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The Minimal Gap Between $\Lambda_2(\Omega)$ and $\Lambda_\infty(\Omega)$ in a Class of Convex Domains

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Dedicated to our friend Thomas Lachand-Robert. We will keep with us forever the memory of a great man and a great mathematician.

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We consider the minimization problem

$$\min_{\Omega \in X} (\Lambda_2 - \Lambda_\infty)(\Omega),$$

where $\Lambda_2(\Omega)$ and $\Lambda_\infty(\Omega)$ are the (square root of the) first eigenvalue of the Laplacian and the first eigenvalue of the $\infty-$Laplacian respectively. X is the class of convex domains with prescribed diameter. We prove existence of a solution, and we provide several geometrical properties of minimizers.

1. Introduction

Let $\mathcal{G}$ be a shape functional which is monotone decreasing with respect to the set inclusion. Consider the minimization problem

$$\inf\{\mathcal{G}(\Omega) : \Omega \text{ convex},\ d(\Omega) = c\} \tag{1}$$

where $d(\Omega)$ stands for the diameter of the set Ω. Suppose that this problem is well posed. In this case, a direct consequence of the geometrical characterization of constant width sets provided in [11] in Theorem 3.4 and the monotonicity of $\mathcal{G}$ is the fact that any optimal shape of (1) is of constant width.

When the set functional is not monotone, the picture may be completely different. One very illuminating example is the famous shape optimization problem

$$\inf\{(\mu_2 - \lambda_2)(\Omega) : \Omega \text{ convex},\ d(\Omega) = c\} \tag{2}$$

where λ_2 and μ_2 are respectively the first and the second eigenvalue of the Laplacian operator Δ with zero Dirichlet data on the boundary. Even in this basic situation the existence of a minimizer for problem (2) is not known. Actually, problem (2) is suspected to be ill-posed and the sequence of rectangles

$$\Omega_\varepsilon = (0, \sqrt{c^2 - \varepsilon^2}) \times (0, \varepsilon)$$

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may be one of the minimizing sequences of this problem (see [6] for more details on extremum problems related to the eigenvalues of the Laplacian).

Recently a lot of attention has been devoted to the infinity-Laplacian operator Δ_∞ which may be defined as the limit when $p \to \infty$ of the $p-$Laplacian operator Δ_p (see [1], [2] and references therein). In opposition to the classical Laplacian, very few is known on the eigenvalues of the $p-$Laplacian and the $\infty-$Laplacian operators [12, 9, 3, 17].

In this paper we study the spectral gap in between Δ and Δ_∞. More precisely we study the extremal problem

$$\inf\{(\Lambda_2 - \Lambda_\infty)(\Omega) : \Omega \text{ convex}, \ d(\Omega) = c\} \tag{3}$$

where $\Lambda_2 = \sqrt{\lambda_2}$, and Λ_∞ is the first eigenvalue of Δ_∞ with zero Dirichlet boundary conditions (see Section 2 for the definition of the first eigenvalue for the nonlinear operator Δ_∞). The choice of the admissible class is dictated by several necessary conditions to have existence: see Section 3.

Let us mention that the link between the first eigenvalue of the Laplacian and the inradius R_∞ (recall the geometrical characterization of $\Lambda_\infty : \Lambda_\infty(\Omega) = 1/R_\infty(\Omega)$, see Section 2) has been previously studied by several authors. As an example, let us reformulate an interesting problem originally settled by Polya (see [8, 15]):

$$\min\left\{\frac{\Lambda_2(\Omega)}{\Lambda_\infty(\Omega)} : \Omega \in X\right\}$$

where X is a suitable class of convex bounded domains. Hersch [8] showed that the infimum ot the functional Λ_2/Λ_∞ is greater or equal than $\pi/2$, when Ω varies among plane convex domains. Protter [15] extended the Hersch's result to bounded convex domains in $\mathbb{R}^n$.

The plan of the paper is as follows: in Section 2 we recall some definitions and known result about Λ_2 and Λ_∞; in Section 3 we prove the existence of a minimum for the functional $\mathcal{F}$ in the admissible class of convex set with prescribed diameter; in Section 4 we obtain several geometrical properties of minimizers; in Section 5 we deduce some analytical first order necessary conditions.

2. Some facts about the infinity-Laplacian operator

Let Ω be an open bounded domain in $\mathbb{R}^N$. The first eigenvalue $\lambda_p(\Omega)$ of the nonlinear operator $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, the $p-$Laplacian, is defined in the following way [12]

$$\lambda_p(\Omega) := \min\left\{\|\nabla u\|_p^p : u \in W_0^{1,p}(\Omega), \ \|u\|_p = 1\right\} \tag{4}$$

for every $1 < p < \infty$. When $p = 2$, (4) reduces to the first eigenvalue $\lambda_2(\Omega)$ of the Laplacian operator. We set, for every $1 < p < \infty$,

$$\Lambda_p := (\lambda_p)^{1/p}. \tag{5}$$

Let us observe that $\Lambda_p(\mu\Omega) = \Lambda_p(\Omega)/\mu$ holds for every $\mu > 0$. It has been shown in [9, 3] that

$$\Lambda_p \to \Lambda_\infty, \quad p \to \infty$$

where

$$\Lambda_\infty(\Omega) := \min\left\{\|\nabla u\|_\infty : u \in W_0^{1,\infty}(\Omega),\ \|u\|_\infty = 1\right\}. \tag{6}$$

The number Λ_∞ introduced in (6) is called (see [12]) the first eigenvalue of the operator

$$\Delta_\infty u = < D^2 u \cdot \nabla u,\, \nabla u >$$

where $D^2 u$ is the Hessian of u.The operator Δ_∞ is the so-called $\infty-$Laplacian: see [2] and the references therein. The differential equation satisfied by a first eigenfunction of Δ_∞ is

$$\begin{cases} \max\{\Lambda_\infty u(x) - |Du(x)|,\, \Delta_\infty u(x)\} = 0 & x \in \Omega \\ u = 0 & x \in \partial\Omega. \end{cases} \tag{7}$$

Equation (7), which have to be intended in the sense of viscosity solutions (see [5]), is obtained in [9] as the limit for $p \to \infty$ of the $p-$eigenvalue equations

$$-\Delta_p u = \lambda_p |u|^{p-2} u, \quad x \in \Omega. \tag{8}$$

Remark that equation (7) can also be obtained as the Euler-Lagrange equation of (6): see [3]. Uniqueness for the solution of (7) is known only for some particular domains: see [3, 17]. A quite remarkable property of the first egenvalue Λ_∞ of Δ_∞ is its geometrical caracterization (see [9, 12]):

$$\frac{1}{\Lambda_\infty(\Omega)} = \max\left\{R : B(x_0, R) \subseteq \Omega\right\}. \tag{9}$$

3. An existence result

In this section we will consider the minimization problem

$$\min_X \mathcal{F}(\Omega) \tag{10}$$

where $\mathcal{F}(\Omega) := \Lambda_2(\Omega) - \Lambda_\infty(\Omega)$ and X stands for some class of admissible domains $\Omega \subseteq I\!R^2$.

In the following, we use the fact that our functional is homogeneous of order -1, namely

$$\mathcal{F}(\lambda\Omega) = \frac{1}{\lambda}\mathcal{F}(\Omega), \quad \forall \lambda > 0. \tag{11}$$

As a preliminary question let us look at the sign of the functional (10). We obtain the following result:

Theorem 3.1 (see [15]). *Let Ω be a plane convex domain. Then*

$$\lambda_2(\Omega) \geq \frac{\pi^2}{4R_\infty^2(\Omega)}, \tag{12}$$

where λ_2 is the first eigenvalue of the Laplacian and R_∞ is the radius of the maximal inscribed ball in Ω.

Theorem 3.1 implies that, for every convex domain Ω,

$$\Lambda_2(\Omega) - \Lambda_\infty(\Omega) > \Lambda_2(\Omega) - \frac{\pi}{2}\Lambda_\infty(\Omega) \geq 0, \tag{13}$$

which implies that our functional $\mathcal{F}$, restricted to this class of domains, is positive.

Corollary 3.2. *Let $\mathcal{C} = \{\Omega \subseteq I\!R^2 : \Omega$ convex and bounded$\}$. Then,*

$$\inf_{\Omega\in\mathcal{C}} (\Lambda_2 - \Lambda_\infty)(\Omega) = 0.$$

Proof. Theorem 3.1 tells us that $(\Lambda_2 - \Lambda_\infty)(\Omega) > 0$ for every convex set Ω. By the homogeneity of degree -1 we have

$$(\Lambda_2 - \Lambda_\infty)(t\Omega) = \frac{1}{t}(\Lambda_2 - \Lambda_\infty)(\Omega) \to 0, \quad t \to \infty.$$

□

Now, let us go back to the well-posedness of the problem (10). Taking into account Corollary 3.2, some additional constraints must be imposed to the admissible class X in order to have existence of a minimizer of problem (10). First we must exclude the rescaling which provides the ill-posedness of Corollary 3.2. Additionally, the number of holes of the admissible sets must be bounded from above. This kind of hypothesis is also necessary: removing a point doesn't change Λ_2 (points have capacity 0 in $I\!R^N$ for $N > 1$), but this operation would reduce dramatically the radius $1/\Lambda_\infty$ of the largest inscribed ball.

In the following we consider the class of convex domain with fixed diameter, namely

$$X_d = \left\{\Omega \subseteq I\!R^2 : \Omega \text{ convex}, \ d(\Omega) = 1\right\}, \tag{14}$$

where $d(\Omega)$ stands for the diameter of Ω, i.e. $d(\Omega) = \sup_{x,y\in\Omega} d(x,y)$. This class obviously satisfies the above requirements.

Notice that the constraint on the diameter may be relaxed to $d(\Omega) \leq 1$. By the homogeneity of $\mathcal{F}$, if $d(\Omega) < 1$ we have

$$\mathcal{F}\left(\frac{1}{d(\Omega)}\Omega\right) = d(\Omega)\mathcal{F}(\Omega) < \mathcal{F}(\Omega).$$

For this class of admissible sets, we have the following existence result.

Theorem 3.3. *The minimum problem*

$$\min_{\Omega\in X_d} \mathcal{F}(\Omega) \tag{15}$$

admits a solution.

Proof. Let (Ω_n) be a minimizing sequence, i.e. $\mathcal{F}(\Omega_n) \to \inf_{X_d} \mathcal{F}(\Omega)$. We can assume that the sequence (Ω_n) converges with respect to the Hausdorff complementary topology [7, 4]. Indeed, since the diameter is bounded, we can suppose that the sets (Ω_n) are all contained in a fixed compact set (the functional is translation invariant) which provides the existence of a convergent sub-sequence. Let Ω^* be the limit set. It is straightforward to establish that Ω^* is convex with diameter less or equal to 1.

We claim that $|\Omega^*| \neq 0$ (where $|\cdot|$ stands for the Lebesgue measure in $I\!R^2$). Suppose by contradiction that there exists a sub-sequence (still indexed by n) (Ω_n) such that the radius (ϵ_n) of the maximal inscribed balls B_{ϵ_n} converges to 0.

From [16], for all convex set $\Omega \subset I\!R^2$, we have the following inequality:

$$\sqrt{3}(w - 2r)d \leq 2wr$$

where d is the diameter of Ω, r its inradius, and w stands for its minimal width. In our situation, we get

$$w_n \leq \frac{2\sqrt{3}}{\sqrt{3} - 2\varepsilon_n}\varepsilon_n.$$

Then, we can enclose the whole set Ω_n in a rectangle R_n of length's sides respectively equal to 1 and $\gamma_n := \frac{2\sqrt{3}}{\sqrt{3}-2\varepsilon_n}\varepsilon_n$, so $d(R_n) = \sqrt{1+\gamma_n^2}$. By the previous inequality, we have

$$\begin{aligned}
\mathcal{F}(\Omega_n) &= \Lambda_2(\Omega_n) - \Lambda_\infty(B_{\epsilon_n}) \\
&\geq \Lambda_2(R_n) - \frac{1}{\epsilon_n} \\
&\geq \left[\pi\epsilon_n\sqrt{1 + \frac{1}{\gamma_n^2}} - 1\right]\frac{1}{\epsilon_n} \\
&= a_n
\end{aligned} \tag{16}$$

where $a_n \to +\infty$. The previous computations shows that $|\Omega^*| \neq 0$. The continuity of $\mathcal{F}$ with respect to the Hausdorff complementary convergence ends the proof. □

4. Geometrical properties of optimal domains

In this section we prove some geometrical properties of the minimizers of problem (15). We will make use of the following definition:

Definition 4.1. Let Ω be a subset of $I\!R^2$. A point $x \in \Omega$ is called diametrical if there exists $y \in \Omega$ such that $d(\Omega) = d(x, y)$.

Let us give whitout proof the following proposition.

Proposition 4.2. *Let Ω be a convex set with $d(\Omega) = 1$ and let x be a diametrical point. Then there exists $y \in \partial\Omega$ such that $\|x - y\| = 1$ and*

$$\nu(x) = -\nu(y) = \frac{y - x}{\|y - x\|}, \tag{17}$$

with $\nu(x)$ in the normal cone of x and $\nu(y)$ in the normal cone of y.

In the next theorem we establish that non diametrical points on the boundary of an optimal set are always inside flat parts.

Theorem 4.3. *Let $P \in \partial\Omega^*$ be a non diametrical point. Then there exists a non empty segment $S \subset \partial\Omega^*$ containing P.*

Proof. In the following B_∞ denotes a ball contained in Ω^* having maximal radius (notice that such a ball is not necessarily unique). We consider the two following cases:

1) $P \in \partial\Omega^*$ and P is not on the boundary of a ball of maximal radius,
2) It exists a ball B_∞ of maximal radius such that $P \in \partial\Omega^* \cap \partial B_\infty$.

In case 1), suppose by contradiction that the non diametrical point P does not belong to a segment of $\partial\Omega^*$. Let $\varepsilon > 0$ and consider a point $Q \in B(P,\varepsilon) \setminus \Omega^*$. We define the convex envelope $\widehat{\Omega^*} := \mathrm{co}\,(\Omega^* \cup Q)$. If ε is small enough, we have that $\widehat{\Omega^*}$ satisfies the following properties:

- $\widehat{\Omega^*}$ is convex,
- by continuity of the distance function, $\widehat{\Omega^*}$ has diameter less or equal to 1,
- $\Lambda_\infty(\Omega^*) = \Lambda_\infty(\widehat{\Omega^*})$ since the radius of maximal inscribed balls does not change,
- $\Lambda_2(\Omega^*) > \Lambda_2(\widehat{\Omega^*})$, since $\Omega^* \subset \widehat{\Omega^*}$.

It follows the contradiction $\mathcal{F}(\Omega^*) > \mathcal{F}(\widehat{\Omega^*})$ and then the thesis.

In case 2), the strategy is similar. Still by contradiction, suppose P belongs in the interior of a non-flat portion of $\partial\Omega^*$. Consider the tangent line to $\partial\Omega^*$ in P (which exists because in this point $\partial\Omega^*$ is internally tangent to B_∞). Then fix two neighboring points $Q_0, Q_1 \neq P$ on this tangent line such that $P \in (Q_0, Q_1)$. As done in case 1), we construct $\widetilde{\Omega^*} = \mathrm{co}(\Omega^* \cup Q_0 \cup Q_1)$. With a proper choice of Q_0, Q_1 we have that the new set $\widetilde{\Omega^*}$ is convex, has diameter less or equal than 1 and $\Omega^* \subset \widetilde{\Omega^*}$. Let us show that the inradius r is not changed. Again, we have two situations: first if Ω^* is contained in the strip defined by the line through (Q_0, Q_1) and its parallel at distance $2r$, then $\widetilde{\Omega^*}$ is still contained in the same strip and by construction has the same inradius. Second, if it exists a triangle T containing Ω^* whose sides are tangent to B_∞, then $\widetilde{\Omega^*} \subset T$ and the inradius is again unchanged. □

If the boundary of Ω^* does not contain flat part, the previous theorem would implies that all the points on $\partial\Omega^*$ are diametrical. As a consequence, Ω^* would be of constant width (see [11]). Theorem 4.6 will show that the boundary of every minimizer must contain at least 2 flat parts.

The statement of the previous theorem can be reversed in the following way:

Theorem 4.4. *Let $P \in \partial\Omega^*$. Suppose P is contained in the interior of a flat neighborhood of $\partial\Omega^*$, then P is non diametrical.*

Proof. It is a direct consequence of Pythagora's theorem: suppose by contradiction that the diametrical point P belongs to the interior of a segment $I = \mathrm{co}(A \cup B) \subset \partial\Omega^*$. Let $\overline{P} \in \partial\Omega^*$ be such that $\|P - \overline{P}\| = 1$. It is straightforward to prove that $\max(|A - \overline{P}|, |B - \overline{P}|) > 1$ which contradicts $\Omega^* \in X_d$. □

We are now able to give some additional informations related to the boundary of an optimal convex set.

Theorem 4.5. *Let $P \in \partial\Omega^*$. Then one and only one of the following eventualities holds:*

a) *P is not a diametrical point and belongs to the interior of a segment of $\partial\Omega^*$,*

b) *P is diametrical and is not in interior of a segment of $\partial\Omega^*$,*

c) *P is located on the extremities of two different segments of $\partial\Omega^*$.*

Proof. It is an immediate consequence of Theorem 4.3 that points which are not in the interior of a segment of $\partial\Omega^*$ are all diametrical points. Additionally, Theorem 4.4 shows that points located in the interior of a segment $\partial\Omega^*$ are all non diametrical. We still have to establish what may happen to a point P which is a vertex of a segment of $\partial\Omega^*$. If we are not in the situation c), it exists a connected part of $\partial\Omega^*$ which is not a segment which contains P . Since all the points of that part are diametrical, by continuity of the distance function, P is also a diametrical point. Thus P is in the situation b). □

Now we state and prove one of the main step in the description of $\partial\Omega^*$.

Theorem 4.6. *Suppose $P_0 \in \partial\Omega^* \cap \partial B_\infty$. Then, there exists a neighborhood U_{P_0} of P_0 such that $\partial\Omega^* \cap U_{P_0}$ is a segment.*

Remark 4.7. At a first guess, a "cutted Reuleaux triangle" (see Figure 4.1) seems to be a natural candidate for optimality. We will see later, as a consequence of Lemma 4.13, that this is not true.

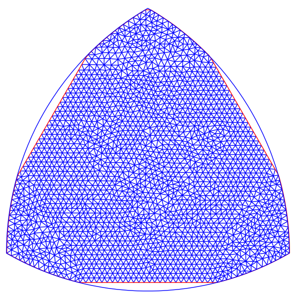


Figure 4.1: A cutted Reuleaux triangle

Proof of Theorem 4.6. Suppose that the thesis is not true, then in a neighborhood of the point $P_0 = (x_0, y_0) \in \partial\Omega^* \cap \partial B_\infty$ the boundary of the minimizer $\partial\Omega^*$ is not a segment. As a consequence of Theorem 4.3, P_0 is a diametrical point. Our claim is the following: there exists a domain variation in the direction of the normal to the boundary in (x_0, y_0) that decreases $\mathcal{F}(\Omega^*)$. This would contradict the minimality of Ω^*.

Let us construct such a domain variation. First we study the functional derivative with respect to the boundary variation of a special family of vector fields (V_ϵ). Then we will establish that among this family, it exists at least one vector field which decreases the cost function.

Fix the origin in the center of B_∞, and suppose the $y-$axis directed as $-\nu(x_0, y_0)$ (the same direction of the normal, but with reversed sign). For all $\epsilon > 0$ consider the vector field

$$V_\epsilon(x, y) := (0, y_0 + \epsilon - y), \quad (x, y) \in I\!R^2$$

and for every $t > 0$, the set

$$\Omega_t^\epsilon := \Omega^* + tV_\epsilon(\Omega^*).$$

For ϵ small enough, it exists a compactly supported vector field equal to V_ϵ in a neighborhood of P_0 such that Ω_t^ϵ is still convex for t small. We still denote by V_ϵ such a vector field. A direct computation gives the following estimate

$$|d(\Omega^*) - d(\Omega_t^\epsilon)| \le kt\epsilon$$

for some constant $k > 0$ independent of t and ϵ. This implies the asymptotic development of Λ_∞

$$\Lambda_\infty(\Omega_t^\epsilon) = \Lambda_\infty(\Omega^*) + ct\epsilon + o(t) \tag{18}$$

holds for some constant $c > 0$ independent of t and ϵ. Following [6] pp. 38, we have to compute the first derivative of the first eigenvalue of the laplacian $\lambda_2(\Omega^*)$ with respect to the domain variation induced by V_ϵ:

$$\left(\frac{d}{dt}\lambda_2(\Omega_t^\epsilon)\right)_{t=0} = -\int_{\partial\Omega^*} |\nabla u_2|^2 V_\epsilon \cdot \nu \, d\sigma, \tag{19}$$

where u_2 is the first eigenfunction of the laplacian. We have that

$$\left|\int_{\partial\Omega^*} |\nabla u_2|^2 V_\epsilon \cdot \nu \, d\sigma\right| \le C\epsilon f(\epsilon), \tag{20}$$

where C is a constant independent of ε and $f(\epsilon)$ is the length of the part of the boundary changed by V_ϵ, that is $(\Omega^* \backslash \Omega_\epsilon) \cap \partial\Omega^*$.

A crucial point is that $f(\epsilon) \to 0$ as $\epsilon \to 0$. This is a direct consequence of the following facts:

- P_0 is a diametrical point;
- the following inclusion holds (we set $D := (x_0, y_0) - \nu(x_0, y_0)$)

$$(\Omega^* \backslash \Omega_\epsilon) \cap \partial\Omega^* \subset (B(D, 1) \setminus B_\infty).$$

Then, the following asymptotic development

$$\begin{aligned} \Lambda_2(\Omega_t^\epsilon) &= \Lambda_2(\Omega^*) + \left[\frac{1}{2}(\lambda_2(\Omega^*))^{-1/2}\left(\frac{d}{dt}\lambda_2(\Omega_t^\epsilon)\right)_{t=0}\right] t + o(t) \\ &\le \Lambda_2(\Omega^*) + C'\epsilon f(\epsilon)t + o(t). \end{aligned} \tag{21}$$

holds for some constant C' independent of t and ϵ. Collecting the developments (18) and (21), we can choose ϵ small enough and then t small such that

$$\begin{aligned} \mathcal{F}(\Omega_t^\epsilon) &= \Lambda_2(\Omega_t^\epsilon) - \Lambda_\infty(\Omega_t^\epsilon) \\ &\le \Lambda_2(\Omega^*) + C'\epsilon f(\epsilon)t - \Lambda_\infty(\Omega^*) - ct\epsilon + o(t) \\ &= \mathcal{F}(\Omega^*) + \epsilon(C'f(\epsilon) - c)t + o(t) \\ &< \mathcal{F}(\Omega^*) \end{aligned} \tag{22}$$

for a suitable choice of t, that is our claim, and so the thesis follows. □

A maximal inscribed ball B_∞ has at least 2 points of contact with $\partial\Omega^*$. The following theorem proves that the number of contact points must be at most 3.

Theorem 4.8. *A ball of maximal radius B_∞ has at most 3 distinct points of contact with $\partial\Omega^*$.*

Proof. Theorem 4.6 implies that $\partial\Omega^*$ is flat around every contact point with B_∞. Suppose there exists just one maximal inscribed ball B_∞ (if not, the number of contact points is obviously 2) with more than 3 points of contact. Consider 4 distinct points of contact. Among those 4 points, we select 3 of them such that their convex envelope contains the center of B_∞. The segment (AB) of the boundary of $\partial\Omega^*$ containing the fourth point $\overline{x}$ is made of non diametrical points (with possibly the exception of A and B). Then there exists a point ω in the exterior of Ω^* - sufficiently close to (AB) - such that $\|\omega - x\| \leq 1$ for every $x \in \Omega^*$. Set $\widehat{\Omega^*} = \mathrm{co}(\omega \cup \Omega^*)$. This set is still admissible because it is convex and has diameter less or equal than 1.

By construction $\Omega^* \subset \widehat{\Omega^*}$ and the maximal inscribed ball B_∞ is unchanged. We have $\Lambda_\infty(\Omega^*) = \Lambda_\infty(\widehat{\Omega^*})$ and $\Lambda_2(\Omega^*) > \Lambda_2(\widehat{\Omega^*})$ which implies that $\mathcal{F}(\Omega^*) > \mathcal{F}(\widehat{\Omega^*})$, a contradiction. □

Theorem 4.9. *Let Ω^* be a minimizer. Suppose that A and B are the extrema of a segment $(AB) \subset \partial\Omega^*$, then A and B must be singular points.*

Proof. We prove the result for one extremal point A. We have three cases.

- A is non diametrical: by c) of Theorem 4.5, this point must be vertex of two segment. Then A is singular.
- A is diametrical and vertex of two non-parallel segments, then we conclude as before.
- A is diametrical and is an extremum of a non-flat region S of Ω. The tangent to S in A must not contains (AB): if not, let $y \in \partial\Omega^*$ be such that $d(A, y) = 1$. There exists $x \in (AB)$ such that $d(x, y) > 1$ (because $(A - y) \cdot (A - B) = 0$), which is clearly a contradiction. This implies that $\partial\Omega^*$ is not differentiable in A, and this concludes the proof. □

Theorem 4.10. *Let Ω^* be a minimizer. Then, the boundary of Ω^*, contains at most 3 flat parts.*

Proof. As a direct consequence of Theorems 4.6 and 4.8, given a ball of maximal radius B_∞ there exist at most 3 flat parts of the boundary touching it. The result will be proved if we show that every flat part of the boundary must be tangent to B_∞.
Suppose by contradiction that there exists a segment $(A, B) \subset \partial\Omega^*$ such that $(A, B) \cap \partial B_\infty = \emptyset$. Theorem 4.9 implies that A and B are both singular points. Also, by Theorem 4.4, there are not diametrical points in the interior of the segment (A, B). Now we argue as in the proof of Theorem 4.3, case 1). Let $P = (A + B)/2$, fix $\varepsilon > 0$ and consider a point $Q \in B(P, \varepsilon) \setminus \Omega^*$. We define the convex envelope $\widehat{\Omega^*} := \mathrm{co}\,(\Omega^* \cup Q)$. If ε is small enough, we have that $\widehat{\Omega^*}$ satisfies the following properties:

- $\widehat{\Omega^*}$ is convex,
- by continuity of the distance function, $\widehat{\Omega^*}$ has diameter less or equal to 1,
- $\Lambda_\infty(\Omega^*) = \Lambda_\infty(\widehat{\Omega^*})$ since the radius of maximal inscribed balls does not change,

- $\Lambda_2(\Omega^*) > \Lambda_2(\widehat{\Omega^*})$, since $\Omega^* \subset \widehat{\Omega^*}$.

We have the contradiction $\mathcal{F}(\Omega^*) > \mathcal{F}(\widehat{\Omega^*})$, and then the thesis follows. □

As a direct consequence of Theorems 4.10 and 4.9, every minimizer has at least 4 and at most 6 singular points on the boundary.

We will distinguish in the following subsections optimality conditions depending on the number of contact points in between the boundary of the optimal set and of its maximal inscribed balls.

4.1. The maximal inscribed ball is not unique

Under this property, we describe completely the shape of the minimizer, see figure 4.2.

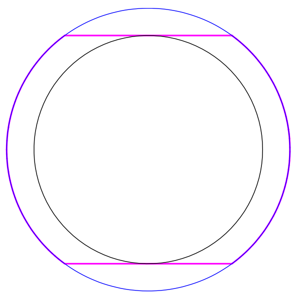


Figure 4.2: Optimal set when the maximal inscribed ball is not unique

Theorem 4.11. *Suppose that an optimal set Ω^* has many maximal inscribed balls. Then Ω^* is the intersection of a disk of diameter 1 and a strip which is symmetric with respect to the center of the disk.*

Proof. Let $x_1 \neq x_2 \in \Omega^*$. If $B(x_i, R) \subset \Omega^*$ for $i = 1, 2$, where $R = \Lambda_\infty^{-1}$ is the maximal inscribed radius, then by convexity $\mathrm{co}\,(B(x_1, R) \cup B(x_2, R)) \subset \Omega^*$. This implies that there exists at least 2 flat parts on the boundary of Ω^*. As in the proof of Theorem 4.8, we can show that the number of flat parts is exactly 2.

Let us suppose that (modulo a rotation and a translation of Ω^*) the x-axis is in between and at equal distance from the flat parts. Considering the Steiner symmetrization [14, 10] of Ω^* with respect to the x-axis, we get that Ω^* is symmetric w.r.t. this axis. In fact, Steiner symmetrization decreases Λ_2 and, in this case, preserves the radius of a maximal inscribed ball.

Analogously, Steiner symmetrization w.r.t. the $y-$axis establishes that Ω^* is also symmetric w.r.t. the $y-$axis.
It remains to show that $S = \partial\Omega^* \setminus ((AB) \cup (CD))$, where (AB) and (CD) are the two parallel segments on $\partial\Omega^*$, is the union of 2 symmetric ars of a unit circle. Consider a regular point $P \in S$ and $\widetilde{P}$ its symmetric w.r.t. the origin O. Since P and $\widetilde{P}$ are diametricals and regular, by Proposition 4.2 we have that $\|P - \widetilde{P}\| = 1$. Consequently $\|P - O\| = 1/2$ for every regular point of S. By density, we obtain that this property holds for every point on S. □

4.2. The maximal inscribed ball is unique

In this situation we provide some properties of a minimizer. The first one is an obvious consequence of Theorem 4.8.

Proposition 4.12. *Suppose that an optimal set Ω^* has a unique maximal inscribed ball. Then the boundary of Ω^* contains exactly 3 flat parts.*

Proof. Theorem 4.10 tell us that there exists at most 3 flat regions in the boundary of Ω^*. If the optimal set Ω^* has just one maximal inscribed ball B_∞, then its number of contact points with $\partial\Omega^*$ must be at least 3. But Theorem 4.8 tells us that this number is at most 3. The thesis now follows from Theorem 4.6 because every contact point must be contained in a flat region of $\partial\Omega^*$. □

Lemma 4.13. *The boundary $\partial\Omega^*$ of an optimal set can not contain both an arc of radius 1 and its center.*

Proof. Let C be one hypothetic arc of circle and P_C its center. The idea of the proof is to construct a domain variation which "push inside" the small portion $\partial\Omega^*$ where the center P_C lies and "push outside" the corresponding arc.

Modulo a rotation and a translation, we assume that P_C and the middle of the arc C are both on the x-axis with coordinates $(-1/2, 0)$ and $(1/2, 0)$ respectively. For every $\epsilon > 0$ we consider $P_\epsilon = (1/2 + \epsilon, 0)$. We define

$$\Omega_\epsilon = \text{co}\,(\Omega^* \cup P_\epsilon) \cap B(P_\epsilon, 1).$$

It is straightforward to prove that for ϵ sufficiently small, the following facts holds

- $\Lambda_\infty(\Omega_\epsilon) = \Lambda_\infty(\Omega^*)$,
- $d(\Omega_\epsilon) \leq 1$,
- $|\partial\Omega^* \setminus \Omega_\epsilon|$ is of order ϵ and $|\partial\Omega_\epsilon \setminus \Omega^*|$ is of order $\sqrt{\epsilon}$.

As in the proof of Theorem 4.6, we can show that it exists a positive c such that

$$\Lambda_2(\Omega_\epsilon) = \Lambda_2(\Omega^*) - c\epsilon^{3/2} + o(\epsilon^{3/2}).$$

The previous development contradicts the minimality of Ω^*. □

As a consequence of Lemma 4.13, a "cutted Reuleaux triangle" is never optimal.

Theorem 4.14. *Suppose that an optimal set Ω^* has a unique maximal inscribed ball. Then Ω^* has at most one axis of symmetry.*

Proof. Suppose, ex absurdum, that there are at least 2 axes of symmetry. Since the maximal inscribed ball is unique, by Theorem 4.12 we have that $B_\infty \cap \partial\Omega^* = \{P_1, P_2, P_3\}$, each one lying inside a flat region. Since the number of flat parts is exactly 3, each axis of symmetry must cross one and only one point of contact. As a consequence also the triangle (P_1, P_2, P_3) has 2 axes of symmetry. This implies the existence of a third axis of symmetry for this triangle. As in the proof of Theorem 4.11, Steiner symmetrization gives that Ω^* is symmetric w.r.t. this new axis. The axes of symmetry are then 3. Geometrical considerations implies that the optimal shape must be a "cutted Reuleaux triangle" (see Figure 4.1), which is not admissible as optimal set because of Lemma 4.13. This contradiction proves the theorem. □

5. First order optimality conditions when the maximal inscribed ball is unique

We already described in Section 4.1 the minimizer's shape if the maximal inscribed ball is not unique. Then, in the present section, we concentrate our attention in deriving first order optimality conditions only when the maximal inscribed ball is unique.

We first prove that any local variation of a flat part of the boundary of optimal sets do not provide more information than linear perturbations. More precisely, for all sets Ω, for all vector fields V and for every $t > 0$ let us define

$$\Omega_V(t) = co(\{x + tV(x) : x \in \Omega\}) \bigcap \left\{ \bigcap_{x\in\Omega} B(x,1) \right\}. \tag{23}$$

Consider a flat part of the boundary of an optimal set S. Up to a rotation and a translation we can assume that $S = [0,1] \times \{0\}$ and that Ω^* is below the x-axis. Let $V_\phi = \phi(x)\nu(x)\chi_S(x)$ be defined for all concave functions $\phi > 0$ on the interior of the segment S, ν is the outward normal vector and χ_S is the usual characteristic function of S. If Ω^* is optimal for problem (15) and the functional $\mathcal{F}$ differentiable in the direction of V_ϕ, we must have

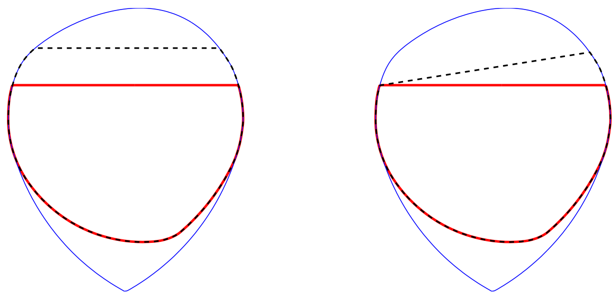
$$\left.\frac{d\mathcal{F}(\Omega_{V_\phi}(t))}{dt}\right|_{t=0^+} \geq 0 \tag{24}$$

for all such vector field V_ϕ.

We define

$$\begin{cases} V_1(x) = \phi_1(x)\nu(x)\chi_S(x), & \phi_1(x) = 1 \quad \forall x \in S \\ V_2(x) = \phi_2(x)\nu(x)\chi_S(x), & \phi_2(x) = x \quad \forall x \in S. \end{cases} \tag{25}$$

The actions of the vector fields V_1 and V_2 on a flat part of Ω are drawn in Figure 5.1(a) and 5.1(b) respectively.



(a) The action of V_1 (b) The action of V_1

Figure 5.1: Linear vector fields

Theorem 5.1. *The set of one side optimality conditions* (24)*, where ϕ is a concave function defined on a flat part S, is equivalent to the* 4 *conditions* (24) *for $V = \pm V_1$ and $V = \pm V_2$.*

Proof. Let $V_\phi(x) = \phi(x)\nu(x)\chi_S(x)$ be defined on the flat part of the boundary $S = [0,1] \times \{0\}$ (modulo a rotation and a translation) of a mimimizer Ω^*, where $\phi \neq \phi_i$, $i = 1,2$. Let us consider $\Omega^*_{V_\phi}(t)$ as defined in (23). For t small enough, it exists at least one point P_t of contact in between the maximal inscribed ball $B_\infty(t)$ and $\partial\Omega^*_{V_\phi}(t)$ along the curve $V_\phi(S)$.

Consider the tangent line to $\Omega^*_{V_\phi}(t)$ in P_t, and construct the new domain $\Omega^*_{\widetilde{V_\phi}}(t) = \Omega^*_{V_{\varphi_t}}$, where $\varphi_t(x)$ is the equation of the tangent line in P_t to $\partial\Omega^*_{V_\phi}(t)$. We have $\Omega^*_{V_\phi}(t) \subset \Omega^*_{V_{\varphi_t}}$. Moreover, since the maximal inscribed ball in Ω^* is unique, we can assume that $\Lambda_\infty\left(\Omega^*_{V_\phi}(t)\right) = \Lambda_\infty\left(\Omega^*_{V_{\varphi_t}}\right)$ with a suitable choice of the point P_t. This imply, for every $t > 0$,

$$\frac{\mathcal{F}\left(\Omega^*_{V_\phi}(t)\right) - \mathcal{F}(\Omega^*)}{t} \geq \frac{\mathcal{F}\left(\Omega^*_{V_{\varphi_t}}(t)\right) - \mathcal{F}(\Omega^*)}{t}.$$

Now for every $t > 0$ there exists $(a(t), b(t))$ such that $\varphi_t(x) = a(t)\phi_1(x) + b(t)\phi_2(x)$. Bolzano-Weierstrass theorem implies that, there exists at least a cluster point (a^*, b^*) of $(a(t), b(t))$. Eventually passing to a subsequence we have that $(a(t), b(t)) \to (a^*, b^*)$ as $t \to 0^+$. Then

$$\left.\frac{d\mathcal{F}(\Omega_{V_\phi}(t))}{dt}\right|_{t=0^+} \geq \left.\frac{d\mathcal{F}(\Omega_{t(a^*\phi_1 + b^*\phi_2)})}{dt}\right|_{t=0^+}$$

which is our claim. □

By Theorem 4.12 there exists only 3 flat regions in $\partial\Omega^*$. Let us call them l_a, l_b and l_c. Assume (ABC) be the triangle enveloping Ω^* containing the three flat regions l_a, l_b and l_c. Suppose that $l_a \subset (BC)$, $l_b \subset (CA)$ and $l_c \subset (AB)$. We set

$$a = |(BC)|, \qquad b = |(CA)|, \qquad c = |(AB)|,$$

$$\widehat{A} = \widehat{BAC}, \qquad \widehat{B} = \widehat{ABC} \quad \text{and} \quad \widehat{C} = \widehat{BCA}.$$

Finally, let $p = a+b+c$ be the perimeter ot the triangle and R_∞ the radius of the maximal inscribed ball B_∞ (which is, by construction, the incircle of the triangle ABC). In this case, the uniqueness of the maximal inscribed ball provides differentiability of $\mathcal{F}(\Omega^*_{V_i}(t))$, $i = 1, 2$, in $t = 0$.

Now we can deduce the first order optimality conditions.

Theorem 5.2. *Suppose Ω^* is a minimizer with a unique maximal inscribed ball B_∞. Then the following relations hold (where V_1, V_2 are defined in* (25)*):*

$$\frac{1}{2}\left(\lambda_2(\Omega^*)\right)^{-1/2} \int_{l_a} |\nabla u_2|^2 V_1 \cdot \nu \, d\sigma = \frac{a}{pR_\infty^2}; \tag{26}$$

$$\frac{1}{2}\left(\lambda_2(\Omega^*)\right)^{-1/2} \int_{l_b} |\nabla u_2|^2 V_1 \cdot \nu \, d\sigma = \frac{b}{pR_\infty^2}; \tag{27}$$

$$\frac{1}{2}\left(\lambda_2(\Omega^*)\right)^{-1/2} \int_{l_c} |\nabla u_2|^2 V_1 \cdot \nu \, d\sigma = \frac{c}{pR_\infty^2}; \tag{28}$$

$$\frac{1}{2}\left(\lambda_2(\Omega^*)\right)^{-1/2} \int_{l_a} |\nabla u_2|^2 V_2 \cdot \nu \, d\sigma = \frac{a}{R_\infty \sin\widehat{C}}\left(\frac{1+\cos\widehat{C}}{p} - \frac{1}{b}\right); \tag{29}$$

$$\frac{1}{2}\left(\lambda_2(\Omega^*)\right)^{-1/2}\int_{l_b}|\nabla u_2|^2 V_2\cdot\nu\,d\sigma = \frac{b}{R_\infty \sin\widehat{C}}\left(\frac{1+\cos\widehat{C}}{p}-\frac{1}{a}\right); \tag{30}$$

$$\frac{1}{2}\left(\lambda_2(\Omega^*)\right)^{-1/2}\int_{l_c}|\nabla u_2|^2 V_2\cdot\nu\,d\sigma = \frac{c}{R_\infty \sin\widehat{B}}\left(\frac{1+\cos\widehat{B}}{p}-\frac{1}{a}\right). \tag{31}$$

Proof. Formulas (26)–(31) are obtained observing that

$$0 = \left.\frac{d\mathcal{F}(\Omega_{V_i}(t))}{dt}\right|_{t=0} = \left.\frac{d\Lambda_2(\Omega_{V_i}(t))}{dt}\right|_{t=0} - \left.\frac{d\Lambda_\infty(\Omega_{V_i}(t))}{dt}\right|_{t=0}, \quad i=1,2. \tag{32}$$

together with elementary geometrical computations. Equalities 32 are obtained applying Theorem 5.1 on all the flat parts of Ω^*. □

6. Concluding remarks

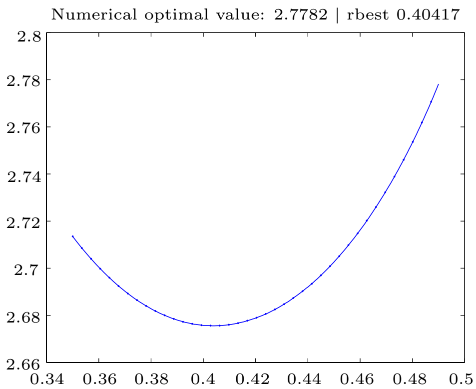


Figure 6.1: Numerical evaluations of the shape functional w.r.t. the width ($d(\Omega) = 1$)

Together with the analysis done in the previous sections, we carried out some numerical experiments computing our cost function for standard convex sets (ellipses, stadium, polygons,...). The eigenvalue was computed by a standard $P1$ finite element method and the inradius was estimated using the *The Convex Geometry toolbox* (see [13]). Those numerical results let us think that the shape of the Figure 4.2 may be the optimal one.

In this case we provide numerical evaluations of the shape functional $\mathcal{F}$ w.r.t. the width of the strip defined by the two flat parts in Figure 6.1.

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