depends on n, N, L, p, μ.

In what follows we will denote by $B_R(x_0)$ the ball $\{x \in I\!R^n : |x - x_0| < R\}$. To simplify the notation, the letter c will denote any positive constant, which may vary throughout the paper. If $u \in L^p$, for any $B_R(x_0)$ we set

$$u_{x_0,R} = \frac{1}{|B_R|} \int_{B_R(x_0)} u(x)dx = ⨍_{B_R(x_0)} u(x)dx. \tag{13}$$

For $\xi \in I\!R^k$, we define the following function

$$V(\xi) = (\mu^2 + |\xi|^2)^{\frac{p-2}{4}} \xi. \tag{14}$$

We begin by giving the following basic inequality .

Lemma 2.2. *For every $\gamma \in (-\frac{1}{2}, 0)$, there exist two positive constants $c_0(\gamma)$ and $c_1(\gamma)$ such that, for every $k \in I\!N$,*

$$c_0(\gamma) \leq \frac{\int_0^1 (\mu^2 + |\eta + t(\xi - \eta)|^2)^\gamma dt}{(\mu^2 + |\eta|^2 + |\xi|^2)^\gamma} \leq c_1(\gamma), \tag{15}$$

for all $\xi, \eta \in I\!R^k$ and $\mu > 0$.

For the proof see Lemma 2.1 in [1] .

Next Lemma contains some useful properties of the function V (see [3]).

Lemma 2.3. *Let $1 < p < 2$ and V be the function defined in (14), then for every $\xi, \eta \in I\!R^k$, $t > 0$*

$$
\begin{aligned}
&(1+\mu^2)^{\frac{p-2}{4}} \min\{|\xi|, |\xi|^{\frac{p}{2}}\} \le |V(\xi)| \le |\xi|^{\frac{p}{2}} \\
&|V(t\xi)| \le \max\{t, t^{\frac{p}{2}}\}|V(\xi)| \\
&|V(\xi+\eta)| \le c(p)\big[|V(\xi)| + |V(\eta)|\big] \\
&\tfrac{p}{2}|\xi-\eta| \le \frac{|V(\xi)-V(\eta)|}{\left(\mu^2+|\xi|^2+|\eta|^2\right)^{\frac{p-2}{4}}} \le c(k,p)|\xi-\eta| \\
&|V(\xi)-V(\eta)| \le c(k,p)|V(\xi-\eta)|.
\end{aligned}
\tag{16}
$$

The next regularity theorem can be found in [1], see Proposition 2.8.

Proposition 2.4. *Let $x_0 \in \Omega$, $B_R(x_0) \subset \Omega$, and let $w \in W^{1,p}(\Omega, I\!R^N)$. If w is a minimizer of the functional*

$$\int_{B_R(x_0)} H(Dw)dx$$

such that

$$w - u \in W_0^{1,p}(B_R(x_0), I\!R^N)$$

then $w \in C^{1,\alpha}(B_R(x_0); I\!R^N)$ for some $0 < \alpha < 1$. Moreover, there exists a constant $c > 0$ such that

$$\sup_{B_{\frac{R}{2}}(x_0)} |Dw|^p \le c \fint_{B_R(x_0)} |Dw|^p dx \tag{17}$$

and

$$
\begin{aligned}
&\int_{B_\rho(x_0)} |V(Dw) - V(Dw)_{x_0,\rho}|^2 dx \\
\le\ & c\left(\frac{\rho}{R}\right)^{n+2\alpha} \int_{B_R(x_0)} |V(Dw) - V(Dw)_{x_0,R}|^2 dx
\end{aligned}
\tag{18}
$$

for every $\rho < R$.

The following Lemma has been proved in [17]. We reproduce the proof here for the sake of completeness.

Lemma 2.5. *Let $1 < p < 2$ and let f, $h : I\!R^{nN} \to I\!R$ be two functions such that, for some $\mu > 0$,*

$$f, h \in C^2(I\!R^{nN}) \tag{19}$$

$$|D^2 f(\xi)|\,, |D^2 h(\xi)| \le L(\mu^2 + |\xi|^2)^{\frac{p-2}{2}}, \quad \text{for all } \xi \in I\!R^{nN} \tag{20}$$

$$\lim_{|\xi|\to\infty} |D^2 f(\xi) - D^2 h(\xi)|(\mu^2 + |\xi|^2)^{\frac{2-p}{2}} = 0. \tag{21}$$

Then for every $\varepsilon > 0$ there exist $\delta(\varepsilon) > 0$, $\omega(\varepsilon) > 0$ with $\lim_{\varepsilon\to 0}\omega(\varepsilon) = 0$, and c depending only on L and p, such that

$$\begin{aligned}&\left|\langle\int_0^1 (1-t)[D^2f(t\xi+(1-t)\xi_0) - D^2h(t\xi+(1-t)\xi_0)]dt(\xi-\xi_0),(\xi-\xi_0)\rangle\right| \\ \le\ & c\,\omega(\varepsilon)(|\xi-\xi_0|^2+\lambda^2)(\mu^2+|\xi|^2+|\xi_0|^2)^{\frac{p-2}{2}}\end{aligned} \tag{22}$$

holds for all $\xi, \xi_0 \in I\!R^{nN}$ and $\lambda \ge 0$ satisfying $|\xi_0|^2 + \lambda^2 > \delta^2(\varepsilon)$.

Proof. Assumption (21) implies that for every $\varepsilon > 0$ there exists $\Lambda(\varepsilon) > 0$ such that if $|\xi| > \Lambda(\varepsilon)$ then

$$|D^2f(\xi) - D^2h(\xi)| \le \varepsilon(\mu^2+|\xi|^2)^{\frac{p-2}{2}} \tag{23}$$

Now, for $\xi_0 \in I\!R^{nN}$, consider the set $I = \{t \in [0,1] :\ |t\xi + (1-t)\xi_0| \le \Lambda(\varepsilon)\}$ and split the integral in the left hand side of (22) into the sum of the integrals on I and $(0,1)\setminus I$, respectively. Thus, thanks to Lemma 2.2 and inequality (23),

$$\begin{aligned}&\left|\langle\int_{(0,1)\setminus I} (1-t)[D^2f(t\xi+(1-t)\xi_0) - D^2h(t\xi+(1-t)\xi_0)]dt(\xi-\xi_0),(\xi-\xi_0)\rangle\right| \\ \le\ & c_1(p)\varepsilon|\xi-\xi_0|^2(\mu^2+|\xi|^2+|\xi_0|^2)^{\frac{p-2}{2}}.\end{aligned} \tag{24}$$

Assumption (20), Hölder's inequality and again Lemma 2.2 yield

$$\begin{aligned}&\left|\langle\int_I (1-t)[D^2f(t\xi+(1-t)\xi_0) - D^2h(t\xi+(1-t)\xi_0)]dt(\xi-\xi_0),(\xi-\xi_0)\rangle\right| \\ \le\ & 2L\int_I(\mu^2+|t\xi+(1-t)\xi_0|^2)^{\frac{p-2}{2}}dt|\xi-\xi_0|^2 \\ \le\ & 2L|I|^{\frac{1}{\alpha}}\left(\int_0^1(\mu^2+|t\xi+(1-t)\xi_0|^2)^{\frac{p-2}{2}\alpha'}dt\right)^{\frac{1}{\alpha'}}|\xi-\xi_0|^2 \\ \le\ & 2Lc(p,\alpha)|I|^{\frac{1}{\alpha}}|\xi-\xi_0|^2(\mu^2+|\xi|^2+|\xi_0|^2)^{\frac{p-2}{2}}\end{aligned} \tag{25}$$

where $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$ and α' is choosen strictly less than $\frac{1}{2-p}$ in order to apply Lemma 2.2 with $\gamma = \frac{p-2}{2}\alpha'$.

Let us now estimate $|I|$. If S denotes the segment with end points ξ, ξ_0, we have

$$|I| = \frac{|S \cap \{\xi \in I\!R^{nN} : |\xi| \le \Lambda(\varepsilon)\}|}{|\xi-\xi_0|} \le \frac{2\Lambda(\varepsilon)}{|\xi-\xi_0|}.$$

Let us choose

$$\delta^2(\varepsilon) = \left(1+\frac{4L}{\varepsilon}\right)^2\Lambda^2(\varepsilon) + \left(\frac{4L}{\varepsilon}\right)^2\Lambda^2(\varepsilon),$$

and note that if

$$|\xi-\xi_0| > \frac{4L}{\varepsilon}\Lambda(\varepsilon)$$

then

$$|I| \leq \frac{\varepsilon}{2L}.$$

On the other hand if

$$|\xi - \xi_0| \leq \frac{4L}{\varepsilon}\Lambda(\varepsilon),$$

let $\lambda^2 + |\xi_0|^2 > \delta^2(\varepsilon)$; then

$$\lambda^2 > \left(\frac{4L}{\varepsilon}\Lambda(\varepsilon)\right)^2 \qquad \text{or} \qquad |\xi_0|^2 > \left(1 + \frac{4L}{\varepsilon}\right)^2 \Lambda^2(\varepsilon).$$

In the first case we have

$$|I|^{\frac{1}{\alpha}}|\xi - \xi_0|^2 \leq 2^{4-\frac{1}{\alpha}} L^{2-\frac{1}{\alpha}} \Lambda^{\frac{1}{\alpha}}(\varepsilon) \left(\frac{\Lambda(\varepsilon)}{\varepsilon}\right)^{2-\frac{1}{\alpha}} \leq (2L)^{-\frac{1}{\alpha}} \lambda^2 \varepsilon^{\frac{1}{\alpha}}.$$

In the second case one can easily see that $|I| = 0$. We conclude the proof combining previous estimate with (25).

Next Lemma has been proved in [18] in case $p \geq 2$, but it holds also in case $1 < p < 2$, with a slight modification.

Lemma 2.6. *Let H be the function defined at* (7) *and V the one defined at* (14). *Then there exist two positive constants c_1 and c_2 such that, for every $\xi, \eta \in I\!R^{nN}$, we have*

$$c_1|V(\xi) - V(\eta)|^2 \leq H(\xi) - H(\eta) - DH(\eta)(\xi - \eta) \leq c_2|V(\xi) - V(\eta)|^2 \tag{26}$$

We shall need also the following

Lemma 2.7. *Let H be the function defined at* (7) *and let f satisfy assumptions* (4), (5) *and* (6). *For every $\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that, for every $\xi \in I\!R^{nN}$, we have*

$$|f(\xi) - H(\xi)| \leq \varepsilon H(\xi) + c_\varepsilon \tag{27}$$

Proof. For every ξ we have

$$\begin{aligned}
&|f(\xi) - H(\xi)| \\
\leq\ & |f(0) - H(0)| + |Df(0) - DH(0)||\xi| + \int_0^1 (1-t)|D^2 f(t\xi) - D^2 H(t\xi)||\xi|^2 dt \\
\leq\ & c + \varepsilon|\xi|^p + c_\varepsilon|Df(0) - DH(0)|^{\frac{p}{p-1}} + \int_0^1 (1-t)|D^2 f(t\xi) - D^2 H(t\xi)||\xi|^2 dt \\
\leq\ & c_\varepsilon + \varepsilon H(\xi) + \int_0^1 (1-t)|D^2 f(t\xi) - D^2 H(t\xi)||\xi|^2 dt.
\end{aligned} \tag{28}$$

Thanks to the growth assumption (5) and since by the assumption (6) for every $\varepsilon > 0$ there exists c'_ε such that if $|\eta| > c'_\varepsilon \Longrightarrow |D^2 f(\eta) - D^2 H(\eta)| \leq \varepsilon(\mu^2 + |\eta|^2)^{\frac{p-2}{2}}$, we easily get that

$$\begin{aligned}
&\int_0^1 (1-t)|D^2 f(t\xi) - D^2 H(t\xi)||\xi|^2 dt \\
= &\int_{\{t|\xi|\le c'_\varepsilon\}} (1-t)|D^2 f(t\xi) - D^2 H(t\xi)||\xi|^2 dt \\
&\quad + \int_{\{t|\xi|> c'_\varepsilon\}} (1-t)|D^2 f(t\xi) - D^2 H(t\xi)||\xi|^2 dt \qquad (29)\\
\le &\int_{\{t|\xi|\le c'_\varepsilon\}} c(1-t)(\mu^2 + t^2|\xi|^2)^{\frac{p-2}{2}}|\xi|^2 dt + \int_{\{t|\xi|> c'_\varepsilon\}} c(1-t)\varepsilon(\mu^2 + t^2|\xi|^2)^{\frac{p-2}{2}}|\xi|^2 dt \\
\le &\ c\int_{\{t|\xi|\le c'_\varepsilon\}} t^{p-2}|\xi|^p dt + c\int_0^1 \varepsilon t^{p-2}|\xi|^p dt \ \le\ c''_\varepsilon|\xi| + \varepsilon|\xi|^p.
\end{aligned}$$

Now inserting (29) in (28) and using again Young's inequality, we get (27).

Let us remark that, by the growth assumptions on f, in the scalar case, if g grows less than p^*, u is locally Hölder continuous and thus locally bounded (see [14]).

Having this Lemma at our disposal, we can prove the following maximum principle in the scalar case.

Theorem 2.8. *Let $w : B_R(x_0) \to I\!R$ be the solution of the problem*

$$\min_{w\in u+W_0^{1,p}(B_R(x_0))} \int_{B_R(x_0)} H(Dw)dx \qquad (30)$$

where $u : \Omega \subset I\!R^n \to I\!R$ is a minimizer of $I(u)$ and $B_R(x_0) \subset\subset \Omega$. Then w is bounded in $B_R(x_0)$ and

$$\|w\|_{L^\infty(B_R(x_0))} \le \|u\|_{L^\infty(B_R(x_0))}.$$

Proof. Fix $B_R(x_0) \subset\subset \Omega$ and set $\|u\|_{L^\infty(B_R(x_0))} = k$. Consider

$$\tilde{w} = \max\{\min\{w(x), k\}, -k\}.$$

Notice that, since $|u(x)| \le k$ on $\partial B_R(x_0)$ and $w = u$ on $\partial B_R(x_0)$, also $\tilde{w} \in w + W_0^{1,p}(B_R(x_0))$. Moreover

$$D\tilde{w} = \begin{cases} Dw & \text{if } -k \le w \le k \\ 0 & \text{otherwise.} \end{cases}$$

By the minimality of w

$$\int_{B_R(x_0)} H(Dw(x))dx \le \int_{B_R(x_0)} H(D\tilde{w}(x))dx$$

hence

$$\int_{B_R(x_0)\cap\{|w|>k\}} H(Dw(x))dx \le \mu^p|B_R(x_0)|.$$

Since w is continuous, this implies that $|\{x \in B_R(x_0) : |w(x)| > k\}| = 0$, hence the result follows.

We conclude this section with a standard decay Lemma (see [14]).

Lemma 2.9. *Let $\varphi(t)$ a non negative and non decreasing function. Suppose that the following inequality*

$$\varphi(\rho) \leq A\left[\left(\frac{\rho}{R}\right)^{\alpha} + \varepsilon\right]\varphi(R) + BR^{\beta} \tag{31}$$

holds for every $\rho \leq R \leq R_0$, for some positive constants A, α, β with $\alpha > \beta$ and for some non negative constants B, ε. Then there exists a constant $\varepsilon_0 = \varepsilon_0(A, \alpha, \beta)$ such that if $\varepsilon < \varepsilon_0$, for every $\rho \leq R \leq R_0$ we have

$$\varphi(\rho) \leq c\left(\frac{\rho}{R}\right)^{\beta}[\varphi(R) + BR^{\beta}] \tag{32}$$

where $c = c(A, \alpha, \beta)$.

3. Hölder regularity

This Section is devoted to the proof of our first regularity result which shows that a minimizer of $I(v)$ has the gradient in the Morrey space $L^{p,\nu}_{loc}(\Omega; I\!R^N)$ for all $0 < \nu < n$ and as a consequence is Hölder continuous. To this aim we recall that the Morrey space $L^{p,\nu}_{loc}(\Omega; I\!R^N)$ is defined as the space of the functions v such that, for every ball $B_R \subset \Omega$, the following quantity

$$\sup_{0<\rho<R} \rho^{-\nu} \int_{B_\rho} |v|^p dx$$

is finite.

We recall also that v is locally α-Hölder continuous if, for every ball $B_R \subset \Omega$,

$$\sup_{x,y \in B_R} \frac{|v(x) - v(y)|}{|x - y|^{\alpha}} < \infty.$$

Lemma 3.1. *For every local minimizers u of $I(v)$ we have*

$$Du \in L^{p,\nu}_{loc}(\Omega; I\!R^N) \quad \forall\, 0 < \nu < n \tag{33}$$

and

$$u \in C^{0,\alpha}_{loc}(\Omega; I\!R^N) \quad \forall\, 0 < \alpha < 1 \tag{34}$$

Proof. Let $x_0 \in \Omega$, $R < R_0 = \min\left\{\frac{1}{2}\,\mathrm{dist}(x_0, \partial\Omega), 1\right\}$ and fix $0 < \rho < R$. Our aim is to prove that

$$\int_{B_\varrho} H(Du)dx \leq c\left(\left(\frac{\rho}{R}\right)^{n} + \epsilon\right)\int_{B_R} H(Du)dx + cR^n.$$

Such inequality is obviously satisfied if

$$\int_{B_\varrho} H(Du)dx \leq R^n$$

then, in what follows, we will assume that

$$\int_{B_\varrho} H(Du)dx > R^n \tag{35}$$

Let w be the minimizer of the problem

$$\int_{B_R(x_0)} H(Dw)dx$$

such that

$$w - u \in W_0^{1,p}(B_R(x_0), I\!R^N)$$

and observe that, from the first estimate of Proposition 2.4, we get

$$\int_{B_\rho} H(Dw)dx \le c\Big(\frac{\rho}{R}\Big)^n \int_{B_R} H(Dw)dx \le c\Big(\frac{\rho}{R}\Big)^n \int_{B_R} H(Du)dx. \tag{36}$$

By the definition of $H(\xi)$, we easily get that

$$\int_{B_\rho} H(Du)\,dx \le c\int_{B_\rho} H(Dw)\,dx + \bar{c}\int_{B_\rho} |Du - Dw|^p\,dx \tag{37}$$

Now, observe that

$$\begin{aligned}
&\|Du - Dw\|^p_{L^p(B_\rho)} \\
\le\ & c\int_{B_\rho} |V(Du) - V(Dw)|^p(|Du|^2 + |Dw|^2 + \mu^2)^{\frac{(2-p)p}{4}}dx \\
\le\ & c\Big(\int_{B_R} |V(Du) - V(Dw)|^2dx\Big)^{\frac{p}{2}}\Big(\int_{B_\rho} |Du|^p + |Dw|^pdx\Big)^{\frac{2-p}{2}} \\
&\quad + c\Big(\int_{B_R} |V(Du) - V(Dw)|^2dx\Big)^{\frac{p}{2}} R^{\frac{n(2-p)}{2}} \\
\le\ & c\Big(\int_{B_R} H(Du) - H(Dw)dx\Big)^{\frac{p}{2}}\Big(\int_{B_\rho} |Du|^p + |Dw|^pdx\Big)^{\frac{2-p}{2}} \\
\le\ & c\int_{B_R} H(Du) - H(Dw)dx + \frac{1}{2\bar{c}}\int_{B_\rho} |Du|^pdx + c\int_{B_\rho} |Dw|^pdx
\end{aligned} \tag{38}$$

where we used Lemmas 2.3 and 2.6 together with (35), the minimality of w and Young's inequality. Then inserting (38) in (37) and using (36) we get

$$\begin{aligned}
&\int_{B_\rho} H(Du)\,dx \\
\le\ & c\int_{B_\rho} H(Dw)\,dx + c\int_{B_R} H(Du) - H(Dw)dx + \frac{1}{2}\int_{B_\rho} |Du|^pdx + c\int_{B_\rho} |Dw|^pdx \\
\le\ & c\Big(\frac{\rho}{R}\Big)^n \int_{B_R} H(Du)\,dx + c\int_{B_R} H(Du) - H(Dw)\,dx + \frac{1}{2}\int_{B_\rho} H(Du)\,dx
\end{aligned} \tag{39}$$

and then

$$\int_{B_\rho} H(Du)\,dx \le c\Big(\frac{\rho}{R}\Big)^n \int_{B_R} H(Du)\,dx + c\int_{B_R} H(Du) - H(Dw)\,dx \tag{40}$$

Let us remark that, by the minimality of w, we have

$$\int_{B_R} DH(Dw)(Du - Dw)dx = 0.$$

By Lemma 2.7 and (10), using the minimality of u, we have

$$\begin{aligned} &\int_{B_R} [H(Du) - H(Dw)]\,dx \\ = &\int_{B_R} [H(Du) - f(Du) + f(Du) + g(x,u) - (f(Dw) + g(x,w)) - g(x,u) \\ &\quad + f(Dw) - H(Dw) + g(x,w)]\,dx \\ \le\ & c_\epsilon R^n + 2\epsilon \int_{B_R} H(Du)dx + \int_{B_R} (|u|^2 + |w|^2 + 1)^{\frac{q}{2}} |u - w| dx. \end{aligned} \tag{41}$$

In the scalar case, using Theorem 2.8, since $q + 1 < p^*$, we immediately get that

$$\int_{B_R} [H(Du) - H(Dw)]\,dx \le c_\epsilon R^n + 2\epsilon \int_{B_R} H(Du)dx + c(\|u\|_{L^\infty(B_R(x_0))})R^n. \tag{42}$$

In the vectorial case, we use Hölder and Sobolev's inequalities and the assumption $q+1 < \frac{p^*}{p}$, to get:

$$\begin{aligned} \int_{B_R} (|u|^2 + |w|^2 + 1)^{\frac{q}{2}} |u - w| dx \ &\le\ c(\|u\|^q_{p*} + \|w\|^q_{p*})\|Du - Dw\|_{L^p(B_R)} R^{n(1-p^{*-1}(q+1))} \\ &\quad + c\|Du - Dw\|_{L^p(B_R)} R^{n(1-p^{*-1})} \\ &\le\ c\|Du - Dw\|_{L^p(B_R)} R^{n(1-p^{*-1}(q+1))} \end{aligned} \tag{43}$$

By calculations similar to those in (38), we obtain

$$\|Du - Dw\|_{L^p(B_R)} \le c\Big(\int_{B_R} H(Du) - H(Dw)dx\Big)^{\frac{1}{2}} \Big(\int_{B_R} |Du|^p dx\Big)^{\frac{2-p}{2p}} \tag{44}$$

Then inserting (43) and (44) in (41), using Young's inequality we obtain

$$\begin{aligned} &\int_{B_R} H(Du) - H(Dw)dx \\ \le\ & c_\epsilon R^n + 2\epsilon \int_{B_R} H(Du)dx + c\Big(\int_{B_R} H(Du) - H(Dw)dx\Big)^{\frac{1}{2}} \|Du\|^{\frac{2-p}{2}}_{L^p(B_R)} R^{n(1-\frac{q+1}{p^*})} \\ \le\ & c_\epsilon R^n + 2\epsilon \int_{B_R} H(Du)dx + \frac{1}{2}\int_{B_R} H(Du) - H(Dw)dx + c\|Du\|^{2-p}_{L^p(B_R)} R^{2n(1-\frac{q+1}{p^*})} \\ \le\ & c_\epsilon R^n + 2\epsilon \int_{B_R} H(Du)dx + \frac{1}{2}\int_{B_R} H(Du) - H(Dw)dx \\ &\quad + \varepsilon \int_{B_R} H(Du) + c_\varepsilon R^{n(1-\frac{q+1}{p^*})(\frac{p}{p-1})} \end{aligned} \tag{45}$$

The assumption $q+1 < \frac{p^*}{p}$ yields that $n(1-\frac{q+1}{p^*})(\frac{p}{p-1}) > n$, and then inequality (45) implies

$$\int_{B_R} H(Du) - H(Dw) \leq c_\epsilon R^n + 3\epsilon \int_{B_R} H(Du)dx + c_\epsilon R^{n+\sigma},$$

for some $\sigma > 0$. Therefore, by (40), we deduce

$$\int_{B_\varrho} H(Du)dx \leq c\left(\left(\frac{\rho}{R}\right)^n + \epsilon\right) \int_{B_R} H(Du)dx + c_\epsilon R^{n+\sigma}.$$

Using Lemma 2.9, we have, in the scalar as in the vectorial case, that

$$\int_{B_\varrho} H(Du)dx \leq c_\sigma \left(\frac{\rho}{R}\right)^n \int_{B_R} H(Du)dx + c_\sigma R^n.$$

The regularity of Du is then proved and by standard results in Morrey-Campanato spaces it implies also that $u \in C^{0,\alpha}$, for every $\alpha < 1$ (see for instance [14]).

4. Proof of the main result

We are now in position to prove Theorem 2.1.

Proof of Theorem 2.1. Let $x_0 \in \Omega$, $R \leq \operatorname{dist}(x_0, \partial\Omega)$. Consider the following problems

$$\begin{cases} \int_{B_R(x_0)} H(Dw)dx \\ w - u \in W_0^{1,p}(B_R(x_0), I\!R^N), \end{cases} \tag{P1}$$

$$\begin{cases} \int_{B_R(x_0)} [H(Dv)dx + g(x,v)]dx \\ v - u \in W_0^{1,p}(B_R(x_0), I\!R^N). \end{cases} \tag{P2}$$

Let w be a minimizer of (P1) and let v be a minimizer of (P2).

Step 1. *Comparison between v and w*

Since w is a minimizers of problem (P1), w solves the corresponding Euler Lagrange equation and then

$$\int_{B_R(x_0)} DH(Dw)(Dv - Dw)dx = 0$$

Lemma 2.6 implies that

$$\begin{aligned} \int_{B_R(x_0)} |V(Dv) - V(Dw)|^2 dx &\leq \int_{B_R(x_0)} H(Dv) - H(Dw)dx \\ \leq \int_{B_R(x_0)} &[H(Dv) - g(x,v) + g(x,v)]dx \\ &+ \int_{B_R(x_0)} [-g(x,w) + g(x,w) - H(Dw)]dx \\ \leq \int_{B_R(x_0)} &[g(x,v) - g(x,w)]dx \end{aligned} \tag{46}$$

where in the last line we used the minimality of v. Then the assumption (10) and an argument similar to the one used to derive (38) yield

$$\begin{aligned}
&\int_{B_R(x_0)} |V(Dw)-V(Dv)|^2 dx \leq \int_{B_R} (|u|^2+|w|^2+1)^{\frac{q}{2}}|u-w|dx \\
\leq\ & c\|Dv-Dw\|_{L^p(B_R)} R^{n(1-p^{*-1}(q+1))} \\
\leq\ & c\Big(\int_{B_R(x_0)} |V(Dw)-V(Dv)|^2 dx\Big)^{\frac{1}{2}} \\
&\cdot\Big(\int_{B_R(x_0)} |Dw|^p+|Dv|^p\,dx\Big)^{\frac{2-p}{2p}} R^{n(1-p^{*-1}(q+1))} \\
\leq\ & c\Big(\int_{B_R(x_0)} |V(Dw)-V(Dv)|^2 dx\Big)^{\frac{1}{2}} \\
&\cdot\Big(\int_{B_R(x_0)} H(Dw)+H(Dv)\,dx\Big)^{\frac{2-p}{2p}} R^{n(1-p^{*-1}(q+1))}
\end{aligned} \tag{47}$$

Since v is a minimizer of (P2) we get

$$\begin{aligned}
&\int_{B_R(x_0)} H(Dv)dx \\
=\ & \int_{B_R(x_0)} H(Dv)+g(x,v)-g(x,v)dx \leq \int_{B_R(x_0)} H(Du)+g(x,u)-g(x,v)dx \\
\leq\ & \int_{B_R(x_0)} H(Du)dx + c\|Dv-Du\|_{L^p(B_R)} R^{n(1-p^{*-1}(q+1))} \\
\leq\ & \int_{B_R(x_0)} H(Du)dx + c\Big(\int_{B_R(x_0)} H(Du)+H(Dv)dx\Big)^{\frac{1}{p}} \cdot R^{n(1-p^{*-1}(q+1))}.
\end{aligned} \tag{48}$$

Using Young's inequality we obtain

$$\begin{aligned}
&\int_{B_R(x_0)} H(Dv)dx \\
\leq\ & (1+\epsilon)\int_{B_R(x_0)} H(Du)dx + \epsilon\int_{B_R(x_0)} H(Dv)dx + c_\epsilon R^{n(1-p^{*-1}(q+1))(\frac{p}{p-1})}.
\end{aligned} \tag{49}$$

Therefore

$$\int_{B_R(x_0)} H(Dv)dx \leq c\int_{B_R(x_0)} H(Du)dx + cR^{n(1-p^{*-1}(q+1))(\frac{p}{p-1})}. \tag{50}$$

Now inserting the above estimate in (47) and using the minimality of w we obtain

$$\begin{aligned}
&\int_{B_R(x_0)} |V(Dw)-V(Dv)|^2 dx \\
\leq\ & c\Big(\int_{B_R(x_0)} H(Du)dx + R^{n(1-p^{*-1}(q+1))(\frac{p}{p-1})}\Big)^{\frac{2-p}{p}} R^{2n(1-p^{*-1}(q+1))} \\
\leq\ & cR^{2n(1-p^{*-1}(q+1))+\nu\frac{2-p}{p}} cR^{n(1-p^{*-1}(q+1))\big(\frac{p}{p-1}\big)\big(\frac{2-p}{p}\big)+2n(1-p^{*-1}(q+1))}
\end{aligned} \tag{51}$$

where we used Lemma 3.1. Let us remark that, $2n(1-p^{*-1}(q+1))+\nu\frac{2-p}{p} > n$. In fact this inequality is equivalent to $\frac{2(n-p)(q+1)-np}{2-p} < \nu$, which is satisfied for some $\nu < n$, since the assumption $q+1 < \frac{p^*}{p}$ implies that $\frac{2(n-p)(q+1)-np}{2-p} < n$. The same assumption $q+1 < \frac{p^*}{p}$ implies also that $n(1-p^{*-1}(q+1))\left(\frac{p}{p-1}\right)\left(\frac{2-p}{p}\right) + 2n(1-p^{*-1}(q+1))$ is strictly greater than n. Then we choose $\frac{2(n-p)(q+1)-np}{2-p} < \nu < n$ in order to have that the exponent of the radius R in the last line is strictly greater than n. Then, inequality (51) can be written as follows

$$\int_{B_R(x_0)} |V(Dw)-V(Dv)|^2 dx \leq cR^{n+\sigma} \tag{52}$$

for some $\sigma > 0$.

Step 2. *The comparison between v and u*

Since v is a minimizer of problem (P2), v solves the corresponding Euler Lagrange equation and then

$$\int_{B_R(x_0)} DH(Dv)(Du-Dv) + Dg(x,v)(u-v)dx = 0$$

Lemma 2.6 and the minimality of u imply that

$$\begin{aligned}
&\int_{B_R(x_0)} |V(Du)-V(Dv)|^2 dx \\
\leq\ & \int_{B_R(x_0)} [H(Du)-H(Dv)-DH(Dv)(Du-Dv)]dx \\
\leq\ & \int_{B_R(x_0)} [H(Du)-H(Dv)+Dg(x,v)(u-v)]dx \\
\leq\ & \int_{B_R(x_0)} [H(Du)-f(Du)+f(Dv)-H(Dv)]dx \\
& \quad + \int_{B_R(x_0)} [g(x,v)-g(x,u)]dx + \int_{B_R(x_0)} Dg(x,v)(u-v)]dx \\
\leq\ & \int_{B_R(x_0)} [g(x,v)-g(x,u)]dx + \int_{B_R(x_0)} Dg(x,v)(u-v)]dx \\
& \quad + \int_{B_R(x_0)} \int_0^1 (1-t)D^2H((1-t)\xi_0+tDu) - D^2 f((1-t)\xi_0+tDu) \\
& \quad \cdot (\xi_0-Du)(\xi_0-Du) \\
& \quad + \int_{B_R(x_0)} \int_0^1 (1-t)D^2H((1-t)\xi_0+tDv) - D^2 f((1-t)\xi_0+tDv) \\
& \quad \cdot (\xi_0-Dv)(\xi_0-Dv),
\end{aligned} \tag{53}$$

where in the last inequality we have used the fact that $v = u$ on $\partial B_R(x_0)$. Let us choose $\xi_{0,R}$ such that $V(\xi_{0,R}) = (V(Du))_{x_0,R}$ and set

$$\lambda^2 = (\mu^2 + |\xi_{0,R}|^2)^{\frac{2-p}{2}} \fint_{B_R(x_0)} |V(Du)-(V(Du))_{x_0,R}|^2 dx. \tag{54}$$

Since $\displaystyle\fint_{B_R(x_0)} |V(Du)-(V(Du))_{x_0,R}|^2dx = \fint_{B_R(x_0)} |V(Du)|^2dx - |V(\xi_{0,R})|^2$, it follows that, for every $\varepsilon > 0$, if

$$(\mu^2 + |\xi_{0,R}|^2)^{\frac{2-p}{2}} \fint_{B_R(x_0)} |V(Du)|^2\, dx > \delta^2(\varepsilon), \tag{55}$$

then

$$|\xi_{0,R}|^2 + \lambda^2 > \delta^2(\varepsilon).$$

Thus, thanks to Lemma 2.5, (53) implies

$$\begin{aligned}
&\int_{B_R(x_0)} |V(Du) - V(Dv)|^2dx \\
\leq\ & c\,\omega(\varepsilon)\Big\{ \int_{B_R(x_0)} |Du - \xi_{0,R}|^2(\mu^2 + |Du|^2 + |\xi_{0,R}|^2)^{\frac{p-2}{2}} dx \\
&\quad +\lambda^2 \int_{B_R(x_0)} (\mu^2 + |Du|^2 + |\xi_{0,R}|^2)^{\frac{p-2}{2}} dx \\
&\quad + \int_{B_R(x_0)} |Dv - \xi_{0,R}|^2(\mu^2 + |Dv|^2 + |\xi_{0,R}|^2)^{\frac{p-2}{2}} dx \\
&\quad +\lambda^2 \int_{B_R(x_0)} (\mu^2 + |Dv|^2 + |\xi_{0,R}|^2)^{\frac{p-2}{2}} dx\Big\} + cR^{n+\sigma}.
\end{aligned}$$

where to estimate the integrals involving g we argued as in (47),. . . ,(52). Since $(\mu^2 + |Du|^2 + |\xi_{0,R}|^2)^{\frac{p-2}{2}}, (\mu^2 + |Dv|^2 + |\xi_{0,R}|^2)^{\frac{p-2}{2}} \leq (\mu^2 + |\xi_{0,R}|^2)^{\frac{p-2}{2}}$, we have by the definition of λ^2 at (54)

$$\lambda^2 \int_{B_R(x_0)} (\mu^2 + |Du|^2 + |\xi_{0,R}|^2)^{\frac{p-2}{2}} dx \leq \int_{B_R(x_0)} |V(Du) - (V(Du))_{x_0,R}|^2dx,$$

$$\lambda^2 \int_{B_R(x_0)} (\mu^2 + |Dv|^2 + |\xi_{0,R}|^2)^{\frac{p-2}{2}} dx \leq \int_{B_R(x_0)} |V(Du) - (V(Du))_{x_0,R}|^2dx.$$

Then, by (16)(iv) we have

$$\begin{aligned}
&\int_{B_R(x_0)} |V(Du) - V(Dv)|^2dx \\
\leq\ & c\,\omega(\varepsilon)\Big\{ \int_{B_R(x_0)} |V(Du) - V(\xi_{0,R})|^2dx + \int_{B_R(x_0)} |V(Du) - V(Dv)|^2dx\Big\} + cR^{n+\sigma}
\end{aligned}$$

Hence, since $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0$, there exists ε_0 such that for every $\varepsilon < \varepsilon_0$, we get

$$\int_{B_R(x_0)} |V(Du) - V(Dv)|^2dx \leq c\,\omega(\varepsilon) \int_{B_R(x_0)} |V(Du) - V(\xi_{0,R})|^2dx + c_\varepsilon R^{n+\sigma}. \tag{56}$$

Now observe that

$$\begin{aligned}
&\int_{B_\rho(x_0)} |V(Du) - V(Du)_{x_0,\rho}|^2 dx \\
\le\ & \int_{B_\rho(x_0)} |V(Du) - V(Dv)|^2 dx + \int_{B_\rho(x_0)} |V(Dv) - V(Dw)|^2 dx \\
&\quad + \int_{B_\rho(x_0)} |V(Dw) - V(Dw)_{x_0,\rho}|^2 dx + \int_{B_\rho(x_0)} |V(Dw)_{x_0,\rho} - V(Du)_{x_0,\rho}|^2 dx \qquad (57) \\
\le\ & c\int_{B_R(x_0)} |V(Du) - V(Dv)|^2 dx + c\int_{B_R(x_0)} |V(Dv) - V(Dw)|^2 dx \\
&\quad + c\int_{B_\rho(x_0)} |V(Dw) - V(Dw)_{x_0,\rho}|^2 dx
\end{aligned}$$

for every $\rho < R$. Then inserting (52) and (56) in (57) and using the second estimate in Proposition 2.4 we get

$$\begin{aligned}
&\int_{B_\rho(x_0)} |V(Du) - (V(Du))_{x_0,\rho}|^2 dx \\
\le\ & c_0\omega(\varepsilon)\int_{B_R(x_0)} |V(Du) - (V(Du))_{x_0,R}|^2 dx + c_0 R^{n+\sigma} \qquad (58) \\
&\quad + \Big(\frac{\rho}{R}\Big)^{n+2\alpha}\int_{B_R(x_0)} |V(Dw) - V(Dw)_{x_0,R}|^2 dx
\end{aligned}$$

for every $0 < \rho < R$ and for a constant $c_0 = c_0(L, p, n, N, \mu) > 1$. On the other hand we also have

$$\begin{aligned}
&\int_{B_R(x_0)} |V(Dw) - (V(Dw))_{x_0,R}|^2 dx \\
\le\ & c\int_{B_R(x_0)} |V(Dw) - V(Dv)|^2 dx + c\int_{B_R(x_0)} |V(Dv) - V(Du)|^2 dx \qquad (59) \\
&\quad + c\int_{B_R(x_0)} |V(Du) - V(Du)_{x_0,R}|^2 dx
\end{aligned}$$

which allows us to conclude that

$$\begin{aligned}
&\int_{B_\rho(x_0)} |V(Du) - (V(Du))_{x_0,\rho}|^2 dx \\
\le\ & c_0\left(\Big(\frac{\rho}{R}\Big)^{n+2\alpha} + \omega(\varepsilon)\right)\int_{B_R(x_0)} |V(Du) - (V(Du))_{x_0,R}|^2 dx + c_0 R^{n+\sigma} \qquad (60)
\end{aligned}$$

for every $0 < \rho < R$ and for a constant $c_0 = c_0(L, p, n, N, \mu) > 1$.

Step 3. *Conclusion*

We conclude as in [18], using a standard iteration argument.

Choose $0 < \tau < 1$, $\varepsilon_1 < \varepsilon_0$ and $\sigma < \gamma < \alpha$ such that

$$2c\tau^{n+\alpha} = \tau^{n+\gamma}, \quad \omega(\varepsilon_1) < \tau^{n+\alpha}.$$

Recall that the choice of ε_1 is possible since the function $\omega(\varepsilon)$ tends to 0 as ε goes to 0. Then for every $R < R_0 = \mathrm{dist}(x_0, \partial\Omega)$ we have

$$(\mu^2 + |\xi_{0,R}|^2)^{\frac{2-p}{2}} \fint_{B_R(x_0)} |V(Du)|^2 \, dx \leq \delta^2(\varepsilon_1) \tag{61}$$

or

$$\Phi(\tau R) \leq \tau^{n+\gamma}\Phi(R) + cR^{n+\sigma} \tag{62}$$

where we set

$$\Phi(R) = \int_{B_R(x_0)} |V(Du) - (V(Du))_{x_0,R}|^2 dx$$

Fix x_0 such that

$$\lim_{r\to 0} \fint_{B_r(x_0)} |V(Du) - (V(Du))(x_0)|^2 \, dx = 0,$$

and set

$$0 \leq R_1 = \inf\{0 < R \leq R_0 : \text{ (61) holds for } R \text{ or } R = R_0\}.$$

If $R_1 = 0$, then (61) holds for a sequence $R_j \to 0$, from which we deduce that

$$|V(Du)(x_0)| \leq \delta(\varepsilon_1)\mu^{\frac{p-2}{2}}. \tag{63}$$

If $0 < R_1 \leq R_0$ let, for $k \in \mathbb{N}$,

$$r_k = \tau^k R_1,$$

which is less than or equal to R_1. From (62), iterating, we get

$$\begin{aligned}
&\fint_{B_{r_{k+1}}(x_0)} |V(Du) - (V(Du))_{x_0,r_k}|^2 dx \\
\leq\ & \tau^{k\gamma} \fint_{B_{R_1}(x_0)} |V(Du) - (V(Du))_{x_0,R_1}|^2 dx + c_0 k \tau^{k\sigma - n} R_1^\sigma \\
\leq\ & \tau^{k\gamma} \fint_{B_{R_1}(x_0)} |V(Du)|^2 dx + c_0 k \tau^{k\sigma - n} R_1^\sigma.
\end{aligned}$$

Since

$$|(V(Du))_{x_0,r_{k+1}} - (V(Du))_{x_0,r_k}| \leq c(\tau) \left(\fint_{B_{r_k}(x_0)} |V(Du) - (V(Du))_{x_0,r_k}|^2 dx \right)^{\frac{1}{2}},$$

we get

$$\begin{aligned}
|V(Du)(x_0)| &\leq \sum_{k=2}^{\infty} |(V(Du))_{x_0,r_{k+1}} - (V(Du))_{x_0,r_k}| + |(V(Du))_{x_0,R_1}| \\
&\leq c \left(\fint_{B_{R_1}(x_0)} |V(Du)|^2 dx \right)^{\frac{1}{2}} + c.
\end{aligned}$$

If $R_1 < R_0$ one obtains by (61)

$$\left(\fint_{B_{R_1}(x_0)} |V(Du)|^2 dx \right)^{\frac{1}{2}} \leq \mu^{\frac{p-2}{2}} \delta(\varepsilon_1). \tag{64}$$

It is worth pointing out that this positive constant depends also on $\Lambda(\varepsilon)$,i. e. the speed of convergence in the limit (21), through the constant $\delta^2(\varepsilon_1)$. If $R_1 = R_0$

$$|V(Du)(x_0)| \leq c\Big(\fint_{B_{R_0}(x_0)} |Du|^p dx \Big)^{\frac{1}{p}} + c,$$

with $c = c(n, N, L, p)$. In conclusion, we proved that

$$|Du(x_0)| \leq c\left(1 + \Big(\fint_{B_{R_0}(x_0)} |Du|^p dx \Big)^{\frac{1}{p}} \right),$$

where c depends on p, L, n, N, μ and on $\Lambda(\varepsilon_1)$. □

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