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Pseudometrizable Bornological Convergence is Attouch-Wets Convergence

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This paper is dedicated to Roger Wets.

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Let $\mathcal{S}$ be an ideal of subsets of a metric space $\langle X, d\rangle$. A net of subsets $\langle A_\lambda\rangle$ of X is called $\mathcal{S}$-*convergent* to a subset A of X if for each $S \in \mathcal{S}$ and each $\varepsilon > 0$, we have eventually $A \cap S \subseteq A_\lambda^\varepsilon$ and $A_\lambda \cap S \subseteq A^\varepsilon$. We identify necessary and sufficient conditions for this convergence to be admissible and topological on the power set of X. We show that $\mathcal{S}$-convergence is compatible with a pseudometrizable topology if and only if $\mathcal{S}$ has a countable base and each member of $\mathcal{S}$ has an ε-enlargement that is again in $\mathcal{S}$. Further, in the case that the ideal is a bornology, we show that $\mathcal{S}$-convergence when pseudometrizable is Attouch-Wets convergence with respect to an equivalent metric.

Keywords: Bornological convergence, Attouch-Wets convergence, bounded Hausdorff convergence, hyperspace, bornology

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1. Introduction

If A is a subset of a metric space $\langle X, d\rangle$ and $\varepsilon > 0$, we write A^ε for the *epsilon enlargement* of A, defined by

$$A^\varepsilon := \{x \in X : d(x, A) < \varepsilon\}$$

where we agree that $d(x, \emptyset) = \infty$. In terms of this notation, a net of sets $\langle A_\lambda\rangle$ in X is declared *Attouch-Wets convergent* to a subset A of X if for each d-bounded set B and each $\varepsilon > 0$, we have eventually

$$A \cap B \subseteq A_\lambda^\varepsilon \quad \text{and} \quad A_\lambda \cap B \subseteq A^\varepsilon.$$

It is hard to overstate the importance of this mode of convergence in convex analysis and, more generally, in variational analysis (see, e.g., [3, 7, 21, 25, 26]). The first result of note for this mode of convergence was the Walkup-Wets isometry theorem [28] that has as a consequence bicontinuity of polarity for convex cones. But their result was a quantitative one, not just a qualitative one, and predicted the importance of this mode of convergence

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in terms of estimation and the development of usable calculus rules, as contrasted with Mosco convergence or Joly/slice convergence (see, e.g., [7]). Precisely, Walkup and Wets showed that if A and C are convex cones in a normed linear space X with polar cones A° and C°, then

$$\begin{aligned} &\sup\{\varepsilon > 0 : C \cap B \subseteq A^\varepsilon \text{ and } A \cap B \subseteq C^\varepsilon\}. \\ &= \sup\{\varepsilon > 0 : C^\circ \cap B^\circ \subseteq (A^\circ)^\varepsilon \text{ and } A^\circ \cap B^\circ \subseteq (C^\circ)^\varepsilon\}, \end{aligned}$$

where B is the closed unit ball of X and B° is the closed unit ball of the dual space (the hypothesis of reflexivity in the original formulation is superfluous). Bicontinuity of Fenchel conjugacy identifying lower semicontinuous convex functions with their epigraphs was established over twenty years later [6, 13, 24]; this implies bicontinuity of polarity for general closed convex sets containing the origin. These results were preceded by similar results for Mosco convergence [22] which of course are valid only in reflexive spaces, along with Joly's seminal work on convex duality and topology [18]. But more fundamentally Attouch-Wets convergence also imposes itself on linear analysis: a sequence of continuous linear functionals converges in the operator norm to a continuous linear functional f if and only if their graphs Attouch-Wets converge to the graph of f (see, e.g., [7, p. 104]).

Recall that a topology τ on a set W is called *pseudometrizable* if there is a pseudometric on W whose balls form a base for the topology, and that a uniformity on W is called a *pseudometrizable uniformity* if it coincides with the uniformity of a pseudometric on W. As is well-known [29], a uniformity is pseudometrizable if and only if the uniformity has a countable base. Attouch-Wets convergence on the power set $\mathcal{P}(X)$ of a metric space $\langle X, d\rangle$ fits within this framework, for if x_0 is a fixed point of X, a compatible uniformity has as a base all sets of the form

$$E_n := \{(A, C) : A \cap \{x_0\}^n \subseteq C^{\frac{1}{n}} \text{ and } C \cap \{x_0\}^n \subseteq A^{\frac{1}{n}}\} \quad (n \in \mathbb{N}).$$

where $\mathbb{N}$ denotes the set of positive integers. Note that this uniformity on $\mathcal{P}(X)$ is not separated since $\forall n \in \mathbb{N}\ \forall A \in \mathcal{P}(X)$, we have $(A, \mathrm{cl}(A)) \in E_n$, and for this reason there is no hope of getting a metrizable topology for the entire hyperspace unless the d-topology is discrete. But restricted to the closed subsets of X, it is easy to check that we get a separated uniformity and hence a metrizable topology τ_{AW} which is completely metrizable provided $\langle X, d\rangle$ is a complete metric space. Further, restricted to the nonempty closed subsets, it can be shown that the Attouch-Wets topology τ_{AW} is the topology of uniform convergence of distance functionals on bounded subsets of X [5, 4, 2, 7].

The nonempty d-bounded subsets $\mathcal{B}_d(X)$ obviously form a *bornology* [16, 8, 20], i.e., a nonempty hereditary family of nonempty subsets of X that is closed under finite unions and that forms a cover of X. Now if we replace $\mathcal{B}_d(X)$ in the definition of Attouch-Wets convergence by an arbitrary bornology $\mathcal{S}$, the resulting convergence need not be topological. It turns out that the convergence we get is topological if and only if the bornology is *stable under small enlargements*, i.e., given $S \in \mathcal{S}\ \exists \varepsilon > 0$ with $S^\varepsilon \in \mathcal{S}$. In this paper we give necessary and sufficient conditions for the topology to be pseudometrizable, and show that when this occurs, there is an equivalent metric for the topology of X so that the convergence is actually Attouch-Wets convergence with respect to the remetrization (see Theorem 3.18 *infra*).

2. Preliminaries

In the sequel, all metric spaces are assumed to contain at least two points.

Let $\mathcal{S}$ be a nonempty family of nonempty subsets of a metric space $\langle X, d\rangle$; following [20] we declare a net $\langle A_\lambda\rangle$ of subsets of X $\mathcal{S}$-*convergent* to a subset A of X if for each $S \in \mathcal{S}$ and each $\varepsilon > 0$, we have eventually

$$A \cap S \subseteq A_\lambda^\varepsilon \quad \text{and} \quad A_\lambda \cap S \subseteq A^\varepsilon.$$

No finer convergence results if we replace $\mathcal{S}$ by subsets of finite unions of members of $\mathcal{S}$, and so in the general case, we may assume without loss of generality that $\mathcal{S}$ is a nonempty hereditary family closed under finite unions, i.e., a so-called *ideal* of subsets of X. Of course, a bornology is an ideal that is a cover. No coarser convergence occurs if we replace our ideal $\mathcal{S}$ by a *base* for $\mathcal{S}$, i.e., a subfamily of $\mathcal{S}$ that is cofinal with respect to inclusion. We mention that bornological convergence has been considered in this way by Di Maio, Meccariello, and Naimpally [14] for bornologies assumed to have a closed base.

Given an ideal $\mathcal{S}$ and $S \in \mathcal{S}$ and $\varepsilon > 0$, form the entourage

$$[S, \varepsilon] := \{(A, B) \in \mathcal{P}(X) \times \mathcal{P}(X) : A \cap S \subseteq B^\varepsilon \text{ and } B \cap S \subseteq A^\varepsilon\},$$

and for $A \subseteq X$ write $[S, \varepsilon](A) = \{B \in \mathcal{P}(X) : (A, B) \in [S, \varepsilon]\}$. Evidently, $\langle A_\lambda\rangle$ is $\mathcal{S}$-convergent to A if and only if for each S and $\varepsilon > 0$, eventually $A_\lambda \in [S, \varepsilon](A)$.

In applications, we often wish to restrict our attention to convergence of nets of closed sets, and we now write $\mathcal{C}(X)$ for the family of closed subsets of X including the empty set. So restricted, a convergence can be better behaved than it is more generally. For example, if $\mathcal{K}(X)$ is the bornology of nonempty subsets of X that have compact closure, then a net $\langle A_\lambda\rangle$ of closed subsets is $\mathcal{K}(X)$-convergent to a closed set A if and only if $\langle A_\lambda\rangle$ is convergent in the *Fell topology* on $\mathcal{C}(X)$ to A [7, p. 141], having as a subbase all sets of the form

$$\{C \in \mathcal{C}(X) : C \cap V \neq \emptyset\} \quad (V \text{ open}),$$

$$\{C \in \mathcal{C}(X) : C \cap K = \emptyset\} \quad (K \text{ compact}).$$

The importance of the Fell topology on $\mathcal{C}(X)$ stems from its compactness without restriction (see, e.g., [1, 7]). As we shall see in Theorem 3.1 below, this kind of convergence on $\mathcal{P}(X)$ may fail to be topological. We will also write $\mathcal{C}_0(X)$ for the nonempty closed subsets of X.

The strongest $\mathcal{S}$-convergence is obtained when our ideal consists of all nonempty subsets $\mathcal{P}_0(X)$, and as $X \in \mathcal{P}_0(X)$, a net $\langle A_\lambda\rangle$ is $\mathcal{P}_0(X)$-convergent to a subset A if and only if $\forall\varepsilon > 0$, we have eventually

$$A \subseteq A_\lambda^\varepsilon \quad \text{and} \quad A_\lambda \subseteq A^\varepsilon.$$

Thus, $\mathcal{P}_0(X)$-convergence is compatible with *Hausdorff distance* on $\mathcal{P}(X)$ [19, 7], an extended real-valued pseudometric defined by

$$H_d(A, B) = \inf\{\varepsilon > 0 : A \subseteq B^\varepsilon \text{ and } B \subseteq A^\varepsilon\}.$$

Note that $H_d(A, B) = 0 \Leftrightarrow \mathrm{cl}(A) = \mathrm{cl}(B)$. In the literature, Attouch-Wets convergence is often called *bounded Hausdorff convergence* (see, e.g., [4, 25]).

One desirable property for a convergence defined on $\mathcal{P}(X)$ or less generally on some subfamily of $\mathcal{P}(X)$ containing the singletons, e.g., the closed convex sets of a normed linear space, is that the assignment $x \mapsto \{x\}$ be an embedding with respect to convergence: for each net $\langle x_\lambda \rangle$ in X, we have

$$\lim d(x_\lambda, x) = 0 \text{ if and only if } \langle \{x_\lambda\} \rangle \text{ converges to } \{x\}.$$

In this case the convergence is called *admissible*. With respect to $\mathcal{S}$-convergence, this amounts to $\mathcal{S}$ almost being a bornology as next described.

Proposition 2.1. *Let $\mathcal{S}$ be an ideal in a metric space $\langle X, d\rangle$. Then $\mathcal{S}$-convergence on $\mathcal{P}(X)$ is admissible if and only if either* (1) *$\mathcal{S}$ is a bornology, or* (2) *$X \backslash \cup \mathcal{S} = \{p\}$ and $\forall \varepsilon > 0\ \exists S \in \mathcal{S}$ with $diam_d(X\backslash S) < \varepsilon$.*

Proof. Whether or not either (1) or (2) hold, since convergence in Hausdorff distance is admissible, if $\lim d(x_\lambda, x) = 0$, then $\lim H_d(\{x_\lambda\}, \{x\}) = 0$ and so {x} = $\mathcal{S}$-lim $\{x_\lambda\}$. Suppose on the other hand that {x} = $\mathcal{S}$-lim $\{x_\lambda\}$. If (1) holds, choose $S_0 \in \mathcal{S}$ with $x \in S_0$; eventually, $\{x\} \cap S_0 \subseteq \{x_\lambda\}^\varepsilon$ and so eventually $d(x, x_\lambda) < \varepsilon$. If (1) fails but (2) holds, we need only deal with the case that $x \notin \cup\mathcal{S}$. Choose $S_1 \in \mathcal{S}$ with $\mathrm{diam}_d(X\backslash S_1) < \varepsilon$. There exists an index λ_0 such that $\lambda \geq \lambda_0 \Rightarrow \{x_\lambda\} \cap S_1 \subseteq \{x\}^\varepsilon$. Thus whether or not $x_\lambda \in S_1$ for $\lambda \geqslant \lambda_0$, we have $d(x, x_\lambda) < \varepsilon$ because $x \notin S_1$.

Conversely suppose neither (1) nor (2) hold. If two distinct points p and q lay outside $\cup\mathcal{S}$, then the sequence $\{p\}, \{p\}, \{p\}, \dots$ is $\mathcal{S}$-convergent to $\{q\}$, violating admissiblity. So we may assume $X \backslash \cup \mathcal{S}$ is a singleton $\{p\}$. If $\forall S \in \mathcal{S}$, $\mathrm{diam}_d(X\backslash S) \geqslant \varepsilon$, then for each $S \in \mathcal{S}, \exists x_S \in X\backslash S$ with $d(x_S, p) \geq \frac{\varepsilon}{2}$. The net $\langle \{x_S\}\rangle_{S\in\mathcal{S}}$ is $\mathcal{S}$-convergent to $\{p\}$ and again admissiblity is violated. □

In the case that condition (2) in Proposition 2.1 holds and $\mathcal{S}$ has a closed base, then $\{(X\backslash S) \cup \{p\} : S \in \mathcal{S}\}$ is a local base for the topology at p. Thus, we can think of X as a one-point extension of $X\backslash\{p\}$ where complements of open neighborhoods of the ideal point $\{p\}$ are closed members of the bornology. It can be shown that any T_1 one-point extension of a T_1 topological space arises exactly in this way from a bornology with closed base (see, e.g., [11, Prop. 2.6]).

More primitively, we can define $\mathcal{S}^-$-convergence and $\mathcal{S}^+$-convergence of a net in $\mathcal{P}(X)$ as follows: $\langle A_\lambda \rangle$ is $\mathcal{S}^-$-convergent (resp. $\mathcal{S}^+$-convergent) to A if for each $S \in \mathcal{S}$ and each $\varepsilon > 0$ we have eventually $A \cap S \subseteq A_\lambda^\varepsilon$ (resp. $A_\lambda \cap S \subseteq A^\varepsilon$). There is not complete symmetry in lower and upper convergence so defined; the reader may consult [20] for details. Generically, all of these convergences are now called *bornological convergences* in the literature, whether or not the ideal is a bornology (see also [10]).

3. Results

When $\mathcal{S}$ is a bornology with closed base, cobbling together Propositions 2.15, Proposition 3.5 and Corollary 3.8 (iii) of [20], one can assert the equivalence of the following statements:

(1) $\mathcal{S}$-convergence is topological on $\mathcal{P}(X)$;
(2) $\{[S, \varepsilon] : S \in \mathcal{S}, \varepsilon > 0\}$ is a base for a uniformity on $\mathcal{P}(X)$;
(3) $\mathcal{S}$ is stable under small enlargements.

As the framework in which this equivalence was obtained involved splitting $\mathcal{S}$-convergence into its upper and lower halves, we find it worthwhile to supply a stand-alone proof here. Further, the equivalence is valid when $\mathcal{S}$ is just an ideal without the closed base assumption.

As is well-known [29, p. 33] one can state properties for a collection of neighborhood bases $\{\mathcal{B}_x : x \in X\}$ for a topological space X that are characteristic. Put differently, in a set X, if a collection $\{\mathcal{B}_x : x \in X\}$ of subsets of X satisfies these properties where of course each x is a member of each element of $\mathcal{B}_x$, then for each x the collection $\mathcal{B}_x$ must form a local base for some topology τ on X at x. Invariably, the crucial property is

$$(\sharp)\ \forall x \in X\ \forall U \in \mathcal{B}_x\ \exists V \in \mathcal{B}_x \text{ such that } \forall y \in V\ \exists W \in \mathcal{B}_y \text{ with } W \subseteq U.$$

If ($\sharp$) fails, one can easily produce a net in X that fails the iterated limit criterion [19, p. 30] necessary for the convergence of $\langle x_\lambda \rangle$ to x defined by

$$\langle x_\lambda \rangle \to x \text{ iff } \forall U \in \mathcal{B}_x,\ x_\lambda \in U \text{ eventually}$$

to be topological

Theorem 3.1. *Let $\mathcal{S}$ be an ideal in a metric space $\langle X, d \rangle$. The following conditions are equivalent:*

(*1*) $\mathcal{S}$*-convergence is topological on* $\mathcal{P}(X)$*;*
(*2*) $\{[S, \varepsilon] : S \in \mathcal{S}, \varepsilon > 0\}$ *is a base for a uniformity* $\triangle_{\mathcal{S}}$ *on* $\mathcal{P}(X)$*;*
(*3*) $\mathcal{S}$ *is stable under small enlargements.*

Proof. (*3*) $\Rightarrow$ (*2*). Only the composition property is at issue. To verify this, let $[S_0, \varepsilon]$ be arbitrary. Choose $0 < \delta < \frac{\varepsilon}{2}$ such that $T = S_0^\delta \in \mathcal{S}$. We intend to show that $[T, \delta] \circ [T, \delta] \subseteq [S_0, \varepsilon]$. To this end suppose $(A, B) \in [T, \delta]$ and $(B, C) \in [T, \delta]$ are arbitrary. We must show $A \cap S_0 \subseteq C^\varepsilon$ and $C \cap S_0 \subseteq A^\varepsilon$, and we only establish the first inclusion. Let $a \in A \cap S_0$ be arbitrary; since $S_0 \subseteq T$, we can choose $b \in B$ with $d(a, b) < \delta$. Now $b \in T$ so there exists $c \in C$ with $d(b, c) < \delta$, and so $a \in (C^\delta)^\delta \subseteq C^\varepsilon$.

(*2*) $\Rightarrow$ (*1*). At each $A \in \mathcal{P}(X)$, $\{[S, \varepsilon](A) : S \in \mathcal{S}\}$ forms a local base for the topology of the uniformity which is compatible with $\mathcal{S}$-convergence.

(*1*) $\Rightarrow$ (*3*). By the remarks preceding the theorem it suffices to show that if $\mathcal{S}$ fails to be stable under small enlargements, then $\{\mathfrak{B}_A : A \in \mathcal{P}(X)\}$ fails to satisfy condition ($\sharp$) where of course

$$\mathfrak{B}_A := \{[S, \varepsilon](A) : S \in \mathcal{S},\ \varepsilon > 0\}.$$

If (*3*) fails, there exists $S_0 \in \mathcal{S}$ such that $\forall \varepsilon > 0,\ S_0^\varepsilon \notin \mathcal{S}$. We show that condition ($\sharp$) fails with respect to $U = [S_0, 1](\emptyset) \in \mathfrak{B}_\emptyset$. We use the following elementary fact: for every $A \in \mathcal{P}(X)$, $S \in \mathcal{S}$, and $\varepsilon > 0$,

$$A \in [S, \varepsilon](\emptyset) \Leftrightarrow A \cap S = \emptyset.$$

Let $V = [S_1, \varepsilon_1](\emptyset)$ be an arbitrary member of $\mathfrak{B}_\emptyset$. Since $\forall n\ S_0^{\frac{1}{n}} \nsubseteq S_1$, we can pick for each n $b_n \in S_0^{\frac{1}{n}} \backslash S_1$. Put

$$B = \{b_n : n \in \mathbb{N}\} \in [S_1, \varepsilon_1](\emptyset).$$

We claim $\forall W \in \mathfrak{B}_B$, we have $W \nsubseteq U$. To see this let $W = [S_2, \delta](B)$ be arbitrary in $\mathfrak{B}_B$. Take $n \in \mathbb{N}$ with $1/n < \delta$ and then $q \in S_0$ with $d(b_n, q) < 1/n$. Then actually $H_d(B, B \cup \{q\}) < 1/n < \delta$ so that $B \cup \{q\} \in [S_2, \delta](B)$. However, $B \cup \{q\} \notin U$ because $q \in S_0$. □

One might object to our proof of *(1)* $\Rightarrow$ *(3)* because we resort to arguing at the empty set. We remark that if we are willing to assume that the ideal has a closed base - which is implied by condition *(3)* - then we can tweak our proof to show that condition *(3)* is necessary for $\mathcal{S}$-convergence to be topological on the nonempty subsets $\mathcal{P}_0(X)$ of X. We leave verification of this claim as an exercise to the interested reader.

Let $\mathcal{F}(X)$ denote the bornology of nonempty finite subsets of X, and again let $\mathcal{K}(X)$ denote the bornology of nonempty subsets with compact closure. The following two corollaries are implicit in [20].

Corollary 3.2. *Let $\langle X, d\rangle$ be a metric space. Then $\mathcal{F}(X)$-convergence is topological on $\mathcal{P}(X)$ if and only if d gives the discrete topology.*

Corollary 3.3. *Let $\langle X, d\rangle$ be a metric space. Then $\mathcal{K}(X)$-convergence is topological on $\mathcal{P}(X)$ if and only if the topology of X is locally compact.*

When $\langle X, d\rangle$ is locally compact, it can be shown that a compatible topology has as a subbase all sets of the form

$$\{A \in \mathcal{P}(X) : A \cap V \neq \emptyset\} \quad (V \text{ open}),$$
$$\{A \in \mathcal{P}(X) : \exists \varepsilon > 0 \text{ with } A \cap K^\varepsilon = \emptyset\} \quad (K \text{ compact}).$$

Generically, a topology of this type is called a *proximal topology* [10]. This last corollary is somewhat unanticipated in that $\mathcal{K}(X)$-convergence restricted to $\mathcal{C}(X)$ is without qualification compatible with the Fell topology as we noted earlier.

Here is a noteworthy negative corollary.

Corollary 3.4. *Let $\langle X, d\rangle$ be a metric space whose topology is not discrete, and let $\mathcal{S}$ be the ideal consisting of the nowhere dense subsets of X. Then $\mathcal{S}$-convergence is never topological on $\mathcal{P}(X)$.*

We next list two corollaries in the domain of functional analysis.

Corollary 3.5. *Let X be a normed linear space and let $\mathcal{S}$ be the bornology having as a base the weakly compact sets. Then $\mathcal{S}$-convergence is topological on $\mathcal{P}(X)$ if and only if X is reflexive.*

Proof. If X is not reflexive, then no ε-enlargement of a point can lie in a weakly compact set as no closed ball is weakly compact. If X is reflexive and A is weakly compact, then for each $\varepsilon > 0, A^\varepsilon \subseteq A + \{x : \|x\| \leq \varepsilon\}$ which is again a weakly compact set. □

Notice that if two-sided bornological convergence as determined by the weakly compact sets is topological, then it is already Attouch-Wets convergence. In a general normed linear space, it follows from the uniform boundedness principle [27] applied to the dual space that the weakly bounded sets coincide with the norm bounded sets. Thus, two-sided bornological convergence as determined by the weakly bounded sets is again Attouch-Wets convergence and is trivially topological. But while weak*- bounded subsets of X^* need not be norm bounded, we may still state

Corollary 3.6. *Let X be a normed linear space and let $\mathcal{S}$ be the bornology on X^* having as a base the weak* bounded sets. Then $\mathcal{S}$-convergence is always topological on $\mathcal{P}(X^*)$.*

Proof. Suppose A is a weak* bounded subset of X^*. Then by definition $\forall x \in X$, $\sup\{a(x) : a \in A\} < \infty$. But for each x,

$$\sup\{a(x) : a \in A^1\} = \sup\{a(x) : a \in A\} + \|x\|,$$

and so the enlargement of radius 1 about A is also weak* bounded. $\square$

In the sequel when $\mathcal{S}$ is an ideal that is closed under small enlargements, we will write $\tau_{\mathcal{S}}$ for the topology of $\mathcal{S}$-convergence on $\mathcal{P}(X)$. This topology is always completely regular because it is induced by a uniformity [29]. In addition to Proposition 2.1, another rationale for restricting our attention here to bornologies is provided by the following proposition.

Proposition 3.7. *Let $\mathcal{S}$ be an ideal in a metric space $\langle X, d\rangle$ that is stable under small enlargements. The following conditions are equivalent:*

(*1*) $\mathcal{S}$ *is a bornology;*
(*2*) $\mathcal{C}(X)$ *equipped with the $\tau_{\mathcal{S}}$-relative topology is Hausdorff;*
(*3*) $\mathcal{C}_0(X)$ *equipped with the $\tau_{\mathcal{S}}$-relative topology is Hausdorff.*

Proof. (*1*) $\Rightarrow$ (*2*). If $\mathcal{S}$ is a cover and A and B are distinct closed sets with $A \nsubseteq B$, chose $a \in A$ and $\varepsilon > 0$ with $d(a, B) > \varepsilon$. As $\mathcal{S}$ is hereditary, we have $\{a\} \in \mathcal{S}$ and $(A, B) \notin [\{a\}, \varepsilon]$. This shows that the trace of the uniformity $\triangle_{\mathcal{S}}$ on $\mathcal{C}(X)$ is separated so that the relative topology is Hausdorff.

(*2*) $\Rightarrow$ (*3*). This is trivial.

(*3*) $\Rightarrow$ (*1*). Suppose (*1*) fails, i.e., $\exists x_0 \notin \cup\mathcal{S}$. Take $x_1 \neq x_0$; then the sequence of nonempty closed sets $\{x_0, x_1\}, \{x_0, x_1\}, \{x_0, x_1\}, \ldots$ is $\mathcal{S}$-convergent to both $\{x_1\}$ and to $\{x_0, x_1\}$, and so the relative topology on $\mathcal{C}_0(X)$ cannot be Hausdorff. $\square$

We note that whether or not $\mathcal{S}$-convergence is topological restricted to $\mathcal{C}(X)$, one can easily see that $\mathcal{S}$-limits are unique in $\mathcal{C}(X)$ (or in $\mathcal{C}_0(X)$) if and only if $\forall x \in X\ \exists \varepsilon > 0$ with $\{x\}^\varepsilon \in \mathcal{S}$ (see [20, Prop. 4.5]).

Theorem 3.8. *Let $\mathcal{S}$ be an ideal in a metric space $\langle X, d\rangle$ that is stable under small enlargements. The following conditions are equivalent:*

(*1*) $\mathcal{S}$ *has a countable cofinal subset with respect to inclusion;*
(*2*) *The uniformity $\triangle_{\mathcal{S}}$ on $\mathcal{P}(X)$ has a countable base;*
(*3*) *The hyperspace $\langle \mathcal{P}(X), \tau_{\mathcal{S}}\rangle$ is pseudometrizable;*
(*4*) *The hyperspace $\langle \mathcal{F}(X), \tau_{\mathcal{S}}\rangle$ is first countable.*

Proof. (*1*) $\Rightarrow$ (*2*). Choose $\langle S_n\rangle$ cofinal and increasing in $\mathcal{S}$; then clearly $\{[S_n, \frac{1}{n}] : n \in \mathbb{N}\}$ is a countable base for $\triangle_{\mathcal{S}}$.

(2) $\Rightarrow$ *(3)*. This is a consequence of a standard fact from general topology (see, e.g., [29, p. 257]).

(3) $\Rightarrow$ *(4)*. This is obvious.

(4) $\Rightarrow$ *(1)*. Let $x_0 \in \cup \mathcal{S}$ be arbitrary, and let $\{\mathcal{F}(X) \cap [S_n, \varepsilon_n](\{x_0\}) : n \in \mathbb{N}\}$ be a countable local base for $\langle \mathcal{F}(X), \tau_{\mathcal{S}} \rangle$ at $\{x_0\}$. Since $\mathcal{S}$ is hereditary, $\exists \delta > 0$ with $\{x_0\}^\delta \in \mathcal{S}$. For each n set $T_n = \{x_0\}^\delta \cup \cup_{j=1}^n S_j$. Since $\mathcal{S}$ is closed under finite unions, $\langle T_n \rangle$ is an increasing sequence in $\mathcal{S}$. We claim that $\{T_n : n \in \mathbb{N}\}$ is cofinal in $\mathcal{S}$. If this fails we can find $S \in \mathcal{S}$ not included in any T_n. For each n, pick $x_n \in S \backslash T_n$. Then for $j \geq n$, we have

$$\{x_0, x_j\} \in [T_n, \varepsilon_n](\{x_0\}),$$

and evidently $\{\mathcal{F}(X) \cap [T_n, \varepsilon_n](\{x_0\}) : n \in \mathbb{N}\}$ is also a local base for the relative topology at $\{x_0\}$. This yields $\{x_0\} = \tau_{\mathcal{S}} - \lim \{x_0, x_j\}$. But on the other hand, for each j,

$$\{x_0, x_j\} \notin [S, \frac{\delta}{2}](\{x_0\}),$$

and this gives the desired contradiction. $\square$

In view of Proposition 3.7, and since finite sets are closed sets, we may state

Corollary 3.9. *Let $\mathcal{S}$ be an ideal in a metric space $\langle X, d \rangle$ that is stable under small enlargements. Then $\langle \mathcal{C}(X), \tau_{\mathcal{S}} \rangle$ is metrizable if and only if $\mathcal{S}$ is a bornology with a countable base.*

Let X be a metrizable space and let $\mathfrak{D}_X$ be the family of metrics that are compatible with the topology. A basic question to ask is this: under what circumstances do two compatible metrics for X yield the same $\mathcal{S}$-convergence? When $\mathcal{S} = \mathcal{P}_0(X)$, it is well-known that the convergences coincide if and only if the metrics define the same uniformity [7, p. 92]. This does not help much in solving the general problem, but the answer given for $\mathcal{S} = \mathcal{B}_d(X)$ as determined by Beer and Di Concilio [9] puts us on the right track. The key idea is provided by the following definition.

Definition 3.10. Let $\langle X, d \rangle$ and $\langle Y, \rho \rangle$ be metric spaces and let S be a subset of X. We say that a function $f : X \to Y$ is *strongly uniformly continuous on* S if $\forall \varepsilon > 0\, \exists \delta > 0$ such that if $d(x, w) < \delta$ and $\{x, w\} \cap S \neq \emptyset$, then $\rho(f(x), f(w)) < \varepsilon$. If $\mathcal{S}$ is a family of subsets, we say f is *strongly uniformly continuous on* $\mathcal{S}$ if it is strongly uniformly continuous on each member of $\mathcal{S}$.

The following facts are easy to verify: (1) if f is strongly uniformly continuous on $\mathcal{S}$, then it is strongly uniformly continuous on the ideal generated by $\mathcal{S}$; (2) f is globally continuous if and only if it is strongly uniformly continuous on $\mathcal{F}(X)$; (3) f is uniformly continuous on X if and only if f is strongly uniformly continuous on $\mathcal{P}_0(X)$; (4) if f is globally continuous, then it is strongly uniformly continuous on $\mathcal{K}(X)$. If $d \in \mathfrak{D}_X$ and $\rho \in \mathfrak{D}_X$, we say ρ is *uniformly stronger than d on an ideal* $\mathcal{S}$ provided the identity map $\mathrm{id} : \langle X, \rho \rangle \to \langle X, d \rangle$ is strongly uniformly continuous on $\mathcal{S}$. If this is also true when the metrics are reversed, we say that d and ρ are *uniformly equivalent with respect to* $\mathcal{S}$. The following lemma is an easy exercise.

Lemma 3.11. *Let $\langle X, d\rangle$ be a metric space and let $\overline{d}$ be a metric that determines the same uniformity as d. Suppose $f : X \to \mathbb{R}$ is a continuous function that is strongly uniformly continuous on an ideal $\mathcal{S}$. Then the metric $\rho \in \mathfrak{D}_X$ defined by*

$$\rho(x, y) := \overline{d}(x, y) + |f(x) - f(y)|$$

is uniformly equivalent to d with respect to $\mathcal{S}$.

Example 3.12. On the real line $\mathbb{R}$, $\rho(x, y) = |x^2 - y^2| + \min\{1, |x - y|\}$ is uniformly equivalent to the usual metric with respect to the bounded subsets of $\mathbb{R}$ but is not uniformly equivalent to the usual metric.

We now look at two-sided bornological convergence defined on a metrizable space depending on two parameters: the ideal $\mathcal{S}$ and the metric d chosen from $\mathfrak{D}_X$. We incorporate the parameter d in our notation as follows: $(\mathcal{S}, d)$-convergence will mean $\mathcal{S}$-convergence for the metric space $\langle X, d\rangle$, and $[S, d, \varepsilon]$ will mean $[S, \varepsilon]$ where the enlargements are taken with respect to the metric d. Further, $\Delta_{\mathcal{S},d}$ will be denote the "pre-uniform structure" consisting of all supersets of all entourages $[S, d, \varepsilon]$.

Theorem 3.13. *Let d and ρ be compatible metrics for a metrizable topological space X, and let $\mathcal{T}$ and $\mathcal{S}$ be two ideals in X. The following conditions are equivalent:*

(1) $(\mathcal{S}, \rho)$*-convergence ensures* $(\mathcal{T}, d)$*-convergence for nets in* $\mathcal{P}(X)$*;*
(2) $(\mathcal{S}, \rho)$*-convergence ensures* $(\mathcal{T}, d)$*-convergence for nets in* $\mathcal{C}_0(X)$*;*
(3) $\mathcal{T} \subseteq \mathcal{S}$ *and ρ is uniformly stronger than d on* $\mathcal{T}$*;*
(4) *For each $T \in \mathcal{T}$ and $\varepsilon > 0$ there exists $S \in \mathcal{S}$ and $\lambda > 0$ with* $[S, \rho, \lambda] \subseteq [T, d, \varepsilon]$*.*

Proof. *(1)* $\Rightarrow$ *(2)*. This is obvious.

(2) $\Rightarrow$ *(3)*. Suppose first that $\mathcal{T} \nsubseteq \mathcal{S}$. Then $\exists T_0 \in \mathcal{T}$ such that $\forall S \in \mathcal{S}, T_0 \backslash S \neq \emptyset$. For each $S \in \mathcal{S}$, pick $x_S \in T_0 \backslash S$. Clearly, $\forall x \in X$, we have $\{x\} = (\mathcal{S}, \rho)\text{-lim}\ \{x_S, x\}$. By *(2)*, we have $\{x\} = (\mathcal{T}, d)\text{-lim}\ \{x_S, x\}$. But since $\langle x_S\rangle$ is a net in T_0, we conclude that for all x, $\lim d(x_S, x) = 0$. This contradicts our groundrule that X is not a singleton, and we now have shown that $\mathcal{T} \subseteq \mathcal{S}$.

It remains to show that ρ is uniformly stronger than d on $\mathcal{T}$. If not, then by definition $\text{id} : \langle X, \rho\rangle \to \langle X, d\rangle$ is not strongly uniformly continuous on $\mathcal{T}$. Thus for some $T \in \mathcal{T}$ and $\varepsilon > 0$ there exists sequences $\langle t_n\rangle$ in T and $\langle w_n\rangle$ in X such that $\forall n \in \mathbb{N}$

$$\rho(t_n, w_n) \leq \frac{1}{n} \text{ but } d(t_n, w_n) > \varepsilon.$$

Notice that since d and ρ are equivalent metrics, neither sequence can have a cluster point. By the Efremovic Lemma [7, 23] by passing to a subsequence we may assume $\forall n, k \in \mathbb{N}$ that $d(t_n, w_k) \geq \varepsilon/4$. For each $k \in \mathbb{N}$ write

$$A_k := \{w_n : n \in \mathbb{N}\} \cup \{t_n : n \geq k\} \in \mathcal{C}_0(X).$$

Clearly the sequence $\langle A_k\rangle$ converges to the closed set $\{w_n : n \in \mathbb{N}\}$ in Hausdorff distance H_ρ and thus is $(\mathcal{S}, \rho)$-convergent, too. But for all k, $(A_k, \{w_n : n \in \mathbb{N}\}) \notin [T, d, \varepsilon/4]$ and so $(\mathcal{T}, d)$-convergence fails. This contradiction shows ρ is uniformly stronger than d on $\mathcal{T}$.

(3) $\Rightarrow$ *(4)*. Fix $T \in \mathcal{T}$ and $\varepsilon > 0$. There exists $\lambda > 0$ such that if $\{x, y\} \cap T \neq \emptyset$ and $\rho(x, y) < \lambda$ then $d(x, y) < \varepsilon$. Then with $S = T$, we have $[S, \rho, \lambda] \subseteq [T, d, \varepsilon]$.

(4) $\Rightarrow$ *(1)*. This is obvious. □

Corollary 3.14. *Let d and ρ be compatible metrics for a metrizable topological space X, and let $\mathcal{T}$ and $\mathcal{S}$ be two ideals in X. The following conditions are equivalent:*

(1) $(\mathcal{S}, \rho)$*-convergence coincides with* $(\mathcal{T}, d)$*-convergence for nets in* $\mathcal{P}(X)$*;*
(2) $(\mathcal{S}, \rho)$*-convergence coincides with* $(\mathcal{T}, d)$*-convergence for nets in* $\mathcal{C}_0(X)$*;*
(3) $\mathcal{T} = \mathcal{S}$ *and ρ is uniformly equivalent to d on* $\mathcal{T}$*;*
(4) *Coincidence of pre-uniform structures holds:* $\Delta_{\mathcal{S},d} = \Delta_{\mathcal{T},\rho}$.

Corollary 3.15. *Let d and ρ be compatible metrics for a metrizable topological space X. Let $\mathcal{T}$ be an ideal consisting of sets all whose closures are compact, and let $\mathcal{S}$ be a second ideal. Then $(\mathcal{S}, \rho)$-convergence ensures $(\mathcal{T}, d)$-convergence if and only if $\mathcal{T} \subseteq \mathcal{S}$.*

Proof. Each continuous function on X is strongly uniformly continuous on $\mathcal{T}$ no matter what compatible metric is used for the domain. In particular, this applies to $\mathrm{id} : \langle X, \rho\rangle \to \langle X, d\rangle$. □

Corollary 3.16. *Let $\mathcal{T}$ be an ideal in a metric space $\langle X, \rho\rangle$ that is stable under small enlargements. Let d be an equivalent metric and let $\mathcal{S}$ be a second ideal. Then $(\mathcal{S}, \rho)$-convergence ensures $(\mathcal{T}, d)$-convergence if and only if $\mathcal{T} \subseteq \mathcal{S}$ and* $\mathrm{id} : \langle X, \rho\rangle \to \langle X, d\rangle$ *restricted to each member of $\mathcal{T}$ is uniformly continuous.*

Proof. If $\mathcal{T}$ is stable under small enlargements, then a function is strongly uniformly continuous on $\mathcal{T}$ if and only if its restriction to each member of $\mathcal{T}$ is uniformly continuous. □

Example 3.17. We present a bornology $\mathcal{T}$ on a metrizable space X and two compatible metrics d and ρ such that $\mathrm{id} : \langle X, \rho\rangle \to \langle X, d\rangle$ restricted to each member of $\mathcal{T}$ is uniformly continuous, yet ρ is not uniformly stronger than d restricted to $\mathcal{T}$. In the plane, let

$$X = \{(x, 0) : x \in \mathbb{R}\} \cup \{(n, \frac{1}{n}) : n \in \mathbb{N}\},$$

and let $\mathcal{T}$ be the bornology on X consisting of all sets of the form $A \cup B$ where $A \in \mathcal{F}(X)$ and $B \subseteq \{(x, 0) : x \in \mathbb{R}\}$. Let ρ be the usual metric for the plane, and let d be the equivalent metric on X defined by

$$d(x, w) = \begin{cases} \rho(x, w) & \text{if } x \text{ and } w \text{ are both on the horizontal axis} \\ 0 & \text{if } x = w \\ 1 & \text{otherwise.} \end{cases}$$

Clearly, $\mathrm{id} : \langle X, \rho\rangle \to \langle X, d\rangle$ fails to be strongly uniformly continuous on $\{(x, 0) : x \in \mathbb{R}\}$. On the other hand let $T = A \cup B \in \mathcal{T}$ be arbitrary where $A = \{(n_j, \frac{1}{n_j}) : j = 1, 2, \ldots, k\}$ and B lies in the horizontal axis. Given $\varepsilon > 0$, if $(t_1, t_2) \in T$ and if $\rho(t_1, t_2) < \min\{\varepsilon, (n_1 + n_2 + \cdots + n_k)^{-1}\}$, then $d(t_1, t_2) < \varepsilon$.

We call a bornology $\mathcal{S}$ on a metric space $\langle X, d\rangle$ a *metric bornology* if $\mathcal{S} = \mathcal{B}_\rho(X)$ for some equivalent metric ρ. By a celebrated theorem of S.-T. Hu [17], this occurs if and only if both of the following conditions hold: (1) $\mathcal{S}$ has a countable base, and (2) $\forall S \in \mathcal{S}\ \exists T \in \mathcal{S}$ with $\mathrm{cl}\,(S) \subseteq \mathrm{int}\,(T)$. It can be shown that we can find a metric ρ uniformly equivalent to d for which $\mathcal{S} = \mathcal{B}_\rho(X)$ if and only if $\exists \delta > 0$ such that for each $S \in \mathcal{S}$, $S^\delta = \{x : d(x, S) < \delta\} \in \mathcal{S}$ [8]. This means that for the initial metric d, $\mathcal{S}$ is stable under small enlargements uniformly over $\mathcal{S}$.

We intend to show using Corollary 3.14 that if $\mathcal{S}$ is a bornology and $(\mathcal{S}, d)$-convergence on $\mathcal{P}(X)$ is pseudometrizable, then it must be Attouch-Wets convergence with respect to a remetrization. Now by Theorems 3.1 and 3.8, such a bornology $\mathcal{S}$ already satisfies the conditions of Hu's Theorem as condition (2) obviously holds if $\mathcal{S}$ is stable under small enlargements. While there is no way in general to produce a metric ρ that is uniformly equivalent to d knowing just that S is stable under small enlargements, Corollary 3.14 fortunately says that something substantially less than that is required. Hu's original construction [17, pp. 312–313] is not adequate to the task, as we need to construct uniformly continuous Urysohn functions along the way.

Theorem 3.18. *Let $\mathcal{S}$ be a bornology in a metric space $\langle X, d\rangle$. The following conditions are equivalent:*

(*1*) $(\mathcal{S}, d)$*-convergence is compatible with a pseudometrizable topology;*
(*2*) $\mathcal{S}$ *is stable under small enlargements and has a countable base;*
(*3*) *there exists an equivalent metric ρ for X such that $(\mathcal{S}, d)$-convergence on $\mathcal{P}(X)$ is Attouch-Wets convergence with respect to ρ;*
(*4*) *there exists an equivalent metric ρ for X such that $\Delta_{\mathcal{S},d} = \Delta_{\mathcal{B}_\rho(X),\rho}$ on $\mathcal{P}(X)$;*
(*5*) *there exists an equivalent metric ρ for X such that $A = (\mathcal{S}, d)$-lim A_λ for nets of nonempty sets if and only if the associated net of distance functionals $\rho(\cdot, A_\lambda)$ converges to $\rho(\cdot, A)$ on ρ-bounded sets.*

Proof. Theorem 3.8 gives the equivalence of conditions (*1*) and (*2*), and Corollary 3.14 gives the equivalence of conditions (*3*) and (*4*). Equivalence of Attouch-Wets convergence with uniform convergence of distance functions for nets in $\mathcal{C}_0(X)$ is a classical result [5, 4, 2, 7], and the proof is easily adapted to general nets of nonempty sets. Thus, conditions (*3*) and (*5*) are equivalent by Corollary 3.14. Since Attouch-Wets convergence is pseudometrizable, (*3*) ensures (*1*). Thus only (*2*) $\Rightarrow$ (*3*) requires proof.

If $\mathcal{S} = \mathcal{P}_0(X)$, then $(\mathcal{S}, d)$-convergence is convergence in Hausdorff distance H_d which is $(\mathcal{B}_\rho(X), \rho)$-convergence where ρ is the uniformly equivalent bounded metric defined by $\rho = \min\{d, 1\}$. Otherwise, for each $S \in \mathcal{S}$, $X\backslash S$ is nonempty. Let $\langle S_n\rangle$ be an increasing sequence of open sets that is cofinal in $\mathcal{S}$. Set $A_1 = S_1$ and choose $\varepsilon_1 > 0$ with $A_1^{\varepsilon_1} \in \mathcal{S}$. Now let $A_2 = S_2 \cup A_1^{\varepsilon_1} \in \mathcal{S}$. Having produced open $A_1, A_2, \ldots, A_n$ in $\mathcal{S}$, choose $\varepsilon_n > 0$ with $A_n^{\varepsilon_n} \in \mathcal{S}$ and set $A_{n+1} = S_{n+1} \cup A_n^{\varepsilon_n}$. The construction produces a new sequence with these properties:

(1) $\forall n \in \mathbb{N},\ A_n$ is open;
(2) $\forall n \in \mathbb{N},\ A_n^{\varepsilon_n} \subseteq A_{n+1}$;
(3) $\forall n \in \mathbb{N},\ X\backslash A_n \neq \emptyset$;
(4) $\{A_n : n \in \mathbb{N}\}$ is cofinal in $\mathcal{S}$.

For each $n \in \mathbb{N}$ define $f_n : X \to \mathbb{R}$ by

$$f_n(x) = \min \{1, \frac{1}{\varepsilon_n} d(x, A_n)\}.$$

Each f_n is d-uniformly continuous, $f_n(x) = 0$ if $x \in A_n$, and by property (2) above $f_n(x) = 1$ if $x \notin A_{n+1}$. Since $\mathcal{S}$ is a bornology, condition (4) says that $\{A_n : n \in \mathbb{N}\}$ is a cover of X, and we may assert

(5) $\forall x \in X\ \exists \delta > 0\ \exists n_0 \in \mathbb{N}$ such that $d(x, w) < \delta \Rightarrow \forall n \geqslant n_0\ f_n(w) = 0.$

By local finiteness, $f := f_1 + f_2 + f_3 + \cdots$ defines a finite-valued nonnegative continuous function. Note also that if $C \subseteq X$, then $f|C$ is bounded if and only if $C \subseteq A_n$ for some n because (i) if $x \in A_n$, then $f(x) \leq n - 1$, and (ii) if $x \notin A_{n+1}$, then $f(x) \geq n$. Next define an equivalent metric ρ on X by

$$\rho(x, w) = \min \{d(x, w), 1\} + |f(x) - f(w)|.$$

By condition (4), $f|C$ is bounded if and only if $C \in \mathcal{S}$ which obviously gives $\mathcal{B}_\rho(X) = \mathcal{S}$. To show that d and ρ are uniformly equivalent with respect to $\mathcal{S}$, it suffices by Lemma 3.11 to show that f is strongly uniformly continuous on $\mathcal{S}$. To see this, fix $S \in \mathcal{S}$ and choose n with $S \subseteq A_n$. Now f restricted to the ε_n-enlargement of S is uniformly continuous as $f = f_1 + f_2 + f_2 + \cdots + f_n$ so restricted. Thus f is strongly uniformly continuous on $\mathcal{S}$, and in view of Corollary 3.14, the proof is complete. □

Corollary 3.19. *Let $\mathcal{S}$ be a bornology in a complete metric space $\langle X, d\rangle$ with a countable base that is stable under small enlargments. Then $(\mathcal{S}, d)$-convergence on $\mathcal{C}_0(X)$ is completely metrizable.*

Proof. The metric ρ in the proof of Theorem 3.18 satisfies $d \leq \rho$, and so ρ is a complete metric. But the Attouch-Wets topology for $\mathcal{C}_0(X)$ as determined by a complete metric is itself completely metrizable [2, 7]. □

Example 3.20. In the plane $\mathbb{R}^2$ let $\mathcal{S}$ be the bornology having as a countable base all strips of the form $\mathbb{R} \times [-n, n]$ where $n \in \mathbb{N}$. Evidently, $\mathcal{S}$ is stable under arbitrary enlargements, not just small ones, and a uniformly equivalent metric for which $\mathcal{S}$-convergence is Attouch-Wets convergence is given by

$$\rho((x_1, y_1), (x_2, y_2)) = \min\{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}, 1\} + ||y_2| - |y_1||.$$

Now it is not hard to show that if $A = \{(0, 1)\}$ and $A_n = \{(0, 1), (n, \log\ n)\}$, then $A = (\mathcal{S}, d) - \lim A_n$. On the other hand, $\langle d(\cdot, A_n)\rangle$ fails to converge uniformly to $d(\cdot, A)$ on $\mathbb{R} \times [-1, 1]$, because for each n,

$$d((n, 1), (0, 1)) = n \ \text{ while } d((n, 1), (n, \log\ n)) = \log\ n - 1.$$

This shows that uniform convergence of distance functionals for nets of sets on a metric bornology need not be preserved by a uniformly equivalent remetrization.

While necessary and sufficient conditions for remetrizations that preserve uniform convergence of distance functionals on finite subsets for nets of sets have been worked out by Costantini, Levi and Zieminska [12], not much is known in the general case. Uniform convergence of distance functionals on finite subsets for nets of sets is nothing more than pointwise convergence and is known as *Wijsman convergence* in the literature [7]. On the other hand, necessary and sufficient conditions for $\mathcal{S}$-convergence to coincide with this stronger form of convergence have recently been presented in [10], where an example is given showing that this can happen outside the setting of Attouch-Wets convergence.

Example 3.21. Consider the open interval $X = (-1, 1)$ equipped with the usual metric d of $\mathbb{R}$ and with the bornology $\mathcal{K}(X)$. Then $\rho(x, w) = |\frac{x}{1-|x|} - \frac{w}{1-|w|}|$ provides a remetrization of X for which $\mathcal{K}(X)$-convergence is Attouch-Wets convergence. More generally, any locally compact separable metric space $\langle X, d\rangle$ admits a metric ρ for which closed and bounded sets are compact [15], and for any such metric $(\mathcal{K}(X), d)$-convergence is $(\mathcal{B}_\rho(X), \rho)$-convergence (see Corollary 3.15 *supra*).

By Corollary 3.15 if $\langle X, d\rangle$ is a metric space and $\mathcal{S} \subseteq \mathcal{K}(X)$ is an ideal, then each remetrization leaves $\mathcal{S}$-convergence invariant. Our final result of the paper shows that if $\mathcal{S}$ is any other kind of ideal, then we can get many different $\mathcal{S}$-convergences by varying metrics. Our proof calls on an often useful theorem of Hausdorff [15, p. 369]: each metric defined on a closed subset of a metrizable space X that is compatible with the relative topology is extendable to an element of $\mathfrak{D}_X$.

Theorem 3.22. *Let X be a metrizable space with compatible metrics $\mathfrak{D}_X$ and let $\mathcal{S}$ be an ideal of subsets of X that contains some set S_0 whose closure is noncompact. Then there exists an uncountable subset $\mathfrak{T}$ of $\mathfrak{D}_X$ whose members ρ yield distinct $(\mathcal{S}, \rho)$-convergences in $\mathcal{C}_0(X)$ and thus in $\mathcal{P}(X)$.*

Proof. We produce an uncountable family of compatible metrics $\{\rho_\alpha : \alpha \in \Omega\}$ such that (S, ρ_α)-convergence differs from (S, ρ_β)-convergence whenever α, β are distinct indices in Ω. Since $\mathrm{cl}(S_0)$ is noncompact, there is a sequence

$$x_1, y_1, x_2, y_2, x_3, y_3, \ldots$$

in S_0 with no cluster point in X. Since $\{x_n : n \in \mathbb{N}\} \cup \{y_n : n \in \mathbb{N}\}$ is a closed discrete set, by Hausdorff's Theorem on the extension of metrics $\exists d \in \mathfrak{D}_X$ such that for all positive integers $n \neq k$,

$$d(x_n, x_k) = |\frac{1}{n} - \frac{1}{k}| \quad \text{and} \quad d(x_n, y_n) = \frac{1}{n}.$$

Let $\Sigma = \{E : E \subseteq \mathbb{N}$ and both E and $\mathbb{N}\backslash E$ are infinite$\}$, and define an equivalence relation $\equiv$ on Σ as follows:

$$E_1 \equiv E_2 \text{ provided their symmetric difference is a finite set.}$$

Let $\{E_\alpha : \alpha \in \Omega\}$ consist of one representative from each equivalence class, so that the index set Ω is uncountable. By the Tietze Extension Theorem, for each $\alpha \in \Omega$, let $f_\alpha : X \to [0, 1]$ be a continuous functions with these three properties:

(1) $\forall \alpha \in \Omega\ \forall n \in \mathbb{N},\ f_\alpha(x_n) = 0;$
(2) $\forall \alpha \in \Omega\ \forall n \in E_\alpha,\ f_\alpha(y_n) = 0;$
(3) $\forall \alpha \in \Omega\ \forall n \notin E_\alpha,\ f_\alpha(y_n) = 1.$

For each $\alpha \in \Omega$ define $\rho_\alpha \in \mathfrak{D}_X$ by the formula

$$\rho_\alpha(x, w) = d(x, w) + |f_\alpha(x) - f_\alpha(w)|.$$

Now let α, β be distinct indices in Ω; without loss of generality, we may assume $E_\alpha \backslash E_\beta$ is infinite. We produce a sequence of nonempty closed sets that is (S, ρ_α)-convergent but not (S, ρ_β)-convergent to $A = \{x_j : j \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, let A_n be defined by

$$A_n := \{y_j : j \in E_\alpha \text{ and } j \geq n\} \cup A.$$

The sequence $\langle A_n \rangle$ is actually convergent to A in ρ_α-Hausdorff distance, for if $j \geq n$ and $j \in E_\alpha$, we have

$$\rho_\alpha(y_j, A) \leq \rho_\alpha(y_j, x_j) = d(y_j, x_j) + |f_\alpha(y_j) - f_\alpha(x_j)| = \frac{1}{j} + 0 \leq \frac{1}{n},$$

so that $H_{\rho_\alpha}(A_n, A) \leq \frac{1}{n}$. But with respect to the metric ρ_β, we claim that $\forall n \in \mathbb{N}$, we have $A_n \cap S_0 \nsubseteq A^{\frac{1}{2}} = \{x : \rho_\beta(x, A) < \frac{1}{2}\}$. To see this, take $j > n$ with $j \in E_\alpha \backslash E_\beta$. Then $y_j \in A_n \cap S_0$, but

$$\rho_\beta(y_j, A) \geq \inf_{n \in \mathbb{N}} |f_\beta(y_j) - f_\beta(x_n)| = 1.$$

This shows that $y_j \notin A^{\frac{1}{2}}$ with respect to ρ_β, and so (S, ρ_β)-convergence fails. $\square$

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