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Irreversible Quasistatic Evolutions by Minimizing Movements

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Dedicated to the memory of Thomas Lachand-Robert.

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We present an abstract framework for irreversible rate independent evolution processes of quasi-static nature. The main tool relies on the minimizing movement theory. In particular situations, stability and energy inequality of Mielke's type are satisfied. Several examples are given, among which the obstacle erosion.

1. Introduction

In this paper we introduce a general model for the quasi-static evolution of some irreversible processes. The main framework relies on the minimizing movement theory introduced by De Giorgi [9] (see also [1]). The irreversibility is modeled through a monotonicity relation among states; typical examples of such irreversible processes are the crack propagation (see [10]) and the debonding membranes (see [4]).

The abstract setting of minimizing movements requires a convergence structure on the space of states and an energy functional. In the examples above the states are closed or open sets of $\mathbb{R}^N$, the energy functional is related to the Dirichlet energy of the membrane, but the topology on the space of states is not a priori imposed. Nevertheless, a good topology should be compact and strong enough to ensure that the energy functional is continuous. The usual topology on the space of states for which the energy functional is continuous (called in this paper γ) is in general not compact (see for instance [2] for a detailed presentation of γ-convergence). Here is a fundamental different point of view with respect to other abstract models, e.g. the models introduced by Mielke [15] and Mielke and Mainik [14]. By acting on the space of states (e.g. making it smaller), one can artificially restrict the movement to some γ-compact classes (e.g. adding some geometric or topological constraints on the moving shapes). Nevertheless, those movements do not, in general, satisfy stability properties or energy conservation laws in the sense of Mielke [15]. In the frame of Mielke, there is only one topology, which is assumed to be compact, while in the situations we consider (like the debonding membrane and obstacle erosion) two topologies are necessary for a good understanding of the problem. The main idea is

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to introduce a second topology called weak γ (and denoted $w\gamma$), which is weaker than γ and compact. The main inconvenient is that the internal energy is, in general, only lower semicontinuous in the $w\gamma$-topology. Our study concentrates on the interplay between the two topologies and the monotonicity assumption in order to achieve stability and energy conservation in the sense of Mielke, for a minimizing movement solution (see also [3]).

The purpose of this paper is to highlight the double topology framework in the context of irreversible processes. We discuss both situations, i.e. γ compact or not and we give several examples among which the debonding membrane and the obstacle erosion.

2. The abstract setting: the compact case

We start by recalling a simplified version of the notion of generalized minimizing movements. We refer the reader to the pioneering paper of De Giorgi [9] (see also [1]). Consider a topological space $\mathcal{S}$, or more in general a set $\mathcal{S}$ endowed with a convergence structure [13], and a functional

$$[0,T] \times \mathcal{S} \times \mathcal{S} \ni (t,v,w) \mapsto \mathcal{F}(t,v,w) \in \overline{\mathbb{R}}.$$

For every fixed $\varepsilon > 0$, we introduce the following Euler scheme of time step ε and initial condition $u_0 \in \mathcal{S}$. We construct a function $u_\varepsilon : [0,T] \to S$ by setting $u_\varepsilon(t) = w([t/\varepsilon])$, where

$$w(0) = u_0, \qquad w(n+1) \in \operatorname{Argmin}\big\{\mathcal{F}\big((n+1)\varepsilon, \cdot, w(n)\big)\big\}.$$

Here $[\cdot]$ stands for the integer part function.

Definition 2.1. We say that $u : [0,T] \to \mathcal{S}$ is a minimizing movement associated to $\mathcal{F}$ with initial condition u_0, and we write $u \in MM(\mathcal{F}, \mathcal{S}, u_0)$, if there exist a sequence $\varepsilon_n \to 0^+$ such that for any $t \in [0,T]$, $u_{\varepsilon_n}(t) \to u(t)$ in $\mathcal{S}$.

The above procedure can be generalized by considering general partitions of $[0,T]$ instead of constant steps in the Euler scheme. More precisely, for every finite partition A of $[0,T]$ (i.e. $0 = t_0 < t_1 < \cdots < t_h = T$) we define

$$u_A(t) = w(t_i) \quad \forall t \in [t_i, t_{i+1}],$$

where

$$w(0) = u_0, \qquad w(t_{i+1}) \in \operatorname{Argmin}\big\{\mathcal{F}\big(t_{i+1}, \cdot, w(t_i)\big)\big\}.$$

We call generalized minimizing movement (simply GMM) associated to $\mathcal{F}$ with initial condition u_0 a function $u : [0,T] \to \mathcal{S}$ for which there exists a family of finite partitions $\{A_n\}_{n\in\mathbb{N}}$ such that $A_n \to [0,T]$ in the Hausdorff metric and $u_{A_n}(t) \to u(t)$ for every $t \in [0,T]$.

We define now the rate independence property for a function $u : [0,T] \to S$.

Definition 2.2. We say that $u : [0,T] \to S$ verifies the rate independence property if for every increasing continuous bijection $\alpha : [0,T] \to [0,T]$, the mapping $t \mapsto u(\alpha(t))$ is a GMM associaterd to $\mathcal{F}(\alpha(\cdot), \cdot, \cdot)$ and initial condition u_0.

We have the following result.

Proposition 2.3. *Every $u \in MM(\mathcal{F}, \mathcal{S}, u_0)$, satisfies the rate independence property.*

Proof. If (ε_n) is the sequence provided by Definition 2.1 and $\alpha : [0,T] \to [0,T]$ is an increasing continuous bijection, it is enough to consider the partitions $A_n = \{\alpha((k+1)\varepsilon_n) : k \in \mathbb{N}\}$ and to verify that $u_{A_n}(t) \to u(t)$ for every $t \in [0,T]$. □

In our setting we fix the following notation.

- The convergence on $\mathcal{S}$ is denoted by γ (by abuse of language we still call it a topology).
- We introduce an order relation $\preceq$ on $\mathcal{S}$ which is compatible with γ, i.e. if $u_n \preceq v_n$ and $u_n \to u$, $v_n \to v$ in γ, then $u \preceq v$.
- The internal energy of the system is given by a functional
$$\mathcal{E} : [0,T] \times \mathcal{S} \to \mathbb{R},$$
such that for every $t \in [0,T]$ the mapping $\mathcal{E}(t,\cdot)$ is γ-lower semicontinuous. In many examples the natural topology for which this functional is continuous is related to the Γ-convergence, and is denoted by γ (see [2, Chapter 3]).
- The dissipation distance we consider is a symmetric functional
$$\mathcal{D} : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$$
such that for every $v \in \mathcal{S}$ the mapping $\mathcal{D}(v,\cdot)$ is γ-lower semicontinuous and satisfies
(i) $\mathcal{D}(u,u) = 0$ for every $u \in \mathcal{S}$;
(ii) $\mathcal{D}(u_1,u_3) \le \mathcal{D}(u_1,u_2) + \mathcal{D}(u_2,u_3)$ for every $u_1,u_2,u_3 \in \mathcal{S}$.
This functional accounts on the quantity of energy which is necessary to spent for switching between two ordered states.
- The functional to which we associate the minimizing movement is
$$\mathcal{F}(t,v,u) = \mathcal{E}(t,v) + \mathcal{D}(u,v) + \chi_{u \preceq v},$$
where $\chi_{u\preceq v} = 0$ if $u \preceq v$ and $+\infty$ otherwise. The term $\chi_{u\preceq v}$ models the irreversibility property of the quasi-static evolution.

Proposition 2.4. *If γ is compact, for every $\varepsilon > 0$ and $u_0 \in \mathcal{S}$ there exists a solution $u_\varepsilon(t)$ of the discrete Euler scheme with initial datum u_0.*

Proof. Indeed, we fix a time step $\epsilon > 0$ and consider the discretized time $t_k^\epsilon = \epsilon k$ for $k \in \mathbb{N}$. We define $u_\epsilon(t_k^\epsilon)$ iteratively, by taking $u_\epsilon(t_{k+1}^\epsilon)$ as the solution of the minimum problem
$$\min\Big\{E\big(t_{k+1}^\epsilon, u\big) + \mathcal{D}\big(u(t_k^\epsilon), u\big) + \chi_{u(t_k^\epsilon)\preceq u} \,:\, u \in \mathcal{S}\Big\}\,. \tag{1}$$
The minimization problem has a solution since the topology γ is compact, both $\mathcal{E}$ and $\mathcal{D}$ are γ-l.s.c. in the second variable, and the order is compatible with the γ-convergence. □

The following hypothesis insures that monotone mappings from $[0,T]$ in $\mathcal{S}$ have an at most countable family of discontinuity points. This hypothesis is also used in [14, 15] being a fundamental assumption for the passage from the discrete to the continuous frame. In all the examples we consider here, this hypothesis is satisfied.

Hypothesis H1. Every nondecreasing mapping $[0,T] \mapsto \mathcal{S}$ has an at most countable family of γ-discontinuity points.

Theorem 2.5. *Under hypothesis* H1*, there exists a minimizing movement* $u : [0,T] \to \mathcal{S}$ *associated to* $\mathcal{F}$*, which satisfies the rate independence property.*

Proof. The proof relies on the Helly theorem and hypothesis H1. Let $\varepsilon_n = 2^{-n}$ and consider the sequence of discrete movements issued from the Euler scheme through Proposition 2.4; for simplicity we denote by u_n the element u_{ε_n}. Using the compactness of the γ-convergence, by an usual diagonal procedure, we can extract a subsequence (still denoted by (u_n)) such that

$$q = k2^{-j} \in [0,T] \quad \text{with } k, j \in \mathbb{N} \;\Rightarrow\; u_n(q) \to u(q).$$

Yet, u is not defined on the full interval $[0,T]$, so we set

$$\overline{u}(t) = \sup\big\{u(q) \;:\; q = k2^{-j} < t\big\}.$$

We notice that $\overline{u}$ is well defined by the compactness of the γ-convergence and its compatibility with the monotonicity assumption. The mapping $\overline{u}$ may differ from u on the points of the form $k2^{-j}$ but, according to hypothesis H1, $\overline{u}$ has an at most countable family I of discontinuity points.

If $t \notin I$, that is t is a continuity point for $\overline{u}$, then $u_n(t) \to \overline{u}(t)$. If $t \in I$, we extract a subsequence (still denoted by (u_n)) such that

$$u_n(t) \to \tilde{u}(t)$$

for a suitable $\tilde{u}(t)$, and since I is countable, the subsequence (u_n) can be chosen independent of $t \in I$. We may now define the minimizing movement for every $t \in [0,T]$ setting

$$u^*(t) = \begin{cases} \overline{u}(t) & \text{if } t \notin I \\ \tilde{u}(t) & \text{if } t \in I \end{cases}$$

and the proof is achieved.

By Proposition 2.3 the function u satisfies the rate independence property. □

As noticed, the minimizing movement $u : [0,T] \to \mathcal{S}$ in the previous theorem is nondecreasing, hence it has an at most countable family of discontinuity points.

We shall further work with the regularization of the minimizing movement u^* which is the mapping $\overline{u}$ defined above, which verifies

$$\overline{u}(t) = \sup_{s<t} u^*(s). \tag{2}$$

The function $\overline{u} : [0,T] \to \mathcal{S}$ is well defined by the compactness of the γ-topology and its compatibility with the monotonicity. Moreover, $\overline{u}$ coincides with u^* on every continuity point of u^*.

The stability property. In the sequel we investigate the stability property of the regularized movement $\overline{u}$ defined in (2). The stability property was introduced in [14, 15] as being one of the relevant mechanical properties which are required by a minimizing movement.

Definition 2.6. We say that the movement $\overline{u}$ is stable if for every $t \in [0,T]$

$$\mathcal{E}(t,\overline{u}(t)) \le \mathcal{E}(t,v) + \mathcal{D}(\overline{u}(t),v) \quad \forall \overline{u}(t) \preceq v. \tag{3}$$

At this point, we may formalize a second hypothesis (which is satisfied in all examples we consider).

Hypothesis H2. Assume that $u \preceq v$ and $u_n \xrightarrow{\gamma} u$. There exists a subsequence (u_{n_k}) and a sequence (v_k) such that $v_k \xrightarrow{\gamma} v$ and $u_{n_k} \preceq v_k$.

From a practical point of view, we notice that hypothesis H2 has to be satisfied only for the elements u such that $u_0 \preceq u$. This observation may be very useful in practice, since well chosen initial conditions may have a direct influence on the properties of the evolution (this is for instance the case of modeling debonding membranes by measures [4]).

Theorem 2.7. *Assume that $\mathcal{E}$ is continuous on $\mathbb{R} \times \mathcal{S}$ and $\mathcal{D}$ is continuous on $\mathcal{S} \times \mathcal{S}$. Then, the regularized minimizing movement $\overline{u}(t)$ defined in (2) is stable.*

Proof. At each time step, using the optimization problem (1) we can write

$$\mathcal{E}(t_k^\varepsilon, u_\varepsilon(t_k^\varepsilon)) + \mathcal{D}(u_\varepsilon(t_k^\varepsilon - \varepsilon), u_\varepsilon(t_k^\varepsilon)) \le \mathcal{E}(t_k^\varepsilon, v) + \mathcal{D}(u_\varepsilon(t_k^\varepsilon - \varepsilon), v),$$

for every v such that $u_\varepsilon(t_k^\varepsilon - \varepsilon) \preceq v$.

For every $t \in (0,T]$, there exists $t_n \uparrow t$ and $\varepsilon_n \to 0$ such that

$$u_{\varepsilon_n}(t_n) \xrightarrow{\gamma} \overline{u}(t).$$

For every v such that $\overline{u}(t) \preceq v$, following hypothesis H2, there exists v_n such that (up to a subsequence)

$$v_n \xrightarrow{\gamma} v, \qquad u_{\varepsilon_n}(t_n) \preceq v_n.$$

Using the subadditivity of $\mathcal{D}$ we have

$$\mathcal{E}(t_n, u_{\varepsilon_n}(t_n)) \le \mathcal{E}(t_n, v_n) + \mathcal{D}(u_{\varepsilon_n}(t_n), v).$$

Passing to the limit and using the continuity properties of $\mathcal{E}$ and $\mathcal{D}$, the stability property follows. □

The energy inequality. The following energy inequality was introduced in [14, 15] as another condition to give a mechanical relevance to a minimizing movement.

Definition 2.8. We say that a movement $\overline{u}(t)$ satisfies the energy inequality, if for every $0 \le s < t \le T$

$$\mathcal{E}(t,\overline{u}(t)) + Diss(\overline{u},[s,t]) \le \mathcal{E}(s,\overline{u}(s)) + \int_s^t \partial_\tau \mathcal{E}(\tau,\overline{u}(\tau))d\tau, \tag{4}$$

where

$$\partial_\tau \mathcal{E}(\tau,v) = \limsup_{h\to 0} \frac{\mathcal{E}(\tau+h,v) - \mathcal{E}(\tau,v)}{h}$$

and

$$Diss_{\mathcal{D}}(\overline{u},[s,t]) = \sup_{N\in\mathbb{N},\ s=t_0<\dots<t_N=t} \sum_{j=1}^{N} \mathcal{D}(\overline{u}(t_{j-1}),\overline{u}(t_j)).$$

In order to investigate the energy inequality property, we start with some preliminary observations. From (1), writing the optimality of $u_{\varepsilon_n}(t_n)$ with respect to $u_{\varepsilon_n}(t_n-\varepsilon_n)$ we have

$$\mathcal{E}(t_n,u_{\varepsilon_n}(t_n)) + \mathcal{D}(u_{\varepsilon_n}(t_n-\varepsilon_n),u_{\varepsilon_n}(t_n)) \le \mathcal{E}(t_n,u_{\varepsilon_n}(t_n-\varepsilon_n)).$$

The following assumption is related to a suitable time differentiability of the energy, and is verifed in all the examples we consider. It is a Leibnitz-Newton type formula:

$$\text{(LN)} \qquad \mathcal{E}(t,v)-\mathcal{E}(s,v) \le \int_s^t \partial\mathcal{E}_\tau(\tau,v)\,d\tau \quad \forall\, 0\le s<t\le T,\ \forall v\in\mathcal{S}.$$

With this assumption, we have

$$\mathcal{E}(t_n,u_{\varepsilon_n}(t_n-\varepsilon_n)) \le \mathcal{E}(t_n-\varepsilon_n,u_{\varepsilon_n}(t_n-\varepsilon_n)) + \int_{t_n-\varepsilon_n}^{t_n} \partial_\tau\mathcal{E}(\tau,u_{\varepsilon_n}(t_n-\varepsilon_n))\,d\tau.$$

Summing the inequalities

$$\begin{aligned}&\mathcal{E}(t_n,u_{\varepsilon_n}(t_n)) + \mathcal{D}(u_{\varepsilon_n}(t_n-\varepsilon_n),u_{\varepsilon_n}(t_n))\\ &\le \mathcal{E}(t_n-\varepsilon_n,u_{\varepsilon_n}(t_n-\varepsilon_n)) + \int_{t_n-\varepsilon_n}^{t_n} \partial_\tau\mathcal{E}(\tau,u_{\varepsilon_n}(t_n-\varepsilon_n))\,d\tau\end{aligned}$$

between s_n and t_n, we get

$$\begin{aligned}&\mathcal{E}(t_n,u_{\varepsilon_n}(t_n)) - \mathcal{E}(s_n,u_{\varepsilon_n}(s_n)) + \sum_k \mathcal{D}(u_{\varepsilon_n}(s_n+k\varepsilon_n),u_{\varepsilon_n}(s_n+(k+1)\varepsilon_n))\\ &\le \sum_k \int_{s_n+k\varepsilon_n}^{s_n+(k+1)\varepsilon_n} \partial_\tau\mathcal{E}(\tau,u_{\varepsilon_n}(s_n+k\varepsilon_n))d\tau.\end{aligned} \tag{5}$$

Theorem 2.9. *Assume* (LN) *is satisfied, that the mapping*

$$(t,v)\mapsto \partial_t\mathcal{E}(t,v)$$

is $\mathbb{R}\times\gamma$-continuous, and that

$$|\partial_t\mathcal{E}(t,v)| \le g(t) \quad \forall t\in(0,T),\ \forall v\in\mathcal{S} \tag{6}$$

for some $g\in L^1(0,T)$. Then the energy inequality (4) *holds true.*

Proof. Passing to the limit in (5) and using the continuity of $\mathcal{D}$ together with the Lebesgue dominated convergence theorem, we get

$$\mathcal{E}(t,\overline{u}(t)) - \mathcal{E}(s,\overline{u}(s)) + Diss(\overline{u},[s,t]) \le \int_s^t \partial_\tau\mathcal{E}(\tau,\overline{u}(\tau))\,d\tau$$

as required. □

Remark 2.10. In fact, inequality (6) is necessary only for $v = u_{\varepsilon_n}(t)$, where u_{ε_n} are the functions obtained in the Euler scheme. In practice, in all examples we consider, this inequality holds for every $v \in \mathcal{S}$, as a consequence of the regularity of the data.

This remark is also valid for Theorem 3.3.

Example 2.11. The debonding membrane. The following example has been studied in [4]. Let $D \subseteq \mathbb{R}^N$ be a bounded open set. We denote by $\mathcal{M}_0$ the class of all nonnegative Borel measures μ on D, possibly $+\infty$ valued, such that $\mu(B) = 0$ for every Borel set $B \subseteq D$ with $\operatorname{cap}(B, D) = 0$. In our setting $\mathcal{S} = \mathcal{M}_0$. We say that

$$\mu_1 \preceq \mu_2 \text{ if for every quasi-open set } A \subseteq D \text{ we have } \mu_1(A) \geq \mu_2(A),$$

or equivalently if

$$\int_D u^2 \, d\mu_2 \leq \int_D u^2 \, d\mu_1 \quad \forall u \in H_0^1(D).$$

Definition 2.12. We say that a sequence (μ_n) of measures in $\mathcal{M}_0$ γ-converges to a measure $\mu \in \mathcal{M}_0$ if and only if

$$R_{\mu_n}(f) \to R_\mu(f) \quad \text{strongly in } L^2(D) \quad \forall f \in H^{-1}(D)$$

where $R_\mu(f)$ is the variational solution of

$$u \in H_0^1(D) \cap L^2(D, \mu), \qquad -\Delta u + \mu u = f \text{ in } [H_0^1(D) \cap L^2(D, \mu)]'$$

i.e.

$$\int_D \nabla u \nabla v \, dx + \int_D uv \, d\mu = \langle f, v \rangle_{H^{-1}(D)} \quad \forall v \in H_0^1(D) \cap L^2(D, \mu). \tag{7}$$

Remark 2.13. It is possible to show (see for instance [2]) that $\mu_n \xrightarrow{\gamma} \mu$ if and only if $R_{\mu_n}(1) \to R_\mu(1)$ in $L^2(D)$, so that the quantity

$$d_\gamma(\mu, \nu) = \|R_\mu(1) - R_\nu(1)\|_{L^2(D)}$$

is a distance on $\mathcal{M}_0$, equivalent to the γ-convergence and which makes it a compact metric space.

Assume now $f \in W^{1,\infty}\big([0, T], L^2(D)\big)$; if we set

$$\mathcal{E}(t, \mu) = \min \Big\{ \frac{1}{2} \int_D |\nabla u|^2 \, dx + \frac{1}{2} \int_D u^2 \, d\mu - \int_D f(t) u \, dx \ : \ u \in H_0^1(D) \cap L^2(D, \mu) \Big\}$$

and

$$\mathcal{D}(\mu_1, \mu_2) = \int_D |R_{\mu_1}(1) - R_{\mu_2}(1)| \, dx,$$

one can check that Theorem 2.7 applies. Hypothesis H1 is a consequence of the fact that if the mapping

$$t \mapsto R_{\mu_t}(1)$$

is increasing (i.e. $s \leq t \Rightarrow R_{\mu_s}(1) \leq R_{\mu_t}(1)$ a.e. on D) then $t \mapsto \mu_t$ is γ-continuous except the discontinuity points of $t \mapsto \int_D R_{\mu_t}(1) \, dx$. Assumption (LN) is a direct consequence of the time Lipschitz continuity of f.

Hypothesis H2 is more technical, and is satisfied by the family of measures $\{\mu \in \mathcal{M}_0 : \mu \preceq \mu_0\}$ where μ_0 is a measure of $\mathcal{M}_0$ which has a finite mass. Hence, provided the initial condition is a measure of finite mass, Theorem 2.9 applies. We refer to [4] for further details on this case and for the proofs of the results stated above.

3. The non-compact case

The non-compact case refers essentially to the situation in which the natural topology on $\mathcal{S}$ (which is often given by the Γ-convergence of the energy functionals and is denoted as previously by γ) is not compact. For this reason the abstract framework of minimizing movements and rate independent processes of Mielke does not apply. In addition, we introduce a weaker convergence structure called weak γ, and denoted $w\gamma$, which is compact and weaker than γ. The main inconvenient is that the internal energy functional is not, in general, continuous in the $w\gamma$-topology. This situation is for example encountered in shape optimization and obstacle problems, where the space of states (shapes and obstacles, respectively) is not a vector space.

We consider an abstract framework where $\mathcal{S}$ is a set and γ and $w\gamma$ two convergence structures on $\mathcal{S}$. We assume:

- $w\gamma$ is sequentially compact and weaker than γ
- $\preceq$ is compatible with $w\gamma$ (and so also with γ)
- hypothesis H1 is satisfied, with respect to the γ-convergence.

Notice that the first two assumptions above imply that if $u_n \preceq v_n \preceq w_n$ and u_n, w_n are $w\gamma$-convergent to u, then v_n is also $w\gamma$-convergent to u.

The functional we consider is still

$$\mathcal{F}(t, v, u) = \mathcal{E}(t, v) + \mathcal{D}(u, v) + \chi_{u \preceq v}.$$

We further assume that

- $\mathcal{E}(t, \cdot)$ is $w\gamma$-lower semicontinuous,
- $\mathcal{D}(u, \cdot)$ is $w\gamma$-lower semicontinuous.

Consequently, we obtain straightforward the existence of a solution for the discrete Euler scheme.

Theorem 3.1. *There exists a minimizing movement $u : [0, T] \to \mathcal{S}$ associated to $\mathcal{F}$ in $(\mathcal{S}, w\gamma)$. This minimizing movement satisfies the rate independence property.*

Proof. The proof is similar to the one of Theorem 2.5. As in the previous section we introduce the regularized movement

$$\overline{u}(t) = \sup_{s<t} u(s).$$

This time, the supremum is constructed by using the $w\gamma$-convergence. The function $\overline{u} : [0, T] \to \mathcal{S}$ is well defined since one uses the compactness of the $w\gamma$-topology and the compatibility with the monotonicity.

The rate independence property is also proved similarly, as a consequence of the compactness of the $w\gamma$-topology and of the irreversibility. □

In order to study the stability and the energy inequality, we formalize some natural hypotheses holding in all cases e consider. We assume that

H3 $\mathcal{E}$ is l.s.c. in $\mathbb{R} \times w\gamma$
H4 $\mathcal{E}$ is continuous in $\mathbb{R} \times \gamma$
H5 If $u \preceq v$ and $u_n \overset{w\gamma}{\to} u$ there exists a subsequence (u_{n_k}) and a sequence (v_k) such that $v_k \overset{\gamma}{\to} v$, $u_{n_k} \preceq v_k$ and

$$\mathcal{D}(u, v) \geq \limsup_k \mathcal{D}(u_{n_k}, v_k).$$

Theorem 3.2. *Under assumptions* H3, H4 *and* H5*, the regularized movement* $t \mapsto \overline{u}(t)$ *is stable in the sense of Definition* 2.6*.*

Proof. At each time step, using the optimization problem (1) we can write

$$\mathcal{E}(t_k^\varepsilon, u_\varepsilon(t_k^\varepsilon)) + \mathcal{D}(u_\varepsilon(t_k^\varepsilon - \varepsilon), u_\varepsilon(t_k^\varepsilon)) \leq \mathcal{E}(t_k^\varepsilon, v) + \mathcal{D}(u_\varepsilon(t_k^\varepsilon - \varepsilon), v),$$

for every v such that $u_\varepsilon(t_k^\varepsilon - \varepsilon) \preceq v$.

For every $t \in (0, T]$, there exists $t_n \uparrow t$ and $\varepsilon_n \to 0$ such that

$$u_{\varepsilon_n}(t_n) \xrightarrow{w\gamma} \overline{u}(t).$$

For every v such that $\overline{u}(t) \preceq v$, according to hypothesis H5 there exist a subsequence of $u_{\varepsilon_n}(t_n)$ (that we denote by the same indices) and a sequence v_n such that

$$v_n \xrightarrow{\gamma} v \qquad u_{\varepsilon_n}(t_n) \preceq v_n.$$

Using the subadditivity of $\mathcal{D}$ we have

$$\mathcal{E}(t_n, u_{\varepsilon_n}(t_n)) \leq \mathcal{E}(t_n, v_n) + \mathcal{D}(u_{\varepsilon_n}(t_n), v).$$

Passing to the limit and using the assumptions H3, H4 and H5, the stability property (3) follows. □

Theorem 3.3. *Assume* LN, H3, H4, H5*, that the mapping*

$$(t, u) \mapsto \partial_t \mathcal{E}(t, u) \tag{8}$$

is $\mathbb{R} \times w\gamma$ *upper semicontinuous, and that*

$$|\partial_t \mathcal{E}(t, u)| \leq g(t) \quad \forall t \in (0, T),\ \forall v \in \mathcal{S}$$

for some $g \in L^1(0, T)$*. Then the regularized movement* $t \mapsto \overline{u}(t)$ *satisfies the weaker form of the energy inequality*

$$\mathcal{E}(t, \overline{u}(t)) + Diss(\overline{u}, [0, t]) \leq \mathcal{E}(0, \overline{u}(0)) + \int_0^t \partial_\tau \mathcal{E}(\tau, \overline{u}(\tau))\, d\tau \quad \forall t \in (0, T]. \tag{9}$$

Proof. The proof follows step by step Theorem 2.9. The passage to the limit is a consequence of the fact that in (9) we have $s = 0$ and of the upper semicontinuity hypothesis on $\partial_t \mathcal{E}(t, u)$. □

In practical situations, the $w\gamma$-upper semicontinuity of the mapping (8) may not occur, as the following examples show. This is true only under further assumptions related to the data (particular initial data and/or right-hand sides, etc.).

Example 3.4. Shape evolution of the debonding membrane. An example fitting the non-compact frame is given in [4] and deals with the debonding membrane (i.e. without relaxation). Precisely, the space $\mathcal{S}$ consists of the family of quasi-open subsets of a bounded open set $D \subseteq \mathbb{R}^2$. A nonnegative force depending on time $f : [0,T] \to L^2(D,\mathbb{R}^+)$ acts on the membrane. The energy of a debonded membrane A at time t is

$$\mathcal{E}(t,A) = \min_{u \in H_0^1(A)} \frac{1}{2}\int_A |\nabla u|^2\,dx - \int_A f(t)u\,dx,$$

and the dissipation distance is proportional to the surface measure of the symmetric difference

$$\mathcal{D}(A_1,A_2) = |A_2 \setminus A_1| + |A_1 \setminus A_2|.$$

The monotonicity relation is the inclusion up to sets of zero capacity.

We identify a quasi-open set A with the measure ∞_A defined by

$$\infty_A(E) = \begin{cases} 0 & \text{if } \operatorname{cap}(A^c \cap E, D) = 0 \\ +\infty & \text{if } \operatorname{cap}(A^c \cap E, D) > 0. \end{cases}$$

With this identification, the γ-convergence is the same as for measures (defined in the previous section) but it is not compact in the family of quasi-open sets; on the contrary, it can be shown (see for instance [7, 2]) that the measures which are associated to domains are γ-dense in the class $\mathcal{M}_0$ of all capacitary measures.

The $w\gamma$-convergence is defined as follows: we say that A_n $w\gamma$-converges to A if $R_{A_n}(1) \xrightarrow{L^2(D)} w$ and $A = \{x \in D : w(x) > 0\}$, where R_{A_n} are the resolvent operators introduced in Definition 2.12.

Hypothesis H5 is satisfied by the couple $(\gamma, w\gamma)$ in relationship with $\mathcal{E}$ and $\mathcal{D}$. This is the main difficulty when dealing with the non compact case, and the choice of a suitable $w\gamma$ convergence is a challenge in all practical problems.

Following [4], Theorem 3.2 applies. Theorem 3.3 is known to be true only under further assumptions, like for example if $f(t)$ and D have both some symmetry properties (e.g. in the sense of Schwarz or Steiner).

4. Obstacle erosion

In this section we discuss the problem of obstacle erosion, i.e. an obstacle which is diminishing its contact with a membrane. Alternatively, one may have in mind an elastic membrane pressing a deformable obstacle.

Let $D \subseteq \mathbb{R}^N$ be a bounded open set ($N \geq 2$); we consider the space of obstacles

$$\mathcal{S} = \{g : D \to \overline{\mathbb{R}} \text{ quasi u.s.c.}, \ g \leq \Psi, \ \int_D g\,dx \geq c\},$$

where $\Psi \in L^1(D)$ is a fixed function.

For a quasi upper semicontinuous function $g : D \to \overline{\mathbb{R}}$ we define the set

$$K_g = \{u \in H_0^1(D) \, : \, u \geq g \text{ q.e.}\}$$

so that, for every $h \in L^2(D)$ the solution $u_{g,h}$ of the obstacle problem associated to h and g is given by

$$\min \Big\{ \int_D \frac{1}{2} |\nabla u|^2 \, dx - \int_D hu \, dx \, : \, u \in K_g \Big\}. \tag{10}$$

The choice of obstacles as quasi u.s.c. functions is natural, since one can replace an arbitrary obstacle by a suitable quasi u.s.c. one (see [5]).

The γ-convergence of obstacles is defined as follows:

g_n γ-converges to g if for every $h \in L^2(D)$ $u_{g_n,h}$ converges in $L^2(D)$ to $u_{g,h}$.

Following [5] (see also [3]), this is equivalent to the γ-convergence of the level sets $\{g_n < t\}$ to $\{g < t\}$ in the sense of quasi-open sets for a countable and dense set $Q \subseteq \mathbb{R}$ of values t.

The $w\gamma$-convergence for obstacles was introduced in [3]. We say that g_n $w\gamma$-converges to g if for a countable and dense set $Q \subseteq \mathbb{R}$ of values t, we have that $\{g_n < t\}$ $w\gamma$-converges to $\{g < t\}$ in the sense of quasi-open sets seen above. We notice that this is a correct definition in the sense that if $g_n \stackrel{w\gamma}{\rightharpoonup} g$ and $g_n \stackrel{w\gamma}{\rightharpoonup} g'$ then $g = g'$. Thanks to this, there is no ambiguity in the choice of the dense set Q. Moreover, following [3] the space $\mathcal{S}$ endowed with $w\gamma$ is sequentially compact and the γ-convergence of obstacles is stronger that the $w\gamma$-convergence.

Let $h : [0, T] \to L^2(D, \mathbb{R})$. We set

$$\mathcal{E}(t, g) = \min\{ \int_D \frac{1}{2} |\nabla u|^2 dx - \int_D h(t) u \, dx : u \in K_g \}.$$

The dissipation distance that we consider is

$$\mathcal{D}(g_1, g_2) = \int_D |g_1 - g_2| dx,$$

and the monotonicity relation on $\mathcal{S}$ is

$$g_1 \preceq g_2 \quad \text{if } g_1(x) \geq g_2(x) \text{ q.e. on } D.$$

We notice that the monotonicity of obstacles is related to the monotonicity of their level sets, so several properties are inherited from quasi-open sets, as for example the compatibility of the γ and $w\gamma$ convergences of obstacles with the order relation as well as with the energy and the dissipation distance.

Following [3], assumption H5 is satisfied for the couple of topologies (γ, $w\gamma$) and the order relation between functions in $\mathcal{S}$. Consequently, Theorem 3.2 applies and the following result holds.

Theorem 4.1. *Let $T > 0$, $h \in W^{1,\infty}\big([0,T], L^2(D,\mathbb{R}^-)\big)$ and $g_0 \in \mathcal{S}$. There exists a minimizing movement associated to $\mathcal{E}$ and with initial condition g_0 in $(\mathcal{S}, w\gamma)$. This minimizing movement satisfies the rate independence property and the stability property.*

We notice that Theorem 3.3 applies as well, provided that further assumptions are made, in order to safisfy the hypothesis concerning the upper $w\gamma$-semicontinuity of $\partial_t\mathcal{E}$. In the case of two dimensional obstacles, provided that the set D is symmetric (in the sense of Schwarz or Steiner) and the force h is positive and symmetric in the same way as D, the discrete evolutions and the minimizing movements do satisfy the symmetry property too. It turns out that the $w\gamma$-convergence of obstacles coincides with the γ-convergence in this family, hence the upper semicontinuity hypothesis of the mapping (8) is satisfied.

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