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Asymptotic Analysis of Periodically Perforated Nonlinear Media Close to the Critical Exponent

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We give a Γ-convergence result for vector-valued nonlinear energies defined on periodically perforated domains. We consider integrands with p-growth for p converging to the space dimension n. We prove that for p close to the critical exponent n there are three regimes, two with a non-trivial size of the perforations (exponential and mixed polynomial-exponential) and one where the Γ-limit is always trivial.

Keywords: Γ-convergence, perforated domains, critical exponent

1. Introduction

Variational problems on perforated domains can be considered the prototype of the class of problems on varying domains. This is a very much studied class of problems and shows interesting implications in homogenization and shape optimization problems (see [1], [8]). A *perforated domain* is obtained from a fixed Ω by removing some periodic set, the simplest of which is a periodic array of closed sets:

$$\Omega_\delta = \Omega \setminus \bigcup_{i\in\mathbb{Z}^n} (\delta i + \varepsilon K), \tag{1}$$

with $\varepsilon = \varepsilon(\delta)$ and K a bounded closed set with non-empty interior. We are interested in the study of problems in which we fix Dirichlet boundary conditions on the boundary of Ω_δ (or on the boundary of Ω_δ interior to Ω). The asymptotic behaviour of such problems is obtained by studying the Γ-convergence of the functionals

$$F_\delta(u) = \begin{cases} \displaystyle\int_\Omega f(Du)\,dx & \text{if } u \in W_0^{1,p}(\Omega;\mathbb{R}^m) \text{ and } u = 0 \text{ on } \Omega\setminus\Omega_\delta, \\ +\infty & \text{otherwise,} \end{cases} \tag{2}$$

where f is an energy density satisfying a growth condition of order $p > 1$.

From early results by Marchenko and Khruslov [14] we know that in the case $f(Du) = |Du|^p$ there is a particular choice for the scaling of the perforations which produces the appearance in the Γ-limit of an *extra term* replacing the internal boundary conditions. The limit functional, indeed, is given by

$$F_0(u) = \int_\Omega |Du|^p\,dx + \kappa_p \int_\Omega |u|^p\,dx,$$

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where κ_p is a positive constant, explicitly calculable. This result was recast in a rigorous variational setting by Cioranescu and Murat [10], who provided an explicit formula for the critical choice of ε according to the space dimension n:

$$\varepsilon = R\delta^{n/n-p} \ \text{ if } p < n, \quad \text{with } R > 0,$$
$$\varepsilon = \exp(-a\delta^{\frac{-n}{n-1}}) \ \text{ if } p = n, \quad \text{with } a > 0.$$

In [2] Ansini and Braides performed a complete analysis in the vector-valued case of the Γ-convergence result for energies with a general integrand f with p-growth, in the case $p < n$. In their setting the form of the extra term is $\int_\Omega \varphi(u)\,dx$, where the function φ is given by a capacitary formula. The case $n = p$, leading to the exponential scaling, was studied in details in [15]; in this case the limit extra term is characterized by a formula of homogenization type.

In this paper we will consider the dependence of the energies in (2) on varying p, in order to better understand the behaviour at the critical scaling and to overcome the discontinuity in the description of the asymptotic analysis at $p = n$. Since we are interested in a scale analysis we will consider integral functionals on periodically perforated domains (1) in which $f(Du) = |Du|^p$ to avoid the technicalities of more general f (for which we refer to [15]). We will see that the behaviour as $\delta \to 0$ and $p \to n$ gives rise to three possible regimes:

- if $n - p = \gamma\delta^{\frac{n}{n-1}} + o(\delta^{\frac{n}{n-1}})$ with $\gamma \in \mathbb{R}$ then the critical radius is exponential; *i.e.*, $\varepsilon = \exp\big(-a\delta^{\frac{-n}{n-1}}\big)$ with $a > 0$;
- if $n - p > 0$ and $n - p \gg \delta^{\frac{n}{n-1}}$ then the critical size of the perforation is given by an interpolation of polynomial and exponential terms: $\varepsilon = R^{\frac{1}{n-p}}\delta^{\frac{n}{n-p}}(n-p)^{\frac{1-n}{n-p}}$, with $R > 0$;
- if $n - p < 0$ and $p - n \gg \delta^{\frac{n}{n-1}}$, then the limit is finite (and null) only on the constant function zero: this situation will be referred to as rigid regime.

2. The three regimes - Heuristics

In all that follows $n > 1$ and $m \geq 1$ are fixed integers. If $E \subset \mathbb{R}^n$ is a Lebesgue-measurable set then $|E|$ is its Lebesgue measure. $B_r(x)$ is the open ball in $\mathbb{R}^n$ of centre x and radius r; if $x = 0$ we will write B_r in place of $B_r(0)$. The letter c denotes a generic strictly positive constant.

Let Ω be a fixed bounded open subset of $\mathbb{R}^n$ with $|\partial\Omega| = 0$. Let $K \subset \mathbb{R}^n$ be a bounded closed set with non-empty interior. Let (δ_j), (ε_j) be two sequences of positive real numbers converging to zero. For all $i \in \mathbb{Z}^n$ and $j \in \mathbb{N}$ we denote by x_i^j the vector $i\delta_j \in \delta_j\mathbb{Z}^n \subset \mathbb{R}^n$. Let Ω_j be the periodically perforated domain

$$\Omega_j = \Omega \setminus \bigcup_{i\in\mathbb{Z}^n} (x_i^j + \varepsilon_j K). \tag{3}$$

Let (η_j) be an infinitesimal sequence of real numbers. Let $p_j = n - \eta_j$. We want to find the critical scaling $\varepsilon_j = \varepsilon_j(\delta_j, \eta_j)$ for the perforations; *i.e.*, the one which gives a non-trivial

Γ-convergence result for the functionals

$$F_j(u) = \begin{cases} \displaystyle\int_\Omega |Du|^{p_j}\,dx & \text{if } u \in W^{1,p_j}(\Omega;\mathbb{R}^m) \text{ and } u = 0 \text{ on } \Omega \setminus \overline{\Omega}_j, \\ +\infty & \text{otherwise.} \end{cases}$$

In other words, taking into account the n-homogeneity properties of F_j, we look for the critical (ε_j) such that the family (F_j) Γ-converges to a functional F_0 of the form

$$F_0(u) = \int_\Omega |Du|^n\,dx + \kappa \int_\Omega |u|^n\,dx \quad \text{for } u \in W^{1,n}(\Omega;\mathbb{R}^m), \tag{4}$$

where κ is a positive constant that we want to calculate explicitly. As is customary, not to overburden the notation all our functionals will be understood to take the value $+\infty$ where not explicitly defined.

In this paper we will show that the critical scaling and the expression of the extra term in the Γ-limit are determined by the behaviour of the sequence (η_j) with respect to (δ_j), as $j \to +\infty$. The three regimes we mentioned in the Introduction emerge from the analysis of the asymptotic behaviour of a family of minimum problems which play a fundamental role in the computation of the Γ-limit. Indeed, the proof of the Γ-convergence result relies on a general argument by Ansini and Braides [2], which allows to reduce the computation of the extra term to an estimate along converging sequences close to the perforations. In order to give a heuristic idea of the crucial lemma in [2], we consider the case $K = \overline{B}_1$ and a sequence $u_j \to u$. The technical argument of the lemma (which is based on De Giorgi's method for matching boundary conditions) allows to make the assumption that the energy 'far from the perforations' gives a term which can be dealt with separately and produces the first integral in (4). Moreover, the lemma enables to treat each perforation $B_{\varepsilon_j}(x_i^j)$ separately. Suppose that u is continuous; since $u_j \to u$ we can assume that u_j is close to the limit value $u(x_i^j)$ close to $B_{\varepsilon_j}(x_i^j)$. In particular the lemma in [2] shows that we may suppose $u_j = u(x_i^j)$ on the boundary of some small ball $B_{c\delta_j}(x_i^j)$ containing $B_{\varepsilon_j}(x_i^j)$. In our case, after a translation and a scaling argument, we get:

$$\begin{aligned}
&\int_{B_{c\delta_j}(x_i^j)} |Du_j|^{p_j}\,dx \\
\geq\ & \inf\Big\{ \int_{B_{c\delta_j}} |Dv|^{p_j}\,dx : v = 0 \text{ on } B_{\varepsilon_j},\ v = u(x_i^j) \text{ on } \partial B_{c\delta_j} \Big\} \\
\geq\ & \varepsilon_j^{\eta_j} \inf\Big\{ \int_{B_{c\delta_j/\varepsilon_j}} |Dv|^{p_j}\,dx : v = 0 \text{ on } B_1,\ v = u(x_i^j) \text{ on } \partial B_{c\delta_j/\varepsilon_j} \Big\} \\
=\ & |u(x_i^j)|^{p_j} \varepsilon_j^{\eta_j} \inf\Big\{ \int_{B_{c\delta_j/\varepsilon_j}} |Dv|^{p_j}\,dx : v = 0 \text{ on } B_1,\ v = \frac{u(x_i^j)}{|u(x_i^j)|} \text{ on } \partial B_{c\delta_j/\varepsilon_j} \Big\}.
\end{aligned}$$

If we sum over the perforations, we obtain

$$\begin{aligned}
&\sum_i \int_{B_{c\delta_j}(x_i^j)} |Du_j|^{p_j}\,dx \\
\geq\ & \sum_i |u(x_i^j)|^{p_j} \varepsilon_j^{\eta_j} \inf\Big\{ \int_{B_{c\delta_j/\varepsilon_j}} |Dv|^{p_j}\,dx : v = 0 \text{ on } B_1,\ v = \frac{u(x_i^j)}{|u(x_i^j)|} \text{ on } \partial B_{c\delta_j/\varepsilon_j} \Big\}.
\end{aligned}$$

We want ε_j to be such that the following quantity is a Riemann sum:

$$\sum_i \delta_j^n |u(x_i^j)|^{p_j} \frac{\varepsilon_j^{\eta_j}}{\delta_j^n} \inf\Big\{ \int_{B_{c\delta_j/\varepsilon_j}} |Dv|^{p_j}\,dx : v = 0 \text{ on } B_1,\ v = \frac{u(x_i^j)}{|u(x_i^j)|} \text{ on } \partial B_{c\delta_j/\varepsilon_j} \Big\}. \quad (5)$$

If there exists $\kappa \in \mathbb{R}^+$ such that

$$\frac{\varepsilon_j^{\eta_j}}{\delta_j^n} \inf\Big\{ \int_{B_{c\delta_j/\varepsilon_j}} |Dv|^{p_j}\,dx : v = 0 \text{ on } B_1,\ v = \frac{u(x_i^j)}{|u(x_i^j)|} \text{ on } \partial B_{c\delta_j/\varepsilon_j} \Big\} \longrightarrow \kappa, \quad (6)$$

then (5) is a Riemann sum converging to the extra term

$$\kappa \int_\Omega |u|^n\,dx \quad (7)$$

as $j \to +\infty$. The argument above will be made rigorous in the following sections.

Our first step consists in the asymptotic analysis of the scaled minimum problems (6). We fix a vector $\nu \in \mathbb{R}^m$ such that $|\nu| = 1$; we will see that the limit is independent of the choice of ν. We want to study

$$\lim_j \delta_j^{-n} \inf\Big\{ \int_{B_{c\delta_j}} |Dv|^{p_j}\,dx : v \in \nu + W_0^{1,p_j}(B_{c\delta_j};\mathbb{R}^m),\ v = 0 \text{ on } B_{\varepsilon_j} \Big\} \quad (8)$$

$$= \lim_j \frac{\varepsilon_j^{\eta_j}}{\delta_j^n} \inf\Big\{ \int_{B_{c\delta_j/\varepsilon_j}} |Dv|^{p_j}\,dx : v \in \nu + W_0^{1,p_j}(B_{c\delta_j/\varepsilon_j};\mathbb{R}^m),\ v = 0 \text{ on } B_1 \Big\} \quad (9)$$

where c is a positive constant. We assume that $\eta_j \not\equiv 0$; for the case $\eta_j \equiv 0$ we refer to [15].

For any unit vector $\nu \in \mathbb{R}^m$ the infimum

$$\inf\Big\{ \int_{B_{c\delta_j/\varepsilon_j}} |Dv|^{p_j}\,dx : v \in \nu + W_0^{1,p_j}(B_{c\delta_j/\varepsilon_j};\mathbb{R}^m),\ v = 0 \text{ on } B_1 \Big\} \quad (10)$$

equals

$$m_j^c := \inf\Big\{ \int_{B_{c\delta_j/\varepsilon_j}} |Dv|^{p_j}\,dx : v \in 1 + W_0^{1,p_j}(B_{c\delta_j/\varepsilon_j};\mathbb{R}),\ v = 0 \text{ on } B_1 \Big\}, \quad (11)$$

where the inf is taken among scalar functions. To check this, we first note that up to rotations it is not restrictive to assume that $\nu = e_1 = (1, 0, \dots, 0)$. On the one hand we can identify each test function v for (11) with a vector-valued test function $\tilde{v}$ for (10) by setting $\tilde{v} = ve_1$, hence we deduce that

$$\inf\Big\{ \int_{B_{c\delta_j/\varepsilon_j}} |Dv|^{p_j}\,dx :\ v \in e_1 + W_0^{1,p_j}(B_{c\delta_j/\varepsilon_j};\mathbb{R}^m),\ v = 0 \text{ on } B_1 \Big\}$$

$$\leq\ \inf\Big\{ \int_{B_{c\delta_j/\varepsilon_j}} |Dv|^{p_j}\,dx :\ v \in 1 + W_0^{1,p_j}(B_{c\delta_j/\varepsilon_j};\mathbb{R}),\ v = 0 \text{ on } B_1 \Big\}.$$

On the other hand, we note that if $\nu = e_1$ in (10), then the minimum must be reached by a function of the form $\tilde{v} = (\tilde{v}^1, 0, \dots, 0)$ (if $\tilde{v}$ has non-zero components $\tilde{v}^j$ for $j \neq 1$ then the energy increases). Taking $\tilde{v}^1 \in 1 + W_0^{1,p_j}(B_{c\delta_j/\varepsilon_j}; \mathbb{R})$ as a test function for (11), we get

$$\begin{aligned} &\inf\Big\{ \int_{B_{c\delta_j/\varepsilon_j}} |Dv|^{p_j}\,dx :\ v \in 1 + W_0^{1,p_j}(B_{c\delta_j/\varepsilon_j}; \mathbb{R}),\ v = 0 \text{ on } B_1 \Big\} \\ \leq\ &\inf\Big\{ \int_{B_{c\delta_j/\varepsilon_j}} |Dv|^{p_j}\,dx :\ v \in e_1 + W_0^{1,p_j}(B_{c\delta_j/\varepsilon_j}; \mathbb{R}^m),\ v = 0 \text{ on } B_1 \Big\}. \end{aligned}$$

Therefore we can restrict our attention to the scalar problem (11) and note that by simmetry reasons the minimum is reached by a radial function $v(x) = w(|x|)$. Now, $w : \mathbb{R}^+ \to \mathbb{R}$ satisfies the Euler equation

$$\frac{\partial}{\partial \rho}\big(|w'(\rho)|^{p_j-2} \rho^{n-1} w'(\rho) \big) = 0$$

and the constraints

$$w(1) = 0, \quad w(c\delta_j/\varepsilon_j) = 1. \tag{12}$$

With no loss of generality we can assume that $w'(\rho) \geq 0$ and we find

$$w(\rho) = \rho^{\frac{-\eta_j}{p_j-1}} \Big(\Big(\frac{\varepsilon_j}{c\delta_j}\Big)^{\frac{\eta_j}{p_j-1}} - 1 \Big)^{-1} + \Big(1 - \Big(\frac{\varepsilon_j}{c\delta_j}\Big)^{\frac{\eta_j}{p_j-1}} \Big)^{-1}.$$

The minimum in (11) then is computed as

$$\begin{aligned} m_j^c &= \omega_{n-1} \int_1^{c\delta_j/\varepsilon_j} |w'(\rho)|^{p_j} \rho^{n-1}\,d\rho \\ &= \omega_{n-1} \int_1^{c\delta_j/\varepsilon_j} \Big| 1 - \Big(\frac{\varepsilon_j}{c\delta_j}\Big)^{\frac{\eta_j}{p_j-1}} \Big|^{1-p_j} \frac{|\eta_j|^{p_j-1}}{(p_j-1)^{p_j-1}} \Big(\rho^{\frac{-\eta_j}{p_j-1}-1} \Big)^{p_j} \rho^{n-1}\,d\rho \\ &= \omega_{n-1} \frac{|\eta_j|^{p_j-1}}{(p_j-1)^{p_j-1}} \Big| 1 - \Big(\frac{\varepsilon_j}{c\delta_j}\Big)^{\frac{\eta_j}{p_j-1}} \Big|^{1-p_j}. \end{aligned} \tag{13}$$

In conclusion the limit in (9) equals

$$\lim_{j\to\infty} \frac{\omega_{n-1}}{(p_j-1)^{p_j-1}} \varepsilon_j^{\eta_j} \delta_j^{-n} |\eta_j|^{p_j-1} \Big| 1 - \Big(\frac{\varepsilon_j}{c\delta_j}\Big)^{\frac{\eta_j}{p_j-1}} \Big|^{1-p_j}. \tag{14}$$

Remark. It is easily seen that the limit

$$\lim_j \varepsilon_j^{\eta_j} \delta_j^{-n} m_j^c$$

is independent of the constant c. Hence it is not restrictive to perform the asymptotic analysis having fixed $c = 1$. To this end we denote by m_j the infimum m_j^1; *i.e.*,

$$\begin{aligned} m_j &= \inf\Big\{ \int_{B_{\delta_j/\varepsilon_j}} |Dv|^{p_j}\,dx :\ v \in 1 + W_0^{1,p_j}(B_{\delta_j/\varepsilon_j}),\ v = 0 \text{ on } B_1 \Big\} \\ &= \inf\Big\{ \int_{B_{\delta_j/\varepsilon_j}} |Dv|^{p_j}\,dx :\ v \in \nu + W_0^{1,p_j}(B_{\delta_j/\varepsilon_j}; \mathbb{R}^m),\ v = 0 \text{ on } B_1 \Big\}. \end{aligned}$$

We know that

$$m_j = \omega_{n-1} \frac{|\eta_j|^{p_j-1}}{(p_j-1)^{p_j-1}} \left| 1 - \left(\frac{\varepsilon_j}{\delta_j}\right)^{\frac{\eta_j}{p_j-1}} \right|^{1-p_j}. \tag{15}$$

We recall that if $n = p$; *i.e.*, $\eta_j \equiv 0$, the critical scaling for the perforations is exponential (see [15] for the details). We expect the exponential scaling to be the critical one also in the case that the sequence (η_j) is 'not too far' from zero: in fact we will find that if $|\eta_j| \approx \delta_j^{\frac{n}{n-1}}$ or $|\eta_j| \ll \delta_j^{\frac{n}{n-1}}$ then the choice $\varepsilon_j = \exp\big(-a\delta_j^{-n/n-1}\big)$ gives an extra term of the form (7) in the Γ-limit.

Afterwards, we will consider $\eta_j > 0$ such that $\eta_j \gg \delta_j^{n/n-1}$. Our *ansatz* is that in (15) the factor

$$\left(1 - \left(\frac{\varepsilon_j}{\delta_j}\right)^{\frac{\eta_j}{p_j-1}}\right)^{1-p_j}$$

converges to some positive constant, hence we can restrict our attention to

$$\lim_{j\to\infty} \frac{\omega_{n-1}}{(p_j-1)^{p_j-1}} \varepsilon_j^{\eta_j} \delta_j^{-n} |\eta_j|^{p_j-1}.$$

We expect the critical scaling to be $\varepsilon_j \backsimeq \delta_j^{n/\eta_j} \eta_j^{\theta/\eta_j}$ for some $\theta > 0$; an explicit calculation will prove that our assumptions are correct.

Finally, we will deal with $\eta_j < 0$ and $|\eta_j| \gg \delta_j^{\frac{n}{n-1}}$. In this case any choice of (ε_j) gives the result we would get if $\eta_j \equiv c < 0$: the Γ-limit is finite (and null) only on the constant function $u \equiv 0$. In this case the compact embedding into continuous functions prevails over the convergence of $p_j \to n^+$.

(1) Exponential regime. Consider the case in which

$$\eta_j = \gamma \delta_j^{\frac{n}{n-1}} + o(\delta_j^{\frac{n}{n-1}}) \tag{16}$$

with $\gamma \in \mathbb{R}$. We will show that if we take

$$\varepsilon_j = \exp\big(-a\delta_j^{\frac{-n}{n-1}}\big),$$

where $a > 0$ is a fixed constant, then the limit in (9) is finite.

In fact, if $\gamma \in \mathbb{R} \setminus \{0\}$ we get

$$\begin{aligned}
\lim_j \frac{\varepsilon_j^{\eta_j} m_j}{\delta_j^n} &= \lim_j \varepsilon_j^{\eta_j} \omega_{n-1} \frac{1}{(p_j-1)^{p_j-1}} \frac{1}{|\eta_j|^{\eta_j}} |\eta_j|^{n-1} \delta_j^{-n} \left| 1 - \left(\frac{\varepsilon_j}{\delta_j}\right)^{\frac{\eta_j}{p_j-1}} \right|^{1-p_j} \\
&= \frac{\omega_{n-1}}{(n-1)^{(n-1)}} e^{-a\gamma} \lim_j \left(\frac{|\eta_j|}{\delta_j^{\frac{n}{n-1}}}\right)^{n-1} \left| 1 - \frac{\exp\Big(-a\delta_j^{-\frac{n}{n-1}} \frac{\eta_j}{p_j-1}\Big)}{\delta_j^{\frac{\eta_j}{p_j-1}}} \right|^{1-p_j} \\
&= \frac{\omega_{n-1}}{(n-1)^{(n-1)}} e^{-a\gamma} \left| \frac{1 - e^{-\frac{a\gamma}{n-1}}}{\gamma} \right|^{1-n} =: \alpha(\gamma).
\end{aligned} \tag{17}$$

If $\gamma = 0$ we have

$$\lim_j \frac{\varepsilon_j^{\eta_j} m_j}{\delta_j^n} = \frac{\omega_{n-1}}{(n-1)^{n-1}} \frac{a^{1-n}}{(n-1)^{1-n}} = \frac{\omega_{n-1}}{a^{n-1}} =: \alpha(0). \tag{18}$$

Note that $\alpha(0)$ equals the limit we get in the case $\eta_j \equiv 0$; note moreover that

$$\lim_{\gamma \to 0} \alpha(\gamma) = \alpha(0).$$

(2) Mixed polynomial-exponential regime. In the case that $\eta_j > 0$ and

$$\eta_j \gg \delta_j^{\frac{n}{n-1}},$$

then the critical scaling is

$$\varepsilon_j = R^{\frac{1}{\eta_j}} \delta_j^{\frac{n}{\eta_j}} \eta_j^{-\frac{n-1}{\eta_j}},$$

with $R > 0$ fixed. The computation of the limit gives

$$\lim_j \frac{\varepsilon_j^{\eta_j} m_j}{\delta_j^n} = \lim_j \frac{\omega_{n-1}}{(p_j-1)^{p_j-1}} R \delta_j^n \eta_j^{-n+1} \delta_j^{-n} \eta_j^{p_j-1} \left(1 - \frac{R^{\frac{1}{p_j-1}} \delta_j^{\frac{n}{p_j-1}} \eta_j^{\frac{1-n}{p_j-1}}}{\delta_j^{\frac{\eta_j}{p_j-1}}} \right)^{1-p_j}.$$

Since

$$\lim_j \frac{R^{\frac{1}{p_j-1}} \delta_j^{\frac{n}{p_j-1}} \eta_j^{\frac{1-n}{p_j-1}}}{\delta_j^{\frac{\eta_j}{p_j-1}}} = 0$$

we have

$$\lim_j \frac{\varepsilon_j^{\eta_j} m_j}{\delta_j^n} = R\omega_{n-1} \lim_j \frac{\eta_j^{-\eta_j}}{(p_j-1)^{p_j-1}} = R \frac{\omega_{n-1}}{(n-1)^{n-1}}.$$

(3) Rigid regime. Finally, we suppose that $\eta_j < 0$ and

$$|\eta_j| \gg \delta_j^{\frac{n}{n-1}}.$$

In this case we will see that for any choice of (ε_j) the functionals (F_j) Γ-converge to the functional $F_\infty : W^{1,n}(\Omega; \mathbb{R}^m) \to [0, +\infty]$ given by

$$F_\infty(u) = \begin{cases} 0 & \text{if } u \equiv 0, \\ +\infty & \text{otherwise.} \end{cases}$$

3. Statement of the main result

The main result of this paper will be stated in Theorem 3.1.

Theorem 3.1. *Let $m, n \in \mathbb{N}$ with $n \geq 2$, $m \geq 1$. Let Ω be a bounded open subset of $\mathbb{R}^n$ with $|\partial\Omega| = 0$. Let $K \subset \mathbb{R}^n$ be a bounded closed set with non-empty interior. Let (δ_j) be a sequence of positive numbers converging to zero; let (η_j) be an infinitesimal sequence of numbers; we set $p_j = n - \eta_j$. Let (ε_j) be a non-negative sequence such that $\varepsilon_j \leq \delta_j/2$. For all $i \in \mathbb{Z}^n$ and $j \in \mathbb{N}$, x_i^j indicates the vector $x_i^j = i\delta_j \in \delta_j\mathbb{Z}^n \subset \mathbb{R}^n$. Let $K_i^{\delta_j} = x_i^j + \varepsilon_j K$. For all $j \in \mathbb{N}$ we denote by Ω_j the periodically perforated domain*

$$\Omega_j = \Omega \setminus \bigcup_{i\in\mathbb{Z}^n} K_i^{\delta_j}. \tag{19}$$

Consider the functionals $F_j : W^{1,p_j}(\Omega;\mathbb{R}^m) \to [0, +\infty]$ defined by

$$F_j(u) = \begin{cases} \int_\Omega |Du|^{p_j}\,dx & \text{if } u = 0 \text{ on } \Omega \setminus \overline{\Omega}_j, \\ +\infty & \text{otherwise.} \end{cases} \tag{20}$$

Let $\varepsilon_j = \varepsilon_j(\delta_j, \eta_j)$ be defined as follows:

(1) exponential regime: *if $\eta_j = \gamma\delta_j^{\frac{n}{n-1}} + o(\delta_j^{\frac{n}{n-1}})$, $\gamma \in \mathbb{R}$, then $\varepsilon_j = \exp(-a\delta_j^{\frac{-n}{n-1}})$, with $a > 0$;*

(2) mixed polynomial-exponential regime: *if $\eta_j > 0$ and $\eta_j \gg \delta_j^{\frac{n}{n-1}}$, then $\varepsilon_j = R^{\frac{1}{\eta_j}}\delta_j^{\frac{n}{\eta_j}}\eta_j^{\frac{1-n}{\eta_j}}$, with $R > 0$.*

Let κ be the positive constant defined by

(1) exponential regime: *if $\eta_j = \gamma\delta_j^{\frac{n}{n-1}} + o(\delta_j^{\frac{n}{n-1}})$ with $\gamma \in \mathbb{R}$, then*

$$\kappa = \frac{\omega_{n-1}}{(n-1)^{(n-1)}} e^{-a\gamma} \left| \frac{1 - e^{-\frac{a\gamma}{n-1}}}{\gamma} \right|^{1-n} \qquad \text{if } \gamma \neq 0,$$

and κ is extended by continuity to the case $\gamma = 0$; i.e., $\kappa = \frac{\omega_{n-1}}{(n-1)^{n-1}}$;

(2) mixed polynomial-exponential regime: if $\eta_j > 0$ and $\eta_j \gg \delta_j^{\frac{n}{n-1}}$, then

$$\kappa = R\frac{\omega_{n-1}}{(n-1)^{n-1}}.$$

Then the functionals (F_j) defined as in (20) Γ-converge (with respect to the strong convergence of $L^1(\Omega;\mathbb{R}^m)$) to the functional $F : W^{1,n}(\Omega;\mathbb{R}^m) \to [0, +\infty]$ given by

$$F(u) = \int_\Omega |Du|^n\,dx + \kappa \int_\Omega |u|^n\,dx. \tag{21}$$

Moreover,

(3) rigid regime: *if $\eta_j < 0$ and $|\eta_j| \gg \delta_j^{\frac{n}{n-1}}$ and (ε_j) is a generic sequence satisfying $0 \leq \varepsilon_j \leq \delta_j/2$,* *then the functionals* (F_j) *defined as in* (20) Γ*-converge (with respect to the strong convergence of* $L^1(\Omega;\mathbb{R}^m)$*) to the functional* $F_\infty : W^{1,n}(\Omega;\mathbb{R}^m) \to [0,+\infty]$ *given by*

$$F_\infty(u) = \begin{cases} 0 & \text{if } u \equiv 0, \\ +\infty & \text{otherwise.} \end{cases} \tag{22}$$

Corollary 3.2 (Convergence of minimum problems). *Let* (F_j) *be a family of functionals of the form* (20)*, and let* $F = \Gamma\text{-}\lim_j F_j$*. Then for all* $\phi \in L^q(\Omega;\mathbb{R}^m)$*, with* $q > \frac{n}{n-1}$*, the minimum values*

$$\mu_j = \inf\Big\{F_j(u) + \langle\phi, u\rangle : u \in W_0^{1,p_j}(\Omega;\mathbb{R}^m)\Big\}$$

converge to

$$\mu = \min\left\{F(u) + \langle\phi, u\rangle : u \in W_0^{1,n}(\Omega;\mathbb{R}^m)\right\}.$$

Moreover, if (u_j) *is such that* $F_j(u_j) + \langle\phi, u_j\rangle = \mu_j + o(1)$ *as* $j \to \infty$*, then it admits a subsequence converging in* $L^1(\Omega;\mathbb{R}^m)$ *to a solution of the problem defining* μ*.*

Theorem 3.1 will be proved in Sections 5 and 6.

Remark. We can rephrase the result in terms of *equivalence by* Γ*-convergence* following the terminology introduced by Braides and Truskinovsky in [7].

Definition 3.3 (Equivalence by Γ-convergence). Let (F_ε), (G_ε) be two families of functionals. We say that (F_ε) and (G_ε) are *equivalent by* Γ*-convergence* if and only if for each sequence (ε_j) there exists a subsequence (ε_{j_k}) such that

$$\Gamma\text{-}\lim_k F_{\varepsilon_{j_k}} = \Gamma\text{-}\lim_k G_{\varepsilon_{j_k}}$$

and these limits are non-trivial; *i.e.*, they are not identically equal to $+\infty$ and they do not assume the value $-\infty$.

In [2] Ansini and Braides dealt with the Γ-convergence of functionals on $W^{1,p}(\Omega;\mathbb{R}^m)$ of the form

$$\mathcal{F}_j(u) = \begin{cases} \displaystyle\int_\Omega f(Du)\,dx & \text{if } u = 0 \text{ on } \displaystyle\bigcup_{i\in\mathbb{Z}^n} K_i^{\delta_j} \cap \Omega, \\ +\infty & \text{otherwise,} \end{cases} \tag{23}$$

with fixed $p < n$ and f a quasiconvex function satisfying a growth condition of order p. They proved that, under general assumptions, the choice $\varepsilon_j = \delta_j^{\frac{n}{n-p}}$ guarantees the Γ-convergence of $\mathcal{F}_j$ to a functional $\mathcal{F} : W^{1,p}(\Omega;\mathbb{R}^m) \to [0,+\infty]$ of the form

$$\mathcal{F}(u) = \int_\Omega f(Du)\,dx + \int_\Omega \varphi(u)\,dx,$$

where $\varphi : \mathbb{R}^m \to [0,+\infty)$ is given by a capacitary formula. This result can be reformulated as follows: the family $(\mathcal{F}_j)$ is equivalent to the functionals $\mathcal{G}_j : W^{1,p}(\Omega;\mathbb{R}^m) \to [0,+\infty]$ defined by

$$\mathcal{G}_j(u) = \int_\Omega f(Du)\,dx + \frac{\varepsilon_j^{n-p}}{\delta_j^n}\int_\Omega \varphi(u)\,dx,$$

with respect to $L^p(\Omega;\mathbb{R}^m)$-convergence.

A similar argument can be applied to the case in which $\mathcal{F}_j$ are defined as in (23) but p equals n, which was developed in [15]. In this case $(\mathcal{F}_j)$ are equivalent to the functionals $\mathcal{G}_j$ given by

$$\mathcal{G}_j(u) = \int_\Omega f(Du)\,dx + \frac{|\log\varepsilon_j|^{1-n}}{\delta_j^n}\int_\Omega \varphi(u)\,dx$$

with respect to $L^n(\Omega;\mathbb{R}^m)$-convergence.

In the case we deal with in this paper, the statement of Theorem 3.1, taking into account (13) and (15), implies that the functionals $F_j : W^{1,p_j}(\Omega;\mathbb{R}^m) \to [0,+\infty]$ in (20) are equivalent to the family (G_j) defined by

$$G_j(u) = \int_\Omega |Du|^n\,dx + \frac{\omega_{n-1}}{(p_j-1)^{p_j-1}}\varepsilon_j^{\eta_j}\delta_j^{-n}|\eta_j|^{p_j-1}\Big|1-\Big(\frac{\varepsilon_j}{\delta_j}\Big)^{\frac{\eta_j}{p_j-1}}\Big|^{1-p_j}\int_\Omega |u|^n\,dx$$

with respect to $L^1(\Omega;\mathbb{R}^m)$-convergence.

4. Preliminary results

4.1. A lemma for varying domains

In this section we recall a technical Lemma by Ansini and Braides (see [2]) which allows to modify sequences of functions close to the perforations.

Lemma 4.1. *Let (u_j) converge strongly to u in $L^1(\Omega;\mathbb{R}^m)$; let $\sup_j F_j(u_j) < \infty$. Let (ρ_j) be a positive sequence of the form $\rho_j = \overline{c}\delta_j$, where $\overline{c} < \frac{1}{2}$. For all $j \in \mathbb{N}$ we define*

$$Z_j = \left\{ i \in \mathbb{Z}^n : dist\big(x_i^j, \mathbb{R}^n \setminus \Omega\big) > \delta_j \right\}.$$

We fix $k \in \mathbb{N}$. Then, for all $i \in Z_j$ there exists $k_i \in \{0,1,\dots,k-1\}$ such that, having set

$$C_i^j = \Big\{x \in \Omega : \frac{1}{2^{k_i+1}}\rho_j < |x - x_i^j| < \frac{1}{2^{k_i}}\rho_j\Big\}, \tag{24}$$

$$u_j^i = |C_i^j|^{-1}\int_{C_i^j} u_j\,dx, \qquad \rho_j^i = \frac{3}{4}2^{-k_i}\rho_j,$$

there exists a sequence (w_j), with $w_j \to u$ in $L^1(\Omega;\mathbb{R}^m)$, such that

$$w_j = u_j \quad \text{on } \Omega \setminus \bigcup_{i\in Z_j} C_i^j, \tag{25}$$

$$w_j(x) = u_j^i \quad \text{if } |x - x_i^j| = \rho_j^i, \tag{26}$$

$$\text{and} \int_\Omega \big||Dw_j|^{p_j} - |Du_j|^{p_j}\big|\,dx \le \frac{c}{k}. \tag{27}$$

Proof. In [2] Ansini and Braides dealt with integral functionals in which the integrands satisfy a growth condition of order p (p fixed). Neverthless, the proof of Lemma [2, 3.1] can be repeated word for word; we only need to notice that the constant which appears in the estimate of the gradients (now depending on p_j) is equi-bounded. □

4.2. A discretization argument

The extra term of the Γ-limit can be obtained through a discretization argument, as explained in the following proposition.

Proposition 4.2. *Let (u_j) be a bounded sequence in $L^\infty(\Omega;\mathbb{R}^m)$ such that $\sup_j F_j(u_j) < \infty$. We assume that $u_j \to u$ in $L^1(\Omega;\mathbb{R}^m)$. Let (ρ_j) be a positive sequence of the form $\rho_j = \overline{c}\delta_j$, where $\overline{c} < 1/2$. We fix $k \in \mathbb{N}$; for all $i \in Z_j$ we consider an annuli C_i^j of the form (24) for an arbitrary choice of $k_i \in \{0, 1, \dots, k-1\}$. We denote by u_j^i the mean value of u_j on C_i^j and by Q_i^j the cube $Q_i^j = x_i^j + \Big(-\frac{\delta_j}{2}, \frac{\delta_j}{2}\Big)^n$; let ψ_j be defined as*

$$\psi_j = \sum_{i\in Z_j} |u_j^i|^{p_j} \chi_{Q_i^j}. \tag{28}$$

Then

$$\lim_{j\to\infty} \int_\Omega |\psi_j - |u|^n|\, dx = 0. \tag{29}$$

Proof. Since $u_j \to u$ in $L^1(\Omega;\mathbb{R}^m)$, the limit in (29) equals the limits

$$\begin{aligned} \lim_j \int_\Omega |\psi_j - |u_j|^{p_j}|\, dx &= \lim_j \int_\Omega |\sum_{i\in Z_j} |u_j^i|^{p_j}\chi_{Q_i^j} - |u_j|^{p_j}|\, dx \\ &= \lim_j \sum_{i\in Z_j} \int_{Q_i^j} ||u_j^i|^{p_j} - |u_j|^{p_j}|\, dx. \end{aligned}$$

We use the Lipschitz condition

$$||u_j^i|^{p_j} - |u_j|^{p_j}| \le c|u_j^i - u_j|\Big(|u_j^i|^{p_j-1} + |u_j|^{p_j-1}\Big)$$

and Hölder's inequality to get

$$\begin{aligned} \int_{Q_i^j} ||u_j^i|^{p_j} - |u_j|^{p_j}|\, dx &\le c\Big(\sup_j \|u_j\|_\infty^{p_j-1}\Big) \int_{Q_i^j} |u_j - u_j^i|\, dx \\ &\le c\delta_j^{n(p_j-1)/p_j}\Big(\int_{Q_i^j} |u_j - u_j^i|^{p_j}\, dx\Big)^{\frac{1}{p_j}}. \end{aligned}$$

We want to estimate the last integral with a quantity independent of i; to this end we apply Poincaré-Wirtinger's inequality in the following form:

Let $A \subset \mathbb{R}^n$ be an open bounded connected set and let B be an open subset of A. Let $\rho > 0$ be fixed. Let (p_j) be a real sequence converging to n as $j \to +\infty$. Then there exists a constant $C = C(n, A, B)$ such that for all $v \in W^{1,p_j}(\rho A;\mathbb{R}^m)$ we have

$$\Big(\int_{\rho A} \Big| v - \frac{1}{|\rho B|}\int_{\rho B} v\, \Big|^{p_j} dx\Big)^{1/p_j} \le \rho C\Big(\int_{\rho A} |Dv|^{p_j}\, dx\Big)^{1/p_j}.$$

We fix $j \in \mathbb{N}$; for all $i \in Z_j$ there exists a positive constant $\alpha = \alpha(n, C_i^j)$ (independent of the exponent p_j) such that

$$\Big(\int_{Q_i^j} |u_j^i - u_j|^{p_j}\, dx\Big)^{\frac{1}{p_j}} \leq \alpha\delta_j\Big(\int_{Q_i^j} |Du_j|^{p_j}\, dx\Big)^{\frac{1}{p_j}}.$$

Note that α depends on C_i^j and hence on the choice of $k_i \in \{0, 1, \dots, k-1\}$; under our assumptions the family of homothetic annuli $\{C_i^j\}$ is finite (for fixed $j \in \mathbb{N}$), hence we can define $\alpha' = \alpha'(n) := \max_i\{\alpha(n, C_i^j)\}$. In conclusion there exists $\alpha' > 0$ such that

$$\Big(\int_{Q_i^j} |u_j - u_j^i|^{p_j}\, dx\Big)^{\frac{1}{p_j}} \leq \alpha'\delta_j\Big(\int_{Q_i^j} |Du_j|^{p_j}\, dx\Big)^{\frac{1}{p_j}}.$$

Now,

$$\begin{aligned} \lim_j \sum_{i\in Z_j} \int_{Q_i^j} \big||u_j^i|^{p_j} - |u_j|^{p_j}\big|\, dx &\leq \lim_j \sum_{i\in Z_j} c\delta_j^{n(p_j-1)/p_j}\Big(\int_{Q_i^j} |u_j - u_j^i|^{p_j}\, dx\Big)^{\frac{1}{p_j}} \\ &\leq \lim_j c\delta_j^{n(p_j-1)/p_j}\delta_j \sum_{i\in Z_j}\Big(\int_{Q_i^j} |Du_j|^{p_j}\, dx\Big)^{\frac{1}{p_j}}. \end{aligned}$$

For all $j \in \mathbb{N}$ the function $y \mapsto y^{\frac{1}{p_j}}$ is concave; in particular, if $\{t_1, \dots, t_N\} \subset \mathbb{R}^+$ are such that $\sum_i t_i = 1$ and $\{y_1, \dots, y_N\} \subset \mathbb{R}^+$, then

$$\sum_i t_i(y_i)^{\frac{1}{p_j}} \leq \Big(\sum_i t_i y_i\Big)^{\frac{1}{p_j}}.$$

Therefore

$$\begin{aligned} \sum_{i\in Z_j} \frac{1}{\# Z_j}\Big(\int_{Q_i^j} |Du_j|^{p_j}\, dx\Big)^{\frac{1}{p_j}} &\leq \Big(\sum_{i\in Z_j} \frac{1}{\# Z_j}\int_{Q_i^j} |Du_j|^{p_j}\, dx\Big)^{\frac{1}{p_j}} \\ &\leq \Big(\frac{1}{\# Z_j}\Big)^{\frac{1}{p_j}}\Big(\int_{\Omega} |Du_j|^{p_j}\, dx\Big)^{\frac{1}{p_j}}. \end{aligned}$$

Since $\# Z_j \simeq |\Omega|/\delta_j^n$, we have $\# Z_j^{(1-1/p_j)}\delta_j^{n(1-1/p_j)} \leq c$, then

$$\begin{aligned} \lim_j \sum_{i\in Z_j} \int_{Q_i^j} \big||u_j^i|^{p_j} - |u_j|^{p_j}\big|\, dx &\leq c\lim_j \delta_j^{n(p_j-1)/p_j+1} \# Z_j \frac{1}{(\# Z_j)^{1/p_j}}\Big(\int_{\Omega} |Du_j|^{p_j}\, dx\Big)^{\frac{1}{p_j}} \\ &\leq c\lim_j \delta_j = 0. \end{aligned}$$

In conclusion

$$\lim_j \int_{\Omega} |\psi_j - |u|^n|\, dx = 0.$$

□

5. Non-degenerate regimes

In this section we will prove the Γ-convergence result for the exponential and the mixed polynomial-exponential regimes; in what follows (ε_j) and κ are defined as in the statement of Theorem 3.1. We will first consider the case $K = \overline{B}_1$, the closure of the unit ball, and then conclude that the results are indeed independent of the form of K, provided it has a non-empty interior.

5.1. Liminf inequality - Spherical perforations

In the case of fixed p, the first term in the limit functional (21) can be dealt with by a simple lower-semicontinuity argument. In our case, with varying p_j, we note that if $u_j \to u$ in $L^1(\Omega;\mathbb{R}^m)$ then

$$\int_\Omega |Du|^n \, dx \leq \liminf_j \int_\Omega |Du_j|^{p_j} \, dx. \tag{30}$$

In fact, let $p < n$ be fixed. By Hölder's inequality we have

$$\begin{aligned}\int_\Omega |Du|^p \, dx &\leq \liminf_j \int_\Omega |Du_j|^p \, dx \leq \liminf_j \Big(\int_\Omega |Du_j|^{p_j} \, dx \Big)^{p/p_j} |\Omega|^{1-p/p_j} \\ &\leq \liminf_j \Big(\int_\Omega |Du_j|^{p_j} \, dx \Big)^{p/n} |\Omega|^{1-p/n}.\end{aligned}$$

If we evaluate the liminf for $p \to n^-$ we get

$$\begin{aligned}\liminf_{p\to n^-} \int_\Omega |Du|^p \, dx &\leq \liminf_{p\to n^-} \Big(\liminf_j \int_\Omega |Du_j|^{p_j} \, dx \Big)^{p/n} |\Omega|^{1-p/n} \\ &= \liminf_j \int_\Omega |Du_j|^{p_j} \, dx.\end{aligned}$$

Fatou's Lemma implies that

$$\liminf_{p\to n^-} \int_\Omega |Du|^p \, dx \geq \int_\Omega \liminf_{p\to n^-} |Du|^p \, dx = \int_\Omega |Du|^n \, dx.$$

In conclusion we get (30):

$$\int_\Omega |Du|^n \, dx \leq \liminf_{p\to n^-} \int_\Omega |Du|^p \, dx \leq \liminf_j \int_\Omega |Du_j|^{p_j} \, dx.$$

We are now ready to prove the liminf inequality by focusing on the effect of the perforations. Let $u \in W^{1,n}(\Omega;\mathbb{R}^m)$ and let $u_j \to u$ in $L^1(\Omega;\mathbb{R}^m)$ be such that $\sup_j F_j(u_j) < \infty$ (note that for all $p < n$ the functions (u_j) are equi-bounded in $W^{1,p}(\Omega;\mathbb{R}^m)$ and hence $u_j \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^m)$). We denote by (ρ_j) a sequence of the form $\rho_j = \overline{c}\delta_j$, with $\overline{c} < 1/2$.

Proposition 5.1 (Liminf inequality). *The following inequality holds:*

$$\liminf_j \int_\Omega |Du_j|^{p_j} \, dx \geq \int_\Omega |Du|^n \, dx + \kappa \int_\Omega |u|^n \, dx.$$

Proof. Let $k \in \mathbb{N}$. By applying Lemma 4.1 to (u_j) we get a sequence $w_j \to u$ which will be used as a technical device to prove the liminf inequality. We recall that in particular $w_j = u_j$ on $\Omega \setminus \bigcup_{i\in Z_j} C_i^j$ and $w_j(x) = u_j^i$ for $|x - x_i^j| = \rho_j^i$, where $\rho_j^i = \frac{3}{4}\rho_j 2^{-k_i}$, for fixed $k_i \in \{0, \dots, k-1\}$.

We denote by E_j the set

$$E_j = \bigcup_{i\in Z_j} B_i^j, \text{ where } B_i^j = B_{\rho_j^i}(x_i^j).$$

We treat separately the contribution of $|Du_j|^{p_j}$ on $\Omega \setminus E_j$ and on E_j (step **A** and **B** respectively).

A. We first deal with the contribution of the integrals on $\Omega \setminus E_j$. We will prove that

$$\liminf_j \int_{\Omega\setminus E_j} |Du_j|^{p_j}\,dx \geq \int_\Omega |Du|^n\,dx. \tag{31}$$

Let

$$v_j(x) = \begin{cases} u_j^i & \text{for } x \in B_i^j,\ i \in Z_j, \\ w_j(x) & \text{for } x \in \Omega \setminus E_j. \end{cases}$$

Note that there exists a function v such that $v_j \to v$ in $L^1(\Omega; \mathbb{R}^m)$ upon passing to subsequences. Let $\chi_j = \chi_{\Omega\setminus \bigcup_{i\in Z_j} B_{\rho_j}(x_i^j)}$; by construction there exists a constant $\gamma \in \mathbb{R}^+$ such that χ_j converges weakly* to γ in L^∞ (see e.g. [6, Example 2.4]). There follows that $v_j\chi_j \rightharpoonup \gamma v$ in L^1 and $u_j\chi_j \rightharpoonup \gamma u$ in L^1. Since $v_j\chi_j \equiv u_j\chi_j$ we can deduce that $u = v$. From Lemma 4.1 we obtain

$$\begin{aligned} \liminf_j \int_{\Omega\setminus E_j} |Du_j|^{p_j}\,dx + \frac{c}{k} &\geq \liminf_j \int_{\Omega\setminus E_j} |Dw_j|^{p_j}\,dx \\ &= \liminf_j \int_\Omega |Dv_j|^{p_j}\,dx \geq \int_\Omega |Du|^n\,dx. \end{aligned}$$

By the arbitrariness of k we get (31).

B. We now turn our attention to the contribution of $|Du_j|^{p_j}$ on E_j. We will prove that

$$\liminf_j \int_{E_j} |Du_j|^{p_j}\,dx \geq \kappa \int_\Omega |u|^n\,dx. \tag{32}$$

1.B We first assume that (u_j) is a bounded sequence in $L^\infty(\Omega; \mathbb{R}^m)$. Lemma 4.1 implies that

$$\begin{aligned} \liminf_j \int_{E_j} |Du_j|^{p_j}\,dx &\geq \liminf_j \int_{E_j} |Dw_j|^{p_j}\,dx - \frac{c}{k} \\ &= \liminf_j \Big(\sum_{i\in Z_j} \int_{B_i^j} |Dw_j|^{p_j}\,dx\Big) - \frac{c}{k}. \end{aligned}$$

We fix $j \in \mathbb{N}$, $i \in Z_j$ and estimate $\int_{B_i^j} |Dw_j|^{p_j}\,dx$. By modifying w_j we define

$$\tilde{w}_j^i(x) = \begin{cases} w_j(x + x_i^j) & \text{for } |x| \le \rho_j^i, \\ u_j^i & \text{otherwise.} \end{cases}$$

Having set $T_j = \frac{\rho_j}{\varepsilon_j}$, we define $\zeta \in u_j^i + W_0^{1,p_j}(B_{T_j}; \mathbb{R}^m)$ as $\zeta(y) = \tilde{w}_j^i(\varepsilon_j y)$; note that ζ vanishes on B_1. Now,

$$\begin{aligned} \int_{B_i^j} |Dw_j(x)|^{p_j}\,dx &= \int_{B_{\rho_j}} |D\tilde{w}_j^i(x)|^{p_j}\,dx = \varepsilon_j^{\eta_j} \int_{B_{T_j}} |D\zeta(y)|^{p_j}\,dy \\ \ge\ & \varepsilon_j^{\eta_j} \inf\Big\{ \int_{B_{T_j}} |Dv(y)|^{p_j}\,dy : v \in u_j^i + W_0^{1,p_j}(B_{T_j}; \mathbb{R}^m),\ v = 0 \text{ on } B_1 \Big\} \\ =\ & |u_j^i|^{p_j} \varepsilon_j^{\eta_j} \inf\Big\{ \int_{B_{T_j}} |Dv(y)|^{p_j}\,dy : v \in \frac{u_j^i}{|u_j^i|} + W_0^{1,p_j}(B_{T_j}; \mathbb{R}^m),\ v = 0 \text{ on } B_1 \Big\} \\ =\ & |u_j^i|^{p_j} \varepsilon_j^{\eta_j} m_j^{\bar{c}}. \end{aligned}$$

In Section 2 we proved that

$$\lim_{j\to\infty} \frac{\varepsilon_j^{\eta_j} m_j^{\bar{c}}}{\delta_j^n} = \lim_{j\to\infty} \frac{\varepsilon_j^{\eta_j} m_j}{\delta_j^n} = \kappa.$$

Summing up all the contributions on B_i^j, we deduce that

$$\begin{aligned} \liminf_j \int_{E_j} |Du_j|^{p_j}\,dx \ &\ge\ \liminf_j \sum_{i\in Z_j} \int_{B_i^j} |Dw_j|^{p_j}\,dx - \frac{c}{k} \\ &\ge\ \liminf_j \sum_{i\in Z_j} |u_j^i|^{p_j} \delta_j^n \frac{\varepsilon_j^{\eta_j} m_j^{\bar{c}}}{\delta_j^n} - \frac{c}{k} \\ &\ge\ \kappa \liminf_j \sum_{i\in Z_j} |u_j^i|^{p_j} \delta_j^n - \frac{c}{k}. \end{aligned}$$

Proposition 4.2 implies that

$$\lim_j \sum_{i\in Z_j} |u_j^i|^{p_j} \delta_j^n = \int_\Omega |u|^n\,dx,$$

hence

$$\liminf_j \int_{E_j} |Du_j|^{p_j}\,dx \ge \kappa \int_\Omega |u|^n\,dx - \frac{c}{k}.$$

Summing up the contributions on E_j and $\Omega \setminus E_j$ and taking into account the arbitrariness of k we get

$$\liminf_j F_j(u_j) \ge \int_\Omega |Du|^n\,dx + \kappa \int_\Omega |u|^n\,dx.$$

2.B We now remove the boundedness assumption on (u_j). By [4, Lemma 3.5], upon passing to a subsequence, for all $M \in \mathbb{N}$ and $\eta > 0$ there exists $R_M > M$ and a Lipschitz function Φ_M of Lipschitz constant 1 such that

$$\begin{cases} \Phi_M(z) = z & \text{if } |z| < R_M, \\ \Phi_M(z) = 0 & \text{if } |z| > 2R_M, \\ \lim_j F_j(u_j) \geq \liminf_j F_j(\Phi_M(u_j)) - \eta. \end{cases}$$

If we apply Lemma 4.1 and Proposition 4.2 to the sequence $(\Phi_M(u_j))$ we get

$$\begin{aligned} \liminf_j \int_{E_j} |D\Phi_M(u_j)|^{p_j}\, dx + \frac{c}{k} &\geq \kappa \liminf_j \sum_{i \in Z_j} \delta_j^n |(\Phi_M(u))_j^i|^{p_j} \\ &= \kappa \int_\Omega |\Phi_M(u)|^n\, dx. \end{aligned}$$

Since k is arbitrary we obtain

$$\liminf_j F_j(\Phi_M(u_j)) \geq \int_\Omega |D(\Phi_M(u))|^n\, dx + \kappa \int_\Omega |\Phi_M(u)|^n\, dx.$$

Now, Lemma [4, 3.5] implies that

$$\lim_j F_j(u_j) + \eta \geq \int_\Omega |D(\Phi_M(u))|^n\, dx + \kappa \int_\Omega |\Phi_M(u)|^n\, dx.$$

We can let $M \to \infty$ and note that $\Phi_M(u) \rightharpoonup u$ in $W^{1,n}(\Omega; \mathbb{R}^m)$ to get

$$\lim_j F_j(u_j) + \eta \geq \int_\Omega |Du|^n\, dx + \kappa \int_\Omega |u|^n\, dx.$$

By letting $\eta \to 0$ we obtain the thesis. □

5.2. Limsup inequality - Spherical perforations

Proposition 5.2 (Limsup inequality). *For all $u \in W^{1,n}(\Omega; \mathbb{R}^m)$ there exists a sequence (u_j) such that $u_j \to u$ in $L^1(\Omega; \mathbb{R}^m)$ and*

$$\limsup_j F_j(u_j) \leq \int_\Omega |Du|^n\, dx + \kappa \int_\Omega |u|^n\, dx.$$

Proof. We will first assume that the target u is a Lipschitz function and then we will deal with the general case.

1. Let $u \in \mathrm{Lip}(\Omega; \mathbb{R}^m)$ (in particular $u \in L^\infty(\Omega; \mathbb{R}^m)$). For fixed $j \in \mathbb{N}$ we denote by $\phi_j(x) = \varphi_j(|x|)$ the radial minimizing function for the problem

$$\min \Big\{ \int_{B_{\bar{c}\delta_j}} |Du_j|^{p_j} :\ v \in 1 + W_0^{1,p_j}(B_{\bar{c}\delta_j}),\ v = 0 \text{ on } B_{\varepsilon_j} \Big\},$$

where $\overline{c} < 1/2$ is fixed. By a simple calculation we get

$$\varphi_j(\rho) = \begin{cases} \rho^{\frac{\eta_j}{1-p_j}} \Big((\overline{c}\delta_j)^{\frac{\eta_j}{1-p_j}} - \varepsilon_j^{\frac{\eta_j}{1-p_j}} \Big)^{-1} - \Big(\Big(\dfrac{\overline{c}\delta_j}{\varepsilon_j} \Big)^{\frac{\eta_j}{1-p_j}} - 1 \Big)^{-1} & \text{for } \rho > \varepsilon, \\ 0 & \text{for } 0 \le \rho \le \varepsilon. \end{cases}$$

We will build a recovery sequence (u_j) for u by dealing separately with the indices $i \in Z_j$ and $i \in Z'_j = \{i \in \mathbb{Z}^n \backslash Z_j : B_{\varepsilon_j}(x_i^j) \cap \Omega \neq \emptyset\}$ (step **1.A** and **1.B** respectively).

1.A We first consider the perforations such that $i \in Z_j$. We denote by u_j^i the average integral $u_j^i = |C_i^j|^{-1} \int_{C_i^j} u\, dx$, where C_i^j is as in Proposition 4.2. For $x \in B_{\overline{c}\delta_j}(x_i^j)$ we set

$$u_j(x) = u(x)\phi_j(x - x_i^j).$$

Let $\lambda > 0,\ p > 1$ be fixed and let $c_\lambda > 0$ be such that for all $a, b > 0$ we have

$$(a+b)^p \le c_\lambda a^p + (1+\lambda) b^p; \tag{33}$$

c_λ is equi-bounded as $\lambda \to 0$ and $p \to n$. We have:

$$\begin{aligned} &\int_{B_{\overline{c}\delta_j}(x_i^j)} |Du_j(x)|^{p_j}\, dx \\ \le\ & c_\lambda \int_{B_{\overline{c}\delta_j}(x_i^j)} |Du(x)|^{p_j}\, dx + (1+\lambda) \int_{B_{\overline{c}\delta_j}(x_i^j)} |u(x)|^{p_j} |D\phi_j(x - x_i^j)|^{p_j}\, dx \\ \le\ & c_\lambda \int_{B_{\overline{c}\delta_j}(x_i^j)} |Du|^{p_j}\, dx + (1+\lambda) \int_{B_{\overline{c}\delta_j}(x_i^j)} |u_j^i|^{p_j} |D\phi_j(x - x_i^j)|^{p_j}\, dx \\ & \quad + (1+\lambda) \int_{B_{\overline{c}\delta_j}(x_i^j)} ||u|^{p_j} - |u_j^i|^{p_j}| |D\phi_j(x - x_i^j)|^{p_j}\, dx. \end{aligned}$$

Since u is Lipschitz we have

$$\begin{aligned} \int_{B_{\overline{c}\delta_j}(x_i^j)} ||u|^{p_j} - |u_j^i|^{p_j}| |D\phi_j(x - x_i^j)|^{p_j}\, dx &\le \int_{B_{\overline{c}\delta_j}(x_i^j)} c\|u\|_\infty^{p_j - 1} |u - u_j^i| |D\phi_j(x - x_i^j)|^{p_j}\, dx \\ &\le \int_{B_{\overline{c}\delta_j}} c\delta_j |D\phi_j|^{p_j}\, dx \end{aligned}$$

and then

$$\begin{aligned} \int_{B_{\overline{c}\delta_j}(x_i^j)} |Du_j(x)|^{p_j}\, dx &\le c_\lambda \int_{B_{\overline{c}\delta_j}(x_i^j)} |Du(x)|^{p_j}\, dx + (1+\lambda) \int_{B_{\overline{c}\delta_j}} c\delta_j |D\phi_j|^{p_j}\, dx \\ &\quad + (1+\lambda) |u_j^i|^{p_j} \int_{B_{\overline{c}\delta_j}} |D\phi_j|^{p_j}\, dx. \end{aligned}$$

We denote by G_j the set

$$G_j = \bigcup_{i \in Z_j} B_{\overline{c}\delta_j}(x_i^j).$$

1.B Let $i \in Z'_j$. For $x \in B_{\overline{c}\delta_j}(x_i^j) \cap \Omega$ we set $u_j(x) = u(x)\phi_j(x - x_i^j)$. By (33) we get

$$\int_{B_{\overline{c}\delta_j}(x_i^j)\cap\Omega} |Du_j|^{p_j}\,dx \;\leq\; c_\lambda \int_{B_{\overline{c}\delta_j}(x_i^j)\cap\Omega} |Du|^{p_j}\,dx + c(1+\lambda)\int_{B_{\overline{c}\delta_j}} |D\phi_j(x - x_i^j)|^{p_j}\,dx.$$

We denote by G'_j the set

$$G'_j = \bigcup_{i\in Z'_j} B_{\overline{c}\delta_j}(x_i^j) \cap \Omega,$$

while Ω'_j indicates

$$\Omega'_j = \bigcup_{i\in Z'_j} Q_i^j.$$

In conclusion we set $u_j(x) = u(x)$ on $\Omega \setminus (G_j \cup G'_j)$ and hence we get a recovery sequence for the target function u. In fact:

$$\begin{aligned}
&\int_\Omega |Du_j|^{p_j}\,dx \\
=\;& \int_{G_j} |Du_j|^{p_j}\,dx + \int_{G'_j} |Du_j|^{p_j}\,dx + \int_{\Omega\setminus(G_j\cup G'_j)} |Du_j|^{p_j}\,dx \\
\leq\;& c_\lambda \sum_{i\in Z_j} \int_{B_{\overline{c}\delta_j}(x_i^j)} |Du|^{p_j}\,dx + c_\lambda \sum_{i\in Z'_j} \int_{B_{\overline{c}\delta_j}(x_i^j)\cap\Omega} |Du|^{p_j}\,dx \\
&\quad + \int_{\Omega\setminus(G_j\cup G'_j)} |Du|^{p_j}\,dx + (1+\lambda)\delta_j^n \sum_{i\in Z_j} |u_j^i|^{p_j}\delta_j^{-n} \int_{B_{\overline{c}\delta_j}} |D\phi_j|^{p_j}\,dx \\
&\quad + c(1+\lambda)\delta_j\delta_j^n \sum_{i\in Z_j} \delta_j^{-n} \int_{B_{\overline{c}\delta_j}} |D\phi_j|^{p_j}\,dx + c(1+\lambda)|\Omega'_j|\delta_j^{-n} \int_{B_{\overline{c}\delta_j}} |D\phi_j|^{p_j}\,dx.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\int_\Omega |Du_j|^{p_j}\,dx \;\leq\;& \int_\Omega |Du|^{p_j}\,dx + c_\lambda \int_{G_j\bigcup G'_j} |Du|^{p_j}\,dx + (1+\lambda)c\delta_j|\Omega| \\
&+ (1+\lambda)\delta_j^n \sum_{i\in Z_j} |u_j^i|^{p_j}\delta_j^{-n} \int_{B_{\overline{c}\delta_j}} |D\phi_j|^{p_j}\,dx + (1+\lambda)|\Omega'_j|c.
\end{aligned}$$

Taking into account that

$$\lim_j \delta_j^{-n} \int_{B_{\overline{c}\delta_j}} |D\phi_j|^{p_j}\,dx = \kappa \quad\text{and}\quad \lim_j |\Omega'_j| = |\partial\Omega| = 0,$$

we get

$$\begin{aligned}
\limsup_j \int_\Omega |Du_j|^{p_j}\,dx \;\leq\;& \limsup_j \int_\Omega |Du|^{p_j}\,dx + (1+\lambda)\kappa \limsup_j \sum_{i\in Z_j} |u_j^i|^{p_j}\delta_j^n \\
&+ c_\lambda \limsup_j \int_{G_j\bigcup G'_j} |Du|^{p_j}\,dx.
\end{aligned}$$

Since $\lim_j |G_j| = \overline{c}|\Omega|$ and $\lim_j |G'_j| = 0$, we obtain

$$c_\lambda \limsup_j \int_{G_j \bigcup G'_j} |Du|^{p_j}\, dx = c_\lambda o(1) \quad \text{as } \overline{c} \to 0.$$

By Fatou's Lemma and Proposition 4.2 we get

$$\limsup_j \int_\Omega |Du_j|^{p_j}\, dx \le \int_\Omega |Du|^n\, dx + (1+\lambda)\kappa \int_\Omega |u|^n\, dx + c_\lambda o(1) \quad \text{as } \overline{c} \to 0.$$

Finally, we let $\overline{c} \to 0$ and then $\lambda \to 0$, and we obtain the desired inequality

$$\limsup_j \int_\Omega |Du_j|^{p_j}\, dx \le \int_\Omega |Du|^n\, dx + \kappa \int_\Omega |u|^n\, dx.$$

2. We now deal with the general case. Let $u \in W^{1,n}(\Omega;\mathbb{R}^m)$; u can be approximated by a sequence $(u_k) \subset \mathrm{Lip}(\Omega;\mathbb{R}^m) \cap W^{1,n}(\Omega;\mathbb{R}^m)$ with respect to the $W^{1,n}$-norm. For fixed $k \in \mathbb{N}$ we proved that $\Gamma\text{-}\limsup_j F_j(u_k) \le F(u_k)$. Since the Γ-lim sup is a lower semicontinuous functional, we get

$$\Gamma\text{-}\limsup_j F(u) \le \liminf_k \Gamma\text{-}\limsup_j F_j(u_k) \le \liminf_k F(u_k) = F(u).$$

□

5.3. Non-spherical perforations

In this section we will deal with the Γ-convergence result for the general case: $K \subset \mathbb{R}^n$ is a bounded closed set with non-empty interior. We will show how in the non-degenerate regimes the results are indeed independent of the form of K. In particular, we will prove that

$$\kappa^K := \lim_j \delta_j^{-n} \inf\Big\{ \int_{B_{\delta_j}} |Dv|^{p_j}\, dx : \ v \in \nu + W_0^{1,p_j}(B_{\delta_j};\mathbb{R}^m),\ v = 0 \text{ on } \varepsilon_j K \Big\} \qquad (34)$$

equals the constant

$$\kappa = \lim_j \delta_j^{-n} \inf\Big\{ \int_{B_{\delta_j}} |Dv|^{p_j}\, dx : \ v \in \nu + W_0^{1,p_j}(B_{\delta_j};\mathbb{R}^m),\ v = 0 \text{ on } B_{\varepsilon_j} \Big\}$$

we computed explicitly (note that $\kappa^K \le \kappa^{K'}$ if $K \subseteq K'$). This is equivalent to the fact that for any compact set K with non-empty interior the functionals $F_j = F_j^K : W^{1,p_j}(\Omega;\mathbb{R}^m) \to [0,+\infty]$, defined by

$$F_j^K(u) = \begin{cases} \displaystyle\int_\Omega |Du|^{p_j}\, dx & \text{if } u = 0 \text{ on } \displaystyle\bigcup_{i\in\mathbb{Z}^n} (x_i^j + \varepsilon_j K) \cap \Omega, \\ +\infty & \text{otherwise,} \end{cases} \qquad (35)$$

Γ-converge to the integral functional in (21). To this end, it suffices to prove that if we consider two closed balls $\overline{B}_{r_1}(x_0)$ and $\overline{B}_{r_2}(x_0)$ such that $\overline{B}_{r_1}(x_0) \subset K \subset \overline{B}_{r_2}(x_0)$, then the functionals $F_j^{\overline{B}_{r_1}(x_0)}$ and $F_j^{\overline{B}_{r_2}(x_0)}$ Γ-converge to the same limit functional.

(1) Exponential regime Let $\eta_j = \gamma\delta_j^{\frac{n}{n-1}} + o(\delta_j^{\frac{n}{n-1}})$, with $\gamma \in \mathbb{R}$. In the case $K = \overline{B}_1$ we proved that if we set $\varepsilon_j = \exp\big(-a\delta_j^{-n/(n-1)}\big)$ then we get

$$\kappa = \alpha(\gamma) = \frac{\omega_{n-1}}{(n-1)^{(n-1)}} e^{-a\gamma} \left| \frac{1 - e^{-\frac{a\gamma}{n-1}}}{\gamma} \right|^{1-n} \qquad \text{if } \gamma \neq 0,$$

extended by continuity as $\gamma \to 0$. If we fix $R > 0$ and set $\varepsilon_j = R\exp(-a\delta_j^{-n/(n-1)}) = \exp(\log R - a\delta_j^{-n/(n-1)})$, then the computation of the limit in (9) still gives $\alpha(\gamma)$. Therefore we can state that $\kappa^{\overline{B}_{r_1}(x_0)} = \kappa^{\overline{B}_{r_2}(x_0)} = \kappa$, hence $\kappa^K = \kappa$.

(2) Mixed polynomial-exponential regime Let $\eta_j > 0$ and $\eta_j \gtrsim \delta_j^{\frac{n}{n-1}}$. Let $R > 0$ be fixed. For all $\xi > 0$ we can note that if j is large enough we have:

$$R^{\frac{1}{\eta_j}} \operatorname{diam} K \le R^{\frac{1}{\eta_j}} r_2 \le \big(R(1+\xi)\big)^{\frac{1}{\eta_j}}$$

and

$$R^{\frac{1}{\eta_j}} \operatorname{diam} K \ge R^{\frac{1}{\eta_j}} r_1 \ge \big(R(1-\xi)\big)^{\frac{1}{\eta_j}}.$$

In the case $K = \overline{B}_1$ we proved that if we set $\varepsilon_j = R^{\frac{1}{\eta_j}} \delta_j^{\frac{n}{\eta_j}} \eta_j^{\frac{1-n}{\eta_j}}$, then we get $\kappa = R\omega_{n-1}(n-1)^{1-n}$. Now, if we replace the constant R by $R(1 \pm \xi)$, we get $\kappa = R(1 \pm \xi)\omega_{n-1}(n-1)^{1-n}$ respectively. By comparison,

$$R(1-\xi)\frac{\omega_{n-1}}{(n-1)^{n-1}} \le \kappa^K \le R(1+\xi)\frac{\omega_{n-1}}{(n-1)^{n-1}};$$

if we let $\xi \to 0$ we get $\kappa^K = R\frac{\omega_{n-1}}{(n-1)^{n-1}}$.

6. The rigid regime

Finally we prove the Γ-convergence result in the rigid case; *i.e.*, $\eta_j < 0$ and $|\eta_j| \gg \delta_j^{n/n-1}$. The proof will be performed in two steps: first we will show that if we fix $\varepsilon_j \equiv 0$ then the functionals (F_j) Γ-converge to F_∞ defined as in (22); then we will prove (by a comparison argument) that the same result holds for any choice of (ε_j).

1. Let $\varepsilon_j \equiv 0$. We denote by F_j^0 the functional (20) in this particular case:

$$F_j^0(u) = \begin{cases} \displaystyle\int_\Omega |Du|^{p_j}\,dx & \text{if } u(x_i^j) = 0, \\ +\infty & \text{otherwise.} \end{cases} \tag{36}$$

Note that the assumption $u(x_i^j) = 0$ makes sense because of the compact embedding of $W^{1,p_j}(\Omega;\mathbb{R}^m)$ into the set of continuous functions.

We will prove that

Proposition 6.1. *Let $u \neq 0$; then for all $u_j \to u$ in $L^1(\Omega;\mathbb{R}^m)$ we have*

$$\liminf_j F_j(u_j) = +\infty.$$

Proof. Upon a truncation argument as in Step **2.B** of Section 5.1 it is not restrictive to suppose that (u_j) is bounded in $L^\infty(\Omega;\mathbb{R}^m)$.

Let $\overline{c} < 1/2$ be a fixed constant. If we apply Lemma 4.1 to (u_j) (with $k \in \mathbb{N}$ arbitrarily fixed) we get a sequence (w_j) such that for all $i \in Z_j$ we have $w_j = u_j$ on $\Omega \setminus \bigcup_{i\in Z_j} C_i^j$, $w_j = u_j^i$ on $\partial B_{\rho_j^i}(x_i^j)$ (where $\rho_j^i = \frac{3}{4}2^{-k_i}\overline{c}\delta_j$) and

$$\liminf_j \int_\Omega |Du_j|^{p_j} + \frac{c}{k} \geq \liminf \int_\Omega |Dw_j|^{p_j}\,dx.$$

We have:

$$\liminf_j \int_\Omega |Du_j|^{p_j}\,dx + \frac{c}{k} \geq \liminf_j \int_\Omega |Dw_j|^{p_j}\,dx \geq \liminf \sum_{i\in Z_j} \int_{B_i^j} |Dw_j|^{p_j}\,dx.$$

Let

$$\tilde{w}_j^i(x) = \begin{cases} w_j(x + x_i^j) & \text{for } |x| \leq \rho_j^i, \\ u_j^i & \text{otherwise,} \end{cases}$$

and note that

$$\int_{B_i^j} |Dw_j|^{p_j}\,dx = \int_{B_{\overline{c}\delta_j}} |D\tilde{w}_j^i|^{p_j}\,dx.$$

There follows that

$$\begin{aligned} \liminf_j \int_\Omega |Du_j|^{p_j}\,dx + \frac{c}{k} &\geq \liminf_j \sum_{i\in Z_j} \int_{B_{\overline{c}\delta_j}} |D\tilde{w}_j^i|^{p_j}\,dx \\ \geq \ \liminf_j \sum_{i\in Z_j} \inf\Big\{ \int_{B_{\overline{c}\delta_j}} |Dv|^{p_j}\,dx : \ & v \in u_j^i + W_0^{1,p_j}(B_{\overline{c}\delta_j};\mathbb{R}^m),\ v(0) = 0\Big\}. \end{aligned}$$

If we focus our attention on the minimum problem above and repeat the computations of Section 2 we get

$$\begin{aligned} & \inf\Big\{ \int_{B_{\overline{c}\delta_j}} |Dv|^{p_j}\,dx : \ v \in u_j^i + W_0^{1,p_j}(B_{\overline{c}\delta_j};\mathbb{R}^m),\ v(0) = 0\Big\} \\ = \ & |u_j^i|^{p_j} \inf\Big\{ \int_{B_{\overline{c}\delta_j}} |Dv|^{p_j}\,dx : \ v \in 1 + W_0^{1,p_j}(B_{\overline{c}\delta_j};\mathbb{R}),\ v(0) = 0\Big\} \\ = \ & |u_j^i|^{p_j} \omega_{n-1} (\overline{c}\delta_j)^{\eta_j} \Big(\frac{|\eta_j|}{p_j - 1}\Big)^{p_j - 1}. \end{aligned}$$

Taking into account the arbitrariness of k and Proposition 4.2 we get

$$\begin{aligned} \liminf_j \int_\Omega |Du_j|^{p_j}\,dx &\geq \liminf_j \sum_{i\in Z_j} |u_j^i|^{p_j} \omega_{n-1} (\overline{c}\delta_j)^{\eta_j} \Big(\frac{|\eta_j|}{p_j - 1}\Big)^{p_j - 1} \\ &\geq \liminf_j c\Big(\sum_{i\in Z_j} \delta_j^n |u_j^i|^{p_j}\Big) \delta_j^{-n} (\overline{c}\delta_j)^{\eta_j} \Big(\frac{|\eta_j|}{p_j - 1}\Big)^{p_j - 1} \\ &\geq c\Big(\int_\Omega |u|^n\,dx\Big) \liminf_j \Big(\frac{|\eta_j|}{\delta_j^{\frac{n}{n-1}}}\Big)^{p_j - 1} \delta_j^{-p_j} \delta_j^{\frac{n(p_j-1)}{n-1}} = +\infty. \end{aligned}$$

□

The limsup inequality is trivial since it has to be checked only for $u \equiv 0$.

2. Let K be a compact subset of $\mathbb{R}^n$ with non-empty interior. Let (ε_j) be a generic real sequence satisfying $0 \le \varepsilon_j \le \delta_j/2$. Let $F_j : W^{1,p_j}(\Omega;\mathbb{R}^m) \to [0,\infty]$ be defined as in (20) and F_j^0 as in (36).

We proved that $\Gamma\text{-}\lim_j F_j^0 = F_\infty$. Note that if $F_j(u) < \infty$ then $F_j^0(u) = F_j(u)$; hence $F_j^0(u) \le F_j(u)$ for all $u \in W^{1,p_j}(\Omega;\mathbb{R}^m)$. By comparison we get $\Gamma\text{-}\liminf F_j \ge F_\infty$ and the converse inequality is trivial for the $\Gamma\text{-}\limsup$. Hence $\Gamma\text{-}\lim_j F_j = F_\infty$. □

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