Journal of Convex Analysis
Volume 15 (2008), No. 4, 707–718

The Schur Geometrical Convexity of the Extended Mean Values*

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Received: September 7, 2007
Revised manuscript received: December 23, 2007

In this paper, we prove that the extended mean values $E(r, s; x, y)$ are Schur geometrically convex (or concave, respectively) with respect to $(x, y) \in (0, \infty) \times (0, \infty)$ if and only if $s + r \geq 0$ (or $s + r \leq 0$, respectively).

Keywords: Extended mean value, Schur convex, Schur concave, Schur geometrically convex, Schur geometrically concave

2000 Mathematics Subject Classification: 26B25

1. Introduction

In 1975, the extended mean values $E(r, s; x, y)$ were defined by K. B. Stolarsky [1] as follows

$$E(r, s; x, y) = \left(\frac{r}{s} \cdot \frac{y^s - x^s}{y^r - x^r}\right)^{\frac{1}{s-r}}, \quad rs(r-s)(x-y) \neq 0, \tag{1}$$

$$E(r, 0; x, y) = \left(\frac{1}{r} \cdot \frac{y^r - x^r}{\log y - \log x}\right)^{\frac{1}{r}}, \quad r(x-y) \neq 0, \tag{2}$$

$$E(r, r; x, y) = \frac{1}{e^{\frac{1}{r}}} \left(\frac{x^{x^r}}{y^{y^r}}\right)^{\frac{1}{x^r - y^r}}, \quad r(x-y) \neq 0, \tag{3}$$

$$E(0, 0; x, y) = \sqrt{xy}, \quad x \neq y, \tag{4}$$

$$E(r, s; x, y) = x, \quad x = y. \tag{5}$$

Here $x, y > 0$ and $r, s \in R$.

*The research is partly supported by 973 Project of China under grant 2006CB708304, N. S. Foundation of China under grant 10771195 and N. S. Foundation of the Zhejiang Province under grant Y607128.

ISSN 0944-6532 / $ 2.50

It is easy to see that the extended mean values $E(r,s;x,y)$ are continuous on the domain $\{(r,s;x,y) \mid r,s \in R; x,y > 0\}$. They are of symmetry between r and s and between x and y, many basic properties have been obtained by E. B. Leach and M. C. Sholander in [2]. Many mean values are special cases of E, for example,

$$\begin{cases} E(r,2r;x,y) = M_r(x,y), & \text{(power mean or Hölder mean)} \\ E(1,p;x,y) = S_p(x,y), & \text{(extended logarithmic mean)} \\ E(1,1;x,y) = I(x,y), & \text{(identric or exponential mean)} \\ E(p,p;x,y) = I_p(x,y), & \text{(extended identric or exponential mean)} \\ E(1,2;x,y) = A(x,y), & \text{(arithmetic mean)} \\ E(0,0;x,y) = G(x,y), & \text{(geometric mean)} \\ E(-2,-1;x,y) = H(x,y), & \text{(harmonic mean)} \\ E(0,1;x,y) = L(x,y), & \text{(logarithmic mean)} \\ E(r,r+1;x,y) = F_r(x,y). & \text{(one-parameter mean)} \end{cases} \tag{6}$$

Study of $E(r,s;x,y)$ is not only interesting but also important, because most of the two-variables mean values are special cases of $E(r,s;x,y)$ and it is challenging to study a function whose formulation is so indeterminate [3].

For convenience of readers, we recall the notations and definitions as follows.

For $x = (x_1, x_2) \in (0,\infty) \times (0,\infty)$ and $\alpha \geq 0$, we denote by

$$\log x = (\log x_1, \log x_2)$$

and

$$x^\alpha = (x_1^\alpha, x_2^\alpha).$$

For $x = (x_1, x_2)$, $y = (y_1, y_2) \in R^2$, we denote by

$$xy = (x_1 y_1, x_2 y_2)$$

and

$$e^x = (e^{x_1}, e^{x_2}).$$

Definition 1.1. A set $E_1 \subseteq R^2$ is called a convex set if $\frac{x+y}{2} \in E_1$ whenever $x, y \in E_1$. A set $E_2 \subseteq (0,\infty) \times (0,\infty)$ is called a geometrically convex set if $x^{\frac{1}{2}} y^{\frac{1}{2}} \in E_2$ whenever $x, y \in E_2$.

It is easy to see that $E \subseteq (0,\infty) \times (0,\infty)$ is a geometrically convex set if and only if $\log E = \{\log x, x \in E\}$ is a convex set, and $F \subseteq R^2$ is a convex set if and only if $e^F = \{e^x : x \in F\}$ is a geometrically convex set.

Definition 1.2. Let $E \subseteq R^2$ be a convex set. A function $f : E \to R$ is said to be a convex function on E if $f(\frac{x+y}{2}) \leq \frac{f(x)+f(y)}{2}$ for all $x, y \in E$. Moreover, f is called a concave function if $-f$ is a convex function.

Definition 1.3. Let $E \subseteq (0,\infty) \times (0,\infty)$ be a geometrically convex set. A function $f : E \to (0,\infty)$ is called a geometrically convex function on E if $f(x^{\frac{1}{2}} y^{\frac{1}{2}}) \leq f^{\frac{1}{2}}(x) f^{\frac{1}{2}}(y)$ for all $x, y \in E$. We say f is a geometrically concave function if $\frac{1}{f}$ is a geometrically convex function.

Definitions 1.2 and 1.3 have the following consequences.

Fact 1.4. *If* $E_1 \subseteq (0,\infty) \times (0,\infty)$ *is a geometrically convex set and* $f : E_1 \to (0,\infty)$ *is a geometrically convex function, then*

$$F(x) = \log f(e^x) : \log E_1 \to R$$

is a convex function. Conversely, if E_2 *is a convex set and* $F : E_2 \to R$ *is a convex function, then*

$$f(x) = e^{F(\log x)} : e^{E_2} \to (0,\infty)$$

is a geometrically convex function.

Definition 1.5. Let $E \subseteq R^2$ be a set. A function $F : E \to R$ is called a Schur convex function on E if

$$F(x_1, x_2) \le F(y_1, y_2)$$

for each two-tuples $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in E, such that $x \prec y$, i.e.

$$x_{[1]} \le y_{[1]}$$

and

$$x_{[1]} + x_{[2]} = y_{[1]} + y_{[2]},$$

where $x_{[i]}$ denotes the *ith* largest component in x. A function F is called a Schur concave function if $-F$ is a Schur convex function.

Definition 1.6. Let $E \subseteq (0,\infty) \times (0,\infty)$ be a set. A function $F : E \to (0,\infty)$ is called Schur geometrically convex function on E if

$$F(x_1, x_2) \le F(y_1, y_2)$$

for each pair $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in E, such that $\log x \prec \log y$, i.e.

$$x_{[1]} = y_{[1]} \quad \text{and} \quad x_{[1]}x_{[2]} = y_{[1]}y_{[2]}.$$

F is called a Schur geometrically concave function if $\frac{1}{F}$ is a Schur geometrically convex function.

Definitions 1.5 and 1.6 have the following consequences.

Fact 1.7. *Let* $E \subseteq (0,\infty) \times (0,\infty)$ *be a set and* $H = \log E = \{\log x : x \in E\}$. *Then* $f : E \to (0,\infty)$ *is a Schur geometrically convex (or concave, respectively) function on E if and only if* $\log f(e^x)$ *is a Schur convex (or concave, respectively) function on* H.

The following well-known result was proved by A. W. Marshall and I. Olkin in [4].

Theorem 1.8. *Let* $E \subseteq R^2$ *be a symmetric convex set with nonempty interior* $\operatorname{int} E$ *and* $\varphi : E \to R$ *be a continuous symmetric function on* E. *If* φ *is differentiable on* $\operatorname{int} E$, *then* φ *is Schur convex (or concave, respectively) on* E *if and only if*

$$(y - x)\left(\frac{\partial \varphi}{\partial y} - \frac{\partial \varphi}{\partial x}\right) \ge 0 \quad (\text{or} \le 0, \text{ respectively})$$

for all $(x, y) \in \operatorname{int} E$.

The following theorem can easily be derived from Fact 1.7 and Theorem 1.8.

Theorem 1.9. *Let* $E \subseteq (0,\infty)\times(0,\infty)$ *be a symmetric geometrically convex set with nonempty interior* $\operatorname{int} E$ *and* $\varphi : E \to (0,\infty)$ *be a continuous symmetric function on* E. *If* φ *is differentiable on* $\operatorname{int} E$, *then* φ *is Schur geometrically convex (or concave, respectively) on* E *if and only if*

$$(\log y - \log x)\left(y\frac{\partial\varphi}{\partial y} - x\frac{\partial\varphi}{\partial x}\right) \geq 0 \quad (\text{or} \leq 0, \text{ respectively})$$

for all $(x,y) \in \operatorname{int} E$.

The theory of convex functions and Schur convex functions is one of the most important theory in the fields of modern analysis and geometry. It can be used in global Riemannian geometry [5, 6], operator inequalities [7], nonlinear PDE of elliptic type [8], combinatorial optimization [9], isoperimetric problem for polytopes [10], linear regression [11], graphs and metrices [12], improperly posed problems [13], inequalities and extremal problems [14], nilpotent groups [15], global surface theory [16] and other related fields.

The notion of geometrical convexity is introduced by P. Montel [17], in a beautiful paper where the analogues of the notion of convex function in n variables are discussed. Once upon a time, the theory of geometrical convexity seemed to be hidden, which is a pity because of its richness. Recently, C. P. Niculescu [18] discussed an attractive class of inequalities, which arise from the notion of geometrically convex functions.

Recently, the Schur convexity of the extended mean values $E(r,s;x,y)$ with respect to (r,s) and (x,y) are investigated in [19–21]. F. Qi [19] first obtained the following result.

Theorem 1.10. *For fixed* $(x,y) \in (0,\infty)\times(0,\infty)$ *with* $x \neq y$, *the extended mean values* $E(r,s;x,y)$ *are Schur concave on* $[0,\infty)\times[0,\infty)$ *and Schur convex on* $(-\infty,0]\times(-\infty,0]$ *with respect to* (r,s).

In [20], F. Qi, J. Sándor, S. S. Dragomir and A. Sofo tried to obtain the Schur convexity of the extended mean values $E(r,s;x,y)$ with respect to (x,y) for fixed (r,s) and declared an incorrect conclusion as follows. For given (r,s) with $r,s \notin (0,\frac{3}{2})$ (or $r,s \in (0,1]$, respectively), the extended mean values $E(r,s;x,y)$ are Schur concave (or Schur convex, respectively) with respect to (x,y) on $(0,\infty)\times(0,\infty)$. H. N. Shi, S. H. Wu and F. Qi [21] observed that the above conclusion is wrong and obtained the following theorem.

Theorem 1.11. *For fixed* $(r,s) \in R^2$,

(*1*) *if* $2 < 2r < s$ *or* $2 \leq 2s \leq r$, *then the extended mean values* $E(r,s;x,y)$ *are Schur convex with respect to* $(x,y) \in (0,\infty)\times(0,\infty)$;

(*2*) *if* $(r,s) \in \{r < s \leq 2r, 0 < r \leq 1\} \cup \{s < r \leq 2s, 0 < s \leq 1\} \cup \{0 < s < r \leq 1\} \cup \{0 < r < s \leq 1\} \cup \{s \leq 2r < 0\} \cup \{r \leq 2s < 0\}$, *then the extended mean values* $E(r,s;x,y)$ *are Schur concave with respect to* $(x,y) \in (0,\infty)\times(0,\infty)$.

The main purpose of this article is to present the Schur geometrical convexity of the extended mean values $E(r,s;x,y)$ with respect to (x,y) for fixed (r,s). Our main result is the following.

Theorem 1.12. *For fixed* $(r,s) \in R^2$,

(*1*) *the extended mean values of* $E(r,s;x,y)$ *are Schur geometrically convex with respect to* $(x,y) \in (0,\infty) \times (0,\infty)$ *if and only if* $s+r \geq 0$;

(*2*) *the extended mean values of* $E(r,s;x,y)$ *are Schur geometrically concave with respect to* $(x,y) \in (0,\infty) \times (0,\infty)$ *if and only if* $s+r \leq 0$.

From Theorem 1.12 we get the following five immediate consequences.

Corollary 1.13. *For* $(r,s) \in R^2$ *and* $(x,y) \in (0,\infty) \times (0,\infty)$,

(*1*) $E(r,s;x,y) \geq \sqrt{xy}$ *if and only if* $s+r \geq 0$;

(*2*) $E(r,s;x,y) \leq \sqrt{xy}$ *if and only if* $s+r \leq 0$.

Corollary 1.14. *The extended logarithmic mean value* $E(1,p;x,y) = S_p(x,y)$ *is Schur geometrically convex with respect to* $(x,y) \in (0,\infty) \times (0,\infty)$ *if and only if* $p \geq -1$; *and Schur geometrically concave if and only if* $p \leq -1$.

Corollary 1.15. *The extended identric mean value* $E(p,p;x,y) = I_p(x,y)$ *is Schur geometrically convex with respect to* $(x,y) \in (0,\infty) \times (0,\infty)$ *if and only if* $p \geq 0$; *and Schur geometrically concave if and only if* $p \leq 0$.

Corollary 1.16. *The Hölder mean value* $E(r,2r;x,y) = M_r(x,y)$ *is Schur geometrically convex with respect to* $(x,y) \in (0,\infty)\times(0,\infty)$ *if and only if* $r \geq 0$; *and Schur geometrically concave if and only if* $r \leq 0$.

Corollary 1.17. *The one-parameter mean value* $E(r,r+1;x,y) = F_r(x,y)$ *is Schur geometrically convex with respect to* $(x,y) \in (0,\infty) \times (0,\infty)$ *if and only if* $r \geq -\frac{1}{2}$; *and Schur geometrically concave if and only if* $r \leq -\frac{1}{2}$.

2. Lemmas

In this section we introduce and establish several lemmas, which are used in the proof of Theorem 1.12.

Lemma 2.1. *Let* $s,r \in R, s \neq 0$ *and* $f(t) = \frac{r}{s}(s-r)(t^{s+r}-1) - \frac{r}{s}(s+r)(t^s - t^r)$. *Then the following statements hold.*

(*1*) *If* $s \geq r$ *and* $s+r \geq 0$, *then* $f(t) \geq 0$ *for* $t \in [1,\infty)$;

(*2*) *If* $s \geq r$ *and* $s+r \leq 0$, *then* $f(t) \leq 0$ *for* $t \in [1,\infty)$.

Proof. Let $g(t) = t^{1-r}f'(t)$, then simple computation yields

$$f(1) = 0, \tag{7}$$

$$f'(t) = \frac{r}{s}(s^2-r^2)t^{s+r-1} - r(s+r)t^{s-1} + \frac{r^2}{s}(s+r)t^{r-1}, \tag{8}$$

$$g(1) = f'(1) = 0 \tag{9}$$

and

$$g'(t) = (s-r)(s+r)(rt^r - r)t^{s-r-1}. \tag{10}$$

If $s \geq r$ and $s+r \geq 0$, then from (10) we see that

$$g'(t) \geq 0 \tag{11}$$

for $t \in [1, \infty)$. Then Lemma 2.1(*1*) follows from (7)-(9) and (11).

If $s \geq r$ and $s+r \leq 0$, then (10) again yields that

$$g'(t) \leq 0 \tag{12}$$

for $t \in [1, \infty)$. Then Lemma 2.1(*2*) follows from (7)-(9) and (12).

Lemma 2.2. *Let $s \in R$ and $f(t) = s(t^s+1)\log t - 2(t^s-2)$. Then the following conclusions are true.*

(*1*) *If $s \geq 0$, then $f(t) \geq 0$ for $t \in [1, \infty)$;*
(*2*) *If $s \leq 0$, then $f(t) \leq 0$ for $t \in [1, \infty)$.*

Proof. Let $g(t) = tf'(t)$, then simple computation yields

$$f(1) = 0, \tag{13}$$

$$g(1) = f'(1) = 0 \tag{14}$$

and

$$g'(t) = s^3 t^{s-1} \log t. \tag{15}$$

If $s \geq 0$, then clearly (15) gives that

$$g'(t) \geq 0 \tag{16}$$

for $t \in [1, \infty)$; and

$$g'(t) \leq 0 \tag{17}$$

for $t \in [1, \infty)$ if $s \leq 0$.

Thus, the proof follows from (13), (14), (16) and (17).

Lemma 2.3. *Let $t \in R$ and $f(t) = t^{2r} - 2rt^r \log t - 1$. Then the following statements are true:*

(*1*) *If $r \geq 0$, then $f(t) \geq 0$ for $t \in [1, \infty)$;*
(*2*) *If $r \leq 0$, then $f(t) \leq 0$ for $t \in [1, \infty)$.*

Proof. Let $g(t) = t^{1-r} f'(t)$, then simple computation yields

$$f(1) = 0, \tag{18}$$

$$g(1) = f'(1) = 0 \tag{19}$$

and

$$g'(t) = \frac{2r^2}{t}(t^r - 1). \tag{20}$$

If $r \geq 0$, from (20) we clearly see that

$$g'(t) \geq 0 \tag{21}$$

for $t \in [1, \infty)$; and

$$g'(t) \leq 0 \tag{22}$$

for $t \in [1, \infty)$ if $r \leq 0$.

Therefore, Lemma 2.3 follows from (18), (19), (21) and (22).

Lemma 2.4. *For fixed $r, s \in R$, $E(r, s; x, y)$ is differentiable with respect to $(x, y) \in (0, \infty) \times (0, \infty)$.*

Proof. By the elementary theory of differential and integral calculus we know that Lemma 2.4 is true once we prove that $\frac{\partial E(r,s;x,y)}{\partial x}$ and $\frac{\partial E(r,s;x,y)}{\partial y}$ are continuous with respect to $(x, y) \in (0, \infty) \times (0, \infty)$ for fixed $r, s \in R$. Since $E(r, s; x, y)$ is symmetric with respect to x and y, it suffices to prove that $\frac{\partial E(r,s;x,y)}{\partial x}$ is continuous with respect to $(x, y) \in (0, \infty) \times (0, \infty)$ for fixed $r, s \in R$.

If $x \neq y$, then the proof of the continuity for $\frac{\partial E(r,s;x,y)}{\partial x}$ is trivial by (1)-(4). Hence we have to prove that $\frac{\partial E(r,s;x,y)}{\partial x}$ is continuous on $\{(x, y) : x = y, x > 0\}$.

Firstly, we shall prove that there exists partial derivative $\frac{\partial E(r,s;x,y)}{\partial x}\Big|_{(r,s;x_0,x_0)}$ for all $x_0 > 0$ and for fixed $s, r \in R$. The proof is divided into five cases.

Case 1. $r = s = 0$. In this case, we see that

$$\lim_{x \to x_0} \frac{\sqrt{x_0 x} - x_0}{x - x_0} = \frac{1}{2}. \tag{23}$$

So, (4), (5) and (23) yield $\frac{\partial E}{\partial x}\big|_{(0,0;x_0,x_0)} = \frac{1}{2}$.

Case 2. $s = 0,\ r \neq 0$. Making use of L'Hospital's rule we get

$$\begin{aligned} &\lim_{x \to x_0} \frac{\left(\frac{1}{r}\frac{x_0^r - x^r}{\log x_0 - \log x}\right)^{\frac{1}{r}} - x_0}{x - x_0} \\ =\ & \frac{1}{r^2} \lim_{x \to x_0} \left[\left(\frac{1}{r}\frac{x_0^r - x^r}{\log x_0 - \log x}\right)^{\frac{1}{r}-1} \frac{-rx^{r-1}(\log x_0 - \log x) + \frac{1}{x}(x_0^r - x^r)}{(\log x_0 - \log x)^2} \right] \\ =\ & \frac{1}{r^2} \lim_{x \to x_0} \left(\frac{1}{r}\frac{-rx^{r-1}}{-\frac{1}{x}}\right)^{\frac{1}{r}-1} \lim_{x \to x_0} \frac{-rx^r(\log x_0 - \log x) + (x_0^r - x^r)}{x_0(\log x_0 - \log x)^2} \\ =\ & \frac{1}{r^2} x_0^{1-r} \lim_{x \to x_0} \frac{-r^2 x^{r-1}(\log x_0 - \log x)}{-\frac{2x_0}{x}(\log x_0 - \log x)} = \frac{1}{2}. \end{aligned} \tag{24}$$

Now, (2) and (5) together with (24) imply that $\frac{\partial E}{\partial x}\big|_{(r,0;x_0,x_0)} = \frac{1}{2}$.

Case 3. $r = 0,\ s \neq 0$. The symmetry of $E(r, s; x, y)$ with respect to (r, s) and Case *2* show that $\frac{\partial E}{\partial x}\big|_{(0,s;x_0,x_0)} = \frac{1}{2}$.

Case 4. $r = s \neq 0$. Using L'Hospital's rule we obtain

$$\begin{aligned}
&\lim_{x\to x_0} \frac{\frac{1}{e^{\frac{1}{r}}}\left(\frac{x^{x^r}}{x_0^{x_0^r}}\right)^{\frac{1}{x^r-x_0^r}} - x_0}{x - x_0} \\
&= \frac{1}{e^{\frac{1}{r}}} \lim_{x\to x_0}\left(\frac{x^{x^r}}{x_0^{x_0^r}}\right)^{\frac{1}{x^r-x_0^r}} \lim_{x\to x_0}\left(\frac{x^r\log x - x_0^r\log x_0}{x^r - x_0^r}\right)' \\
&= x_0 \lim_{x\to x_0}\frac{-rx_0^r x^{r-1}\log x + x^{2r-1} - x_0^r x^{r-1} + rx_0^r x^{r-1}\log x_0}{(x^r - x_0^r)^2} \\
&= \lim_{x\to x_0}\frac{-r(r-1)x_0^r\log x - (2r-1)x_0^r + (2r-1)x^r + r(r-1)x_0^r\log x_0}{2r(x^r - x_0^r)} \\
&= \lim_{x\to x_0}\frac{-r(r-1)x_0^r\frac{1}{x} + (2r-1)rx^{r-1}}{2r^2x^{r-1}} = \frac{1}{2}.
\end{aligned} \tag{25}$$

Then (3), (5) and (25) lead to $\frac{\partial E}{\partial x}\Big|_{(r,r;x_0,x_0)} = \frac{1}{2}$.

Case 5. $rs(r - s) \neq 0$. Using L'Hospital's rule we obtain

$$\begin{aligned}
&\lim_{x\to x_0}\frac{\left(\frac{r}{s}\frac{x_0^s - x^s}{x_0^r - x^r}\right)^{\frac{1}{s-r}} - x_0}{x - x_0} \\
&= \frac{r}{s(s-r)}\lim_{x\to x_0}\left(\frac{r}{s}\frac{x_0^s - x^s}{x_0^r - x^r}\right)^{\frac{1}{s-r}-1}\lim_{x\to x_0}\left(\frac{x_0^s - x^s}{x_0^r - x^r}\right)' \\
&= \frac{r}{s(s-r)}x_0^{1-s+r}\lim_{x\to x_0}\frac{-sx_0^r x^{s-1} + (s-r)x^{s+r-1} + rx_0^s x^{r-1}}{(x_0^r - x^r)^2} \\
&= \frac{rx_0^{1-s+r}}{s(s-r)}\lim_{x\to x_0}\frac{s(1-s)x_0^r x^{s-2} + [(s-r)(s+r-1)x^s + r(r-1)x_0^s]x^{r-2}}{-2rx^{r-1}(x_0^r - x^r)} \\
&= \frac{x_0^{2-s}}{2s(r-s)}\lim_{x\to x_0}\frac{s(1-s)x_0^r x^{s-2} + [(s-r)(s+r-1)x^s + r(r-1)x_0^s]x^{r-2}}{x_0^r - x^r} \\
&= \frac{-s(s-1)(s-2) + (s-r)(s+r-1)(s+r-2) + r(r-1)(r-2)}{2sr(s-r)} \\
&= \frac{1}{2}.
\end{aligned} \tag{26}$$

So, (1) and (5) together with (26) imply that $\frac{\partial E}{\partial x}\Big|_{(r,s;x_0,x_0)} = \frac{1}{2}$.

Next we shall prove that $\frac{\partial E(r,s;x,y)}{\partial x}$ is continuous with respect to $(x, y) \in (0,\infty) \times (0,\infty)$ for any fixed $(r, s) \in R \times R$. The proof is again divided into five cases.

Case I. $r = s = 0$. In this case, (4), (5) and (23) imply that

$$\frac{\partial E(0,0;x,y)}{\partial x} = \begin{cases} \frac{1}{2}\sqrt{\frac{y}{x}}, & x \neq y, \\ \frac{1}{2}, & x = y. \end{cases}$$

It is easy to see that $\lim_{\substack{x\to x_0\\ y\to x_0}} \frac{\partial E(0,0;x,y)}{\partial x} = \frac{1}{2}$ for any $x_0 > 0$, and hence $\frac{\partial E(0,0;x,y)}{\partial x}$ is continuous with respect to $(x, y) \in (0, \infty) \times (0, \infty)$.

Case II. $s = 0,\ r \neq 0$. In this case, (2) and (5) together with (24) give that

$$\frac{\partial E(r,0;x,y)}{\partial x} = \begin{cases} \frac{1}{r^2}\left(\frac{1}{r}\frac{y^r - x^r}{\log y - \log x}\right)^{\frac{1}{r}-1} \frac{-rx^r(\log y - \log x) + y^r - x^r}{x(\log y - \log x)^2}, & x \neq y, \\ \frac{1}{2}, & x = y. \end{cases}$$

Since

$$\begin{aligned} &\lim_{\substack{x\to x_0\\ y\to x_0}} \left[\frac{1}{r^2}\left(\frac{1}{r}\frac{y^r - x^r}{\log y - \log x}\right)^{\frac{1}{r}-1} \frac{-rx^r(\log y - \log x) + y^r - x^r}{x(\log y - \log x)^2}\right] \\ = \ & \frac{1}{r^2} x_0^{1-r} \lim_{\substack{x\to x_0\\ y\to x_0}} \frac{-r\log\frac{y}{x} + (\frac{y}{x})^r - 1}{(\log\frac{y}{x})^2} \lim_{\substack{x\to x_0\\ y\to x_0}} x^{r-1} \\ = \ & \frac{1}{r^2}\lim_{t\to 1} \frac{-r\log t + t^r - 1}{(\log t)^2} = \frac{1}{2} \end{aligned}$$

for any $x_0 > 0$, we see that $\frac{\partial E(r,0;x,y)}{\partial x}$ is continuous with respect to $(x, y) \in (0, \infty) \times (0, \infty)$.

Case III. $r = 0,\ s \neq 0$. The symmetry of $E(r, s; x, y)$ with respect to (r, s) and Case *II* show that $\frac{\partial E(0,s;x,y)}{\partial x}$ is continuous with respect to $(x, y) \in (0, \infty) \times (0, \infty)$.

Case IV. $r = s \neq 0$. In this case, (3), (5) and (25) yield

$$\frac{\partial E(r,r;x,y)}{\partial x} = \begin{cases} \frac{rx^{r-1}y^r(\log y - \log x) + x^{r-1}(x^r - y^r)}{(x^r - y^r)^2} E(r,r;x,y), & x \neq y, \\ \frac{1}{2}, & x = y. \end{cases}$$

Since

$$\begin{aligned} &\lim_{\substack{x\to x_0\\ y\to x_0}} \left[\frac{rx^{r-1}y^r(\log y - \log x) + x^{r-1}(x^r - y^r)}{(x^r - y^r)^2} E(r,r;x,y)\right] \\ = \ & \lim_{\substack{x\to x_0\\ y\to x_0}} E(r,r;x,y) \lim_{\substack{x\to x_0\\ y\to x_0}} \frac{1}{x} \lim_{\substack{x\to x_0\\ y\to x_0}} \frac{r(\frac{y}{x})^r\log\frac{y}{x} + [1 - (\frac{y}{x})^r]}{[1 - (\frac{y}{x})]^2} \\ = \ & \lim_{t\to 1} \frac{rt^r\log t + (1 - t^r)}{(1 - t^r)^2} = \frac{1}{2} \end{aligned}$$

for all $x_0 > 0$, we find that $\frac{\partial E(r,r;x,y)}{\partial x}$ is continuous with respect to $(x, y) \in (0, \infty) \times (0, \infty)$.

Case V. $rs(r - s) \neq 0$. In this case, (1), (5) and (26) imply that

$$\frac{\partial E(r,s;x,y)}{\partial x} = \begin{cases} \frac{E(r,s;x,y)}{s - r}\left(\frac{rx^{r-1}}{y^r - x^r} - \frac{sx^{s-1}}{y^s - x^s}\right), & x \neq y, \\ \frac{1}{2}, & x = y. \end{cases}$$

Now, for all $x_0 > 0$ we compute

$$\begin{aligned}
&\lim_{\substack{x\to x_0\\ y\to x_0}} \left[\frac{E(r,s;x,y)}{s-r}\left(\frac{rx^{r-1}}{y^r-x^r}-\frac{sx^{s-1}}{y^s-x^s}\right)\right]\\
= & \frac{1}{s-r}\lim_{\substack{x\to x_0\\ y\to x_0}} E(r,s;x,y)\cdot \lim_{\substack{x\to x_0\\ y\to x_0}}\frac{1}{x}\cdot \lim_{\substack{x\to x_0\\ y\to x_0}}\left[\frac{r}{(\frac{y}{x})^r-1}-\frac{s}{(\frac{y}{x})^s-1}\right]\\
= & \frac{1}{s-r}\lim_{t\to 1}\left[\frac{r}{t^r-1}-\frac{s}{t^s-1}\right]=\frac{1}{2}.
\end{aligned}$$

Hence $\frac{\partial E(r,s;x,y)}{\partial x}$ is continuous with respect to $(x,y) \in (0,\infty)\times(0,\infty)$.

For a set $E \subset R^2$, let $\bar{E}$ be the closure of E. From the continuity of the extended mean values $E(r,s;x,y)$ and the definitions of Schur geometrically convex and Schur geometrically concave, the following lemma is obvious.

Lemma 2.5. *Let E be a set in rs-plane with nonempty interior, if the extended mean values $E(r,s;x,y)$ are Schur geometrically convex (or Schur geometrically concave, respectively) with respect to $(x,y) \in (0,\infty)\times(0,\infty)$ for $(r,s) \in E$. Then $E(r,s;x,y)$ is Schur geometrically convex (or Schur geometrically concave, respectively) with respect to $(x,y) \in (0,\infty)\times(0,\infty)$ for $(r,s) \in \bar{E}$.*

3. Proof of Theorem 1.12

We use Theorem 1.9 and Lemma 2.4 to the discuss the nonpositivity and nonnegativity of $(\log y - \log x)(y\frac{\partial E}{\partial y} - x\frac{\partial E}{\partial x})$ for all $(x,y) \in (0,\infty)\times(0,\infty)$ and for fixed $(r,s) \in R^2$. Since $(\log y - \log x)(y\frac{\partial E}{\partial y} - x\frac{\partial E}{\partial x}) = 0$ for $x = y$ and $(\log y - \log x)(y\frac{\partial E}{\partial y} - x\frac{\partial E}{\partial x})$ is symmetric with respect to x and y, w.l.g. we assume $y > x$ in the following discussion.

Let

$$\begin{aligned}
E_1 &= \{(r,s) : s+r>0\},\\
E_2 &= \{(r,s) : s+r<0\},\\
E_{11} &= \{(r,s) : s>r, s+r>0\},\\
E_{12} &= \{(r,s) : s<r, s+r>0\},\\
E_{13} &= \{(r,s) : s=r>0\},\\
E_{21} &= \{(r,s) : s>r, s+r<0\},\\
E_{22} &= \{(r,s) : s<r, s+r<0\}
\end{aligned}$$

and

$$E_{23} = \{(r,s) : s=r<0\}.$$

Then

$$E_1 = E_{11}\cup E_{12}\cup E_{13}, \qquad E_{11}\cap E_{12} = E_{11}\cap E_{13} = E_{12}\cap E_{13} = \emptyset \tag{27}$$

and

$$E_2 = E_{21}\cup E_{22}\cup E_{23}, \qquad E_{21}\cap E_{22} = E_{21}\cap E_{23} = E_{22}\cap E_{23} = \emptyset. \tag{28}$$

By Lemma 2.5, we see that Theorem 1.12 is true if we prove that $(\log y - \log x)(y\frac{\partial E}{\partial y} - x\frac{\partial E}{\partial x}) \geq 0$ for $y > x > 0$ and $(r,s) \in E_1$, and $(\log y - \log x)(y\frac{\partial E}{\partial y} - x\frac{\partial E}{\partial x}) \leq 0$ for $y > x > 0$ and $(r,s) \in E_2$. We divide our proof into three cases.

Case 1. $(r,s) \in E_{11}$. We divide the discussion of this case into two subcases.

Subcase 1.1. $(r,s) \in E_{11}$ and $r \neq 0$. We note that (1) leads to the following identity.

$$\begin{aligned}
&(\log y - \log x)(y\frac{\partial E}{\partial y} - x\frac{\partial E}{\partial x}) \\
= \;& \frac{r}{s}\left(\frac{y^s - x^s}{y^r - x^r}\right)^{\frac{1}{s-r}-1} \frac{\log y - \log x}{(y^r - x^r)^2} x^{s+r} \frac{1}{s-r} \\
\times \;& \left\{\frac{r}{s}(s-r)[(\frac{y}{x})^{s+r} - 1] - \frac{r}{s}(s+r)[(\frac{y}{x})^s - (\frac{y}{x})^r]\right\}. \qquad (29)
\end{aligned}$$

Hence, (29), $y > x$ and Lemma 2.1(*1*) yield

$$(\log y - \log x)(y\frac{\partial E}{\partial y} - x\frac{\partial E}{\partial x}) \geq 0.$$

Subcase 1.2. $(r,s) \in E_{11}$ and $r = 0$. The relations $r = 0$, $s > 0$ and (2) lead to

$$\begin{aligned}
&(\log y - \log x)(y\frac{\partial E}{\partial y} - x\frac{\partial E}{\partial x}) \\
= \;& \frac{1}{s^2}\left(\frac{1}{s}\frac{y^s - x^s}{\log y - \log x}\right)^{\frac{1}{s}-1} \frac{x^s}{\log y - \log x} \\
\times \;& \left\{s[(\frac{y}{x})^s + 1]\log\frac{y}{x} - 2[(\frac{y}{x})^s - 1]\right\}. \qquad (30)
\end{aligned}$$

Since $y > x$, by Lemma 2.2(*1*) we see that (30) gives

$$(\log y - \log x)\left(y\frac{\partial E}{\partial y} - x\frac{\partial E}{\partial x}\right) \geq 0.$$

Case 2. $(r,s) \in E_{12}$. From Case *1* and the symmetry of $E(r,s;x,y)$ with respect to r and s, we get

$$(\log y - \log x)\left(y\frac{\partial E}{\partial y} - x\frac{\partial E}{\partial x}\right) \geq 0.$$

Case 3. $(r,s) \in E_{13}$. In this case, (3) leads to

$$\begin{aligned}
&(\log y - \log x)\left(y\frac{\partial E}{\partial y} - x\frac{\partial E}{\partial x}\right) \\
= \;& \frac{\log y - \log x}{(x^r - y^r)^2} x^{2r} E \left[\left(\frac{y}{x}\right)^{2r} - 2r\left(\frac{y}{x}\right)^r \log\frac{y}{x} - 1\right].
\end{aligned}$$

This identity gives by Lemma 2.3(*1*) that

$$(\log y - \log x)\left(y\frac{\partial E}{\partial y} - x\frac{\partial E}{\partial x}\right) \geq 0,$$

since $y > x$. Making use of Lemma 2.1(*2*), Lemma 2.2(*2*), Lemma 2.3(*2*) and the similar discussions as in the above cases, we conclude that

$$(\log y - \log x)\left(y\frac{\partial E}{\partial y} - x\frac{\partial E}{\partial x}\right) \leq 0.$$

for $(r,s) \in E_2$.

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