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Subbases for N-ary Convexities

Evgeny Sharikov
Centre for Informatics and Applied Optimization,
School of Information Technology and Mathematical Sciences,
University of Ballarat, Victoria, Australia

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We introduce a notion of N-connectedness of a topological space with respect to a convexity on this space. For a given collection $\mathcal{H}$ of subsets of a set X we introduce different convexities and topologies on both $\mathcal{H}$ and X. Then we prove that the convexity $\mathcal{G}$ generated by $\mathcal{H}$ is of arity N whenever $\mathcal{H}$ is connected and X is N-connected.

Keywords: Axiomatic convexity, convexities of arity N, subbases for convexities and topologies

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1. Introduction

Axiomatic convexity deals with families of sets, which have some properties of usual convex sets. A general theory of convex structures can be found, for example, in [4] and [5]. Here we will use the following definition (see [5], p. 3).

A collection $\mathcal{G}$ of subsets of a set X is called a convexity on X if

(1) $\emptyset, X \in \mathcal{G}$,
(2) $\bigcap \mathcal{A} \in \mathcal{G}$ for every $\mathcal{A} \subset \mathcal{G}$,
(3) $\bigcup \mathcal{A} \in \mathcal{G}$ whenever $\mathcal{A} \subset \mathcal{G}$.

Members of $\mathcal{G}$ are called convex sets and the pair $(X, \mathcal{G})$ is called a convexity space.

There are two main ways to introduce a convexity on a set. First, we can say that a set $G \subset X$ is convex if it satisfies certain properties. In this case we should require that the collection $\mathcal{G}$ of all such sets $G \subset X$ satisfies axioms (1)–(3). Another way is based on a notion of a subbase for convexity.

It is clear that the intersection of any family of convexities on a given set X is a convexity as well. This fact allows us to talk about subbases for convexities. A set $\mathcal{H} \subset \mathcal{G}$ is called a subbase for the convexity $\mathcal{G}$ if $\mathcal{G}$ is the intersection of all convexities, which contain $\mathcal{H}$ (we will say also that $\mathcal{G}$ is generated by $\mathcal{H}$). Note that topologies enjoy the same property: intersection of any family of topologies on a given set X is also a topology on X. So we can consider subbases for topologies as well.

Let $\mathcal{H}$ be a subbase for topology $\mathcal{T}$. Then open sets can be described in the following

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way. First we construct the collection $\mathcal{B}$ of all intersections of finite subfamilies of $\mathcal{H}$. Then $\mathcal{T}$ consists of the empty set, whole X and all unions of subfamilies of $\mathcal{B}$.

If A is a subset of X then its convex hull $\mathrm{conv}_{\mathcal{G}}A$ with respect to the convexity $\mathcal{G}$ is defined as follows:

$$\mathrm{conv}_{\mathcal{G}}A = \bigcap\{G \in \mathcal{G} :\ A \subset G\}.$$

For any points $x, y \in X$ denote by $[x, y]_{\mathcal{G}}$ their convex hull $\mathrm{conv}_{\mathcal{G}}\{x, y\}$. We will also use symbol $[A]^{<\omega}$ for the collection of all finite subsets of A.

Recall two results of axiomatic convexity. The following one is well known as the *finitary property.*

Proposition 1.1 ([5], p. 31, Proposition 2.1). *Let $(X, \mathcal{G})$ be a convexity space. Then for every subset $A \subset X$*

$$\mathrm{conv}_{\mathcal{G}}A = \bigcup_{F \in [A]^{<\omega}} \mathrm{conv}_{\mathcal{G}}F. \tag{1}$$

Proposition 1.2 ([5], p. 10, Proposition 1.7.3). *Let $(X, \mathcal{G})$ be a convexity space. If $\mathcal{H}$ is a subbase for the convexity $\mathcal{G}$ then for every finite subset $F \subset X$*

$$\mathrm{conv}_{\mathcal{G}}F = \bigcap\{H \in \mathcal{H} :\ F \subset H\}. \tag{2}$$

In the right-hand side of (2) it is assumed that the intersection over the empty set is equal to X. In other words, if $F \not\subset H$ for any $H \in \mathcal{H}$ then we set $\mathrm{conv}_{\mathcal{G}}F = X$.

It follows from the formulas (1) and (2) that for every $A \subset X$ its convex hull $\mathrm{conv}_{\mathcal{G}}A$ can be described via elements of $\mathcal{H}$ in the following way:

$$\mathrm{conv}_{\mathcal{G}}A = \bigcup_{F \in [A]^{<\omega}} \bigcap\{H \in \mathcal{H} :\ F \subset H\}. \tag{3}$$

Due to Proposition 1.1, a set $A \subset X$ is convex (belongs to convexity $\mathcal{G}$) if and only if

$$A = \bigcup_{F \in [A]^{<\omega}} \mathrm{conv}_{\mathcal{G}}F.$$

This means that $A \in \mathcal{G}$ whenever $\mathrm{conv}_{\mathcal{G}}F \subset A$ for all $F \in [A]^{<\omega}$.

The so-called N-ary convexities form one of the most important subclasses of convexities.

Let N be a positive integer. Let $[A]^{\leq N}$ denote the collection of all subsets $F \subset A$, which contain no more than N points. A convexity $\mathcal{G}$ is called N-ary (or of arity N) (see [5]) if $A \in \mathcal{G}$ whenever $\mathrm{conv}_{\mathcal{G}}F \subset A$ for all $F \in [A]^{\leq N}$.

In this paper we are interested in subbases for N-ary convexities. For a given collection $\mathcal{H}$ of subsets of a set X we present some conditions, which guarantee that the convexity $\mathcal{G}$ generated by $\mathcal{H}$ is of arity N. For this purpose we will need special types of convexities and topologies on both $\mathcal{H}$ and X. Then we introduce a notion of N-connectedness of a topological space with respect to a convexity on this space and prove that $\mathcal{G}$ is of arity N provided that $\mathcal{H}$ is connected and X is N-connected. After the main result we consider two particular cases, where the sets $H \in \mathcal{H}$ are expressed via real-valued functions.

Applications of axiomatic convexity are mainly based on separation properties. In general, we have the following weak version of the separation property, which follows directly from (3): if $\mathcal{H}$ is a subbase for convexity $\mathcal{G}$ and $G \in \mathcal{G}$ then for every $x \notin G$ and for every finite subset $F \subset G$ a set $H \in \mathcal{H}$ exists such that $F \subset H$ and $x \notin H$.

So if $\mathcal{G}$ is N-ary and the number N is not very large (as a rule, we are interested in the cases, when $N = 2$) then we have a sufficiently simple description of convex sets. At the same time, if $\mathcal{G}$ is generated by $\mathcal{H}$ then our weak separation property can be applied for every convex set G.

An important special case is when all sets $H \in \mathcal{H}$ are epigraphs of certain real-valued functions. This leads to the notion of abstract convex functions ([1]). Description of abstract convex functions is very important for many applications. For instance, various numerical methods for global minimization of abstract convex functions were considered in ([1], Chapter 9). Subdifferential calculus for abstract convex functions was investigated in [2]. Another interesting application is the *Principle of Preservation of Inequalities* ([1], Proposition 6.10). Within the framework of the given paper, due to the weak separation property, we give a description of abstract convex functions on finite subsets of their domain (see Proposition 4.11).

In order to have a stronger version of the separation property via elements of a subbase, we need to indicate a property (P) of subsets of X such that a set $G \subset X$ is convex and enjoys (P) if and only if for every $x \notin G$ a set $H \in \mathcal{H}$ exists with $G \subset H$ and $x \notin H$. This is the theme of the paper [3].

2. Preliminaries

Let $\mathcal{H}$ be a collection of subsets of a set X. In this paper we will use the following notations:

- $\mathcal{H}' = \{X \backslash H : H \in \mathcal{H}\}$ is the collection of all complements of sets $H \in \mathcal{H}$;
- $\mathcal{H}_x = \{H \in \mathcal{H} : x \in H\}$ for every $x \in X$;
- $\mathcal{H}^*$ is the collection of all sets $\mathcal{H}_x$ with $x \in X$;
- $\mathcal{H}^{*\prime} = \{\mathcal{H} \backslash \mathcal{H}_x : x \in X\}$ is the collection of all complements of sets $\mathcal{H}_x \in \mathcal{H}^*$.

We introduce also the following convexities and topologies:

- $\mathcal{G}$ is the convexity on X generated by $\mathcal{H}$;
- $\bar{\mathcal{G}}$ is the convexity on X generated by the union $\mathcal{H} \cup \mathcal{H}'$;
- $\mathcal{T}_X$ is the topology on X generated by $\mathcal{H}$;
- $\mathcal{T}'_X$ is the topology on X generated by $\mathcal{H}'$;
- $\bar{\mathcal{G}}^*$ is the convexity on $\mathcal{H}$ generated by the union $\mathcal{H}^* \cup \mathcal{H}^{*\prime}$;
- $\mathcal{T}_{\mathcal{H}}$ is the topology on $\mathcal{H}$ generated by $\mathcal{H}^*$;
- $\mathcal{T}'_{\mathcal{H}}$ is the topology on $\mathcal{H}$ generated by $\mathcal{H}^{*\prime}$.

We give a description of convex hulls $\mathrm{conv}_{\bar{\mathcal{G}}}$ and $\mathrm{conv}_{\bar{\mathcal{G}}^*}$ of finite subsets of X and $\mathcal{H}$ respectively.

Proposition 2.1. *Let F be a finite subset of X. Then a point $x \in X$ belongs to* $\mathrm{conv}_{\bar{\mathcal{G}}} F$

if and only if for every set $H \in \mathcal{H}$ the following implications hold

$$\begin{aligned} F \subset H &\implies x \in H, \\ x \in H &\implies F \cap H \neq \emptyset. \end{aligned}$$

Proof. Since $\mathcal{H} \cup \mathcal{H}'$ is a subbase for convexity $\bar{\mathcal{G}}$ and F contains a finite number of points of X then its convex hull $\mathrm{conv}_{\bar{\mathcal{G}}} F$ can be described via elements of $\mathcal{H} \cup \mathcal{H}'$ (see Proposition 1.2):

$$\mathrm{conv}_{\bar{\mathcal{G}}} F = \left(\bigcap \{H \in \mathcal{H} : \, F \subset H\} \right) \bigcap \left(\bigcap \{X \backslash H : \, H \in \mathcal{H}, \, F \subset (X \backslash H)\} \right).$$

So a point $x \in X$ belongs to $\mathrm{conv}_{\bar{\mathcal{G}}} F$ if and only if for any $H \in \mathcal{H}$

$$(x \in H \quad \text{whenever } F \subset H) \quad \text{and} \quad (x \notin H \quad \text{whenever } F \cap H = \emptyset).$$

□

Proposition 2.2. *Let $\mathcal{E}$ be a finite subset of $\mathcal{H}$. Then*

$$\mathrm{conv}_{\bar{\mathcal{G}}^*} \mathcal{E} = \left\{ H \in \mathcal{H} : \, \bigcap_{E \in \mathcal{E}} E \subset H \subset \bigcup_{E \in \mathcal{E}} E \right\}. \tag{4}$$

Proof. Since $\mathcal{E}$ is a finite subset of $\mathcal{H}$ and $\mathcal{H}^* \cup \mathcal{H}^{*\prime}$ is a subbase for $\bar{\mathcal{G}}^*$ then

$$\begin{aligned} \mathrm{conv}_{\bar{\mathcal{G}}^*} \mathcal{E} &= \bigcap \{\mathcal{A} : \, \mathcal{A} \in \mathcal{H}^* \cup \mathcal{H}^{*\prime}, \, \mathcal{E} \subset \mathcal{A}\} \\ &= \left(\bigcap \{\mathcal{H}_x : \, x \in X, \, \mathcal{E} \subset \mathcal{H}_x\} \right) \bigcap \left(\bigcap \{\mathcal{H}'_x : \, x \in X, \, \mathcal{E} \subset \mathcal{H}'_x\} \right). \end{aligned} \tag{5}$$

We have

$$\mathcal{E} \subset \mathcal{H}_x \iff x \in \bigcap_{E \in \mathcal{E}} E, \qquad \mathcal{E} \subset \mathcal{H}'_x \iff x \notin \bigcup_{E \in \mathcal{E}} E.$$

Hence for every set $H \in \mathcal{H}$

$$\begin{aligned} H \in \bigcap \{\mathcal{H}_x : \, x \in X, \, \mathcal{E} \subset \mathcal{H}_x\} &\iff \bigcap_{E \in \mathcal{E}} E \subset H, \\ H \in \bigcap \{\mathcal{H}'_x : \, x \in X, \, \mathcal{E} \subset \mathcal{H}'_x\} &\iff H \subset \bigcup_{E \in \mathcal{E}} E. \end{aligned} \tag{6}$$

Thus, the required formula (4) follows from (5) and (6). □

3. Main result

First we define N-connectedness of a topological space with respect to a convexity on this space.

Definition 3.1. Let $(X, \mathcal{T})$ be a topological space and $\mathcal{G}$ be a convexity on X. We say that $(X, \mathcal{T})$ is N-connected with respect to $\mathcal{G}$ if N subsets $X_1, \ldots, X_N \subset X$ exist such that $X = X_1 \cup \cdots \cup X_N$ and for each $i = 1, \ldots, N$ every interval $[x, y]_{\mathcal{G}}$ with $x, y \in X_i$ is connected in topology $\mathcal{T}$. We say that $(X, \mathcal{T})$ is connected with respect to $\mathcal{G}$ in the case, when $N = 1$.

It should be mentioned that the number N above is not required to be minimal. Thus if $(X,\mathcal{T})$ is N-connected with respect to $\mathcal{G}$ then it is also n-connected for any $n>N$.

Remark 3.2. It is easy to see that N-connectedness of a topological space with respect to a convexity on this space remains valid if the topology decreases. This means the following. Assume that $(X,\mathcal{T})$ is N-connected with respect to $\mathcal{G}$. Let $\mathcal{T}_1$ be a topology on X such that $\mathcal{T}_1\subset\mathcal{T}$. Then $(X,\mathcal{T}_1)$ is N-connected with respect to $\mathcal{G}$ as well.

Remark 3.3. If $(X,\mathcal{T})$ is connected with respect to $\mathcal{G}$ then every set $G\in\mathcal{G}$ is connected in topology $\mathcal{T}$. Indeed, let $G=U\cup U'$. Take an arbitrary $u\in U$ and $u'\in U'$. Since $G\in\mathcal{G}$ then $[u,u']_{\mathcal{G}}\subset G=U\cup U'$. It follows from the connectedness of $[u,u']_{\mathcal{G}}$ that either U has a limit point of U' or U' has a limit point of U. Hence G is also connected.

Theorem 3.4. *Assume that one of the spaces $(\mathcal{H},\mathcal{T}_{\mathcal{H}})$ or $(\mathcal{H},\mathcal{T}'_{\mathcal{H}})$ is connected with respect to the convexity $\bar{\mathcal{G}}^*$. Let F be a finite subset of X. Then for any points $x,y\in F$ and for each $z\in[x,y]_{\bar{\mathcal{G}}}$ the following holds:*

$$\mathrm{conv}_{\mathcal{G}}F=\mathrm{conv}_{\mathcal{G}}(\{z\}\cup(F\backslash\{x\}))\bigcup\mathrm{conv}_{\mathcal{G}}(\{z\}\cup(F\backslash\{y\})). \tag{7}$$

Proof. Since the set F is finite then

$$\mathrm{conv}_{\mathcal{G}}F=\bigcap\{H\in\mathcal{H}:\ F\subset H\}. \tag{8}$$

Let $F_1=\{z\}\cup(F\backslash\{x\})$ and $F_2=\{z\}\cup(F\backslash\{y\})$. Since $z\in[x,y]_{\bar{\mathcal{G}}}\subset[x,y]_{\mathcal{G}}\subset\mathrm{conv}_{\mathcal{G}}F$ then

$$\mathrm{conv}_{\mathcal{G}}F\supset\mathrm{conv}_{\mathcal{G}}F_1\bigcup\mathrm{conv}_{\mathcal{G}}F_2.$$

Now we need to check the inclusion $\mathrm{conv}_{\mathcal{G}}F\subset\mathrm{conv}_{\mathcal{G}}F_1\bigcup\mathrm{conv}_{\mathcal{G}}F_2$. We have

$$\begin{aligned}\mathrm{conv}_{\mathcal{G}}F_1\bigcup\mathrm{conv}_{\mathcal{G}}F_2 &= \Big(\bigcap\{H_1\in\mathcal{H}:\ F_1\subset H_1\}\Big)\bigcup\Big(\bigcap\{H_2\in\mathcal{H}:\ F_2\subset H_2\}\Big)\\ &= \bigcap\{H_1\cup H_2:\ H_1,H_2\in\mathcal{H},\ F_1\subset H_1,\ F_2\subset H_2\}.\end{aligned}$$

If either $\{H_1:\ \ H_1\in\mathcal{H},\ F_1\subset H_1\}=\emptyset$ or $\{H_2:\ \ H_2\in\mathcal{H},\ F_2\subset H_2\}=\emptyset$ then $\mathrm{conv}_{\mathcal{G}}F_1\bigcup\mathrm{conv}_{\mathcal{G}}F_2=X$ and the inclusion $\mathrm{conv}_{\mathcal{G}}F\subset\mathrm{conv}_{\mathcal{G}}F_1\bigcup\mathrm{conv}_{\mathcal{G}}F_2$ becomes trivial.

So we need to show that $\mathrm{conv}_{\mathcal{G}}F\subset H_1\cup H_2$ whenever $F_1\subset H_1$ and $F_2\subset H_2$ $(H_1,H_2\in\mathcal{H})$. Take such H_1 and H_2 and consider the sets $\mathcal{H}_x=\{H\in\mathcal{H}:\ x\in H\}$ and $\mathcal{H}_y=\{H\in\mathcal{H}:\ y\in H\}$. They cover the interval $[H_1,H_2]_{\bar{\mathcal{G}}^*}$. Indeed, since $z\in H_1\cap H_2$ then, by Proposition 2.2, each $H\in[H_1,H_2]_{\bar{\mathcal{G}}^*}$ contains the point z. Since $z\in[x,y]_{\bar{\mathcal{G}}}$ then it follows from Proposition 2.1 that $H\cap\{x,y\}\neq\emptyset$. Hence $[H_1,H_2]_{\bar{\mathcal{G}}^*}\subset\mathcal{H}_x\cup\mathcal{H}_y$. Since one of the spaces $(\mathcal{H},\mathcal{T}_{\mathcal{H}})$ or $(\mathcal{H},\mathcal{T}'_{\mathcal{H}})$ is connected with respect to $\bar{\mathcal{G}}^*$ then the interval $[H_1,H_2]_{\bar{\mathcal{G}}^*}$ is connected in one of the topologies $\mathcal{T}_{\mathcal{H}}$ or $\mathcal{T}'_{\mathcal{H}}$. Note that $\mathcal{H}_x$ and $\mathcal{H}_y$ are open in topology $\mathcal{T}_{\mathcal{H}}$ and closed in $\mathcal{T}'_{\mathcal{H}}$. Note also that $H_1\in\mathcal{H}_y$ and $H_2\in\mathcal{H}_x$. Consequently, a set $H\in[H_1,H_2]_{\bar{\mathcal{G}}^*}$ exists such that $H\in\mathcal{H}_x\cap\mathcal{H}_y$. In other words, H contains both x and y. Moreover, due to (4), $H_1\cap H_2\subset H\subset H_1\cup H_2$. Since $F\backslash\{x,y\}\subset H_1\cap H_2$ and $x,y\in H$ then $F\subset H$. This implies $\mathrm{conv}_{\mathcal{G}}F\subset H\subset H_1\cup H_2$. $\square$

Theorem 3.5. *Assume that one of the spaces $(\mathcal{H}, \mathcal{T}_{\mathcal{H}})$ or $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to the convexity $\bar{\mathcal{G}}^*$. Let $\mathcal{T}$ be a topology on X such that for any $F \in [X]^{<\omega}$ and $Z \subset X$*

$$\bigcap_{z \in Z} \mathrm{conv}_{\mathcal{G}}(F \cup \{z\}) = \mathrm{conv}_{\mathcal{G}} F \quad \textit{whenever } Z \textit{ has a limit point in } F. \tag{9}$$

Let F be a finite subset of X and $x, y \in F$. Assume that $[x, y]_{\bar{\mathcal{G}}}$ is connected in $\mathcal{T}$. Then

$$\mathrm{conv}_{\mathcal{G}} F = \bigcup_{z \in [x,y]_{\bar{\mathcal{G}}}} \mathrm{conv}_{\mathcal{G}}(\{z\} \cup (F \backslash \{x, y\})). \tag{10}$$

Proof. Inclusion

$$\bigcup_{z \in [x,y]_{\bar{\mathcal{G}}}} \mathrm{conv}_{\mathcal{G}}(\{z\} \cup (F \backslash \{x, y\})) \subset \mathrm{conv}_{\mathcal{G}} F$$

is obvious because $[x, y]_{\bar{\mathcal{G}}} \subset [x, y]_{\mathcal{G}} \subset \mathrm{conv}_{\mathcal{G}} F$.

Let $a \in \mathrm{conv}_{\mathcal{G}} F$. We need to find a point $z \in [x, y]_{\bar{\mathcal{G}}}$ such that

$$a \in \mathrm{conv}_{\mathcal{G}}(\{z\} \cup (F \backslash \{x, y\})). \tag{11}$$

It follows from the Theorem 3.4 that

$$a \in \mathrm{conv}_{\mathcal{G}}(\{z\} \cup (F \backslash \{x\})) \bigcup \mathrm{conv}_{\mathcal{G}}(\{z\} \cup (F \backslash \{y\})) \ \text{ for each } z \in [x, y]_{\bar{\mathcal{G}}}. \tag{12}$$

Consider two sets

$$Z_1 = \{z \in X : a \in \mathrm{conv}_{\mathcal{G}}(\{z\} \cup (F \backslash \{x\}))\},$$
$$Z_2 = \{z \in X : a \in \mathrm{conv}_{\mathcal{G}}(\{z\} \cup (F \backslash \{y\}))\}.$$

Due to (12), they cover the interval $[x, y]_{\bar{\mathcal{G}}}$. Moreover, since $a \in \mathrm{conv}_{\mathcal{G}} F$ then $x \in Z_1$ and $y \in Z_2$. Condition (9) implies that Z_1 and Z_2 are closed in topology $\mathcal{T}$. For example, if z is a limit point of Z_1 then $a \in \bigcap_{z_1 \in Z_1} \mathrm{conv}_{\mathcal{G}}(\{z_1\} \cup \{z\} \cup (F \backslash \{x\}))$, and, by (9), we have $a \in \mathrm{conv}_{\mathcal{G}}(\{z\} \cup (F \backslash \{x\}))$. Since $[x, y]_{\bar{\mathcal{G}}}$ is connected in $\mathcal{T}$ then we conclude that the intersection $[x, y]_{\bar{\mathcal{G}}} \cap Z_1 \cap Z_2$ is nonempty. In other words, a point $z \in [x, y]_{\bar{\mathcal{G}}}$ exists such that

$$a \in \mathrm{conv}_{\mathcal{G}}(\{z\} \cup (F \backslash \{x\})) \bigcap \mathrm{conv}_{\mathcal{G}}(\{z\} \cup (F \backslash \{y\})). \tag{13}$$

Check the inclusion (11). Since $\{z\} \cup (F \backslash \{x, y\})$ is a finite subset of X then it is sufficient to show that $a \in H$ whenever $H \in \mathcal{H}$ and $\{z\} \cup (F \backslash \{x, y\}) \subset H$. So let $\{z\} \cup (F \backslash \{x, y\}) \subset H$. Since $z \in [x, y]_{\bar{\mathcal{G}}}$ and $z \in H$ then $\{x, y\} \cap H \neq \emptyset$ (see Proposition 2.1). If $x \in H$ then $\{z\} \cup (F \backslash \{y\}) \subset H$, and therefore $\mathrm{conv}_{\mathcal{G}}(\{z\} \cup (F \backslash \{y\})) \subset H$. If $y \in H$ then $\mathrm{conv}_{\mathcal{G}}(\{z\} \cup (F \backslash \{x\})) \subset H$. In any event the point a belongs to H (see (13)). $\square$

Now we can formulate the main result of this paper.

Theorem 3.6. *Assume that one of the spaces $(\mathcal{H}, \mathcal{T}_{\mathcal{H}})$ or $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to the convexity $\bar{\mathcal{G}}^*$. Let $\mathcal{T}$ be a topology on X such that (9) holds true. Let $N \geq 2$. Assume that the space $(X, \mathcal{T})$ is N-connected with respect to the convexity $\bar{\mathcal{G}}$. Then the convexity $\mathcal{G}$ is of arity N.*

Proof. Let A be a subset of X such that $\mathrm{conv}_{\mathcal{G}}F \subset A$ whenever $F \in [A]^{\leq N}$. We need to check that A belongs to the convexity $\mathcal{G}$. Due to Proposition 1.1, we have

$$A \in \mathcal{G} \iff \mathrm{conv}_{\mathcal{G}}A \subset A \iff (\mathrm{conv}_{\mathcal{G}}F \subset A \text{ for each } F \in [A]^{<\omega}).$$

Let F be a finite subset of A. If $F \in [A]^{\leq N}$ then the inclusion $\mathrm{conv}_{\mathcal{G}}F \subset A$ is valid.

Now assume that F consists of n different points of A and $n > N$. We will show that for each point $a \in \mathrm{conv}_{\mathcal{G}}F$ a set $F_{n-1} \in [A]^{\leq(n-1)}$ exists such that $a \in \mathrm{conv}_{\mathcal{G}}F_{n-1}$. Then, by induction, we can find a set $F_N \in [A]^{\leq N}$ such that $a \in \mathrm{conv}_{\mathcal{G}}F_N$. Therefore $\mathrm{conv}_{\mathcal{G}}F \subset A$.

So take a point $a \in \mathrm{conv}_{\mathcal{G}}F$. Since the space $(X, \mathcal{T})$ is N-connected with respect to $\bar{\mathcal{G}}$ and F contains more than N points of $A \subset X$ then there exists a pair of points $x, y \in F$ $(x \neq y)$ such that the interval $[x,y]_{\bar{\mathcal{G}}}$ is connected in $\mathcal{T}$. Then, by Theorem 3.5, there is a point $z \in [x,y]_{\bar{\mathcal{G}}}$ with $a \in \mathrm{conv}_{\mathcal{G}}(\{z\} \cup (F\backslash\{x,y\}))$.

Consider the set $F_{n-1} = \{z\} \cup (F\backslash\{x,y\})$. Since $z \in [x,y]_{\bar{\mathcal{G}}} \subset [x,y]_{\mathcal{G}}$ and $\{x,y\} \in [A]^{\leq 2} \subset [A]^{\leq N}$ then $z \in A$. This implies $F_{n-1} \in [A]^{\leq(n-1)}$. □

Below we will show that the estimate of arity number in Theorem 3.6 is sharp (see Example 4.9).

Since condition (9) is not easy for verification, we present a simpler condition, which implies (9).

Proposition 3.7. *Let $\mathcal{T}$ be a topology on X such that for any $F \in [X]^{<\omega}$ we have*

$$\text{if } a \notin \mathrm{conv}_{\mathcal{G}}F \text{ then a set } H \in \mathcal{H} \text{ exists with } F \subset \mathrm{int}\, H \text{ and } a \notin H, \tag{14}$$

where $\mathrm{int}\, H$ *is the interior of H in topology $\mathcal{T}$. Then condition* (9) *holds true for $\mathcal{T}$. In particular,* (9) *holds for any topology $\mathcal{T}$ such that all sets $H \in \mathcal{H}$ are open in $\mathcal{T}$.*

Proof. Let F be a finite subset of X and $Z \subset X$ has a limit point in F. We need to check the inclusion $\bigcap_{z\in Z} \mathrm{conv}_{\mathcal{G}}(F \cup \{z\}) \subset \mathrm{conv}_{\mathcal{G}}F$, which means that $a \notin \bigcap_{z\in Z} \mathrm{conv}_{\mathcal{G}}(F \cup \{z\})$ whenever $a \notin \mathrm{conv}_{\mathcal{G}}F$. So let $a \notin \mathrm{conv}_{\mathcal{G}}F$. Then, by (14), a set $H \in \mathcal{H}$ exists with $F \subset \mathrm{int}\, H$ and $a \notin H$. Since $F \subset \mathrm{int}\, H$ and Z has a limit point in F then H contains a point $z' \in Z$. Hence $a \notin \mathrm{conv}_{\mathcal{G}}(F \cup \{z'\})$, because $a \notin H$ and $F \cup \{z'\} \subset H$. And therefore $a \notin \bigcap_{z\in Z} \mathrm{conv}_{\mathcal{G}}(F \cup \{z\})$.

If all sets $H \in \mathcal{H}$ are open in topology $\mathcal{T}$ (i.e. $\mathrm{int}\, H = H$) then (14) obviously holds. □

Corollary 3.8. *Assume that $(\mathcal{H}, \mathcal{T}_{\mathcal{H}})$ is connected with respect to $\bar{\mathcal{G}}^*$ and $(X, \mathcal{T}_X)$ is N-connected with respect to $\bar{\mathcal{G}}$, where $N \geq 2$. Then the convexity $\mathcal{G}$ is of arity N.*

Proof. It follows directly from Theorem 3.6 and Proposition 3.7 because all sets $H \in \mathcal{H}$ are open in the topology $\mathcal{T}_X$. □

Unfortunately, condition (9) does not necessarily hold for the topology $\mathcal{T} = \mathcal{T}'_X$. To show this consider a simple example.

Example 3.9. Let $X = \mathbb{R}$. Let $\mathcal{H}$ be the collection of all segments $[c, +\infty)$ with $c \in \mathbb{R}$. It is easy to see that the topology $\mathcal{T}'_X$ consists of empty set, whole real line and all segments $(-\infty, c)$ with $c \in \mathbb{R}$. Then $a \in \mathbb{R}$ is a limit point of $Z \subset \mathbb{R}$ (in topology $\mathcal{T}'_X$)

if and only if $\inf_{z\in Z} z \le a$. For example, if $z < a$ then a is a limit point of $Z = \{z\}$. At the same time, $\mathrm{conv}_{\mathcal{G}}\{z\} = [z, +\infty) \not\subset \mathrm{conv}_{\mathcal{G}}\{a\} = [a, +\infty)$. So, condition (9) does not hold in this case.

However, it can be convenient to use a topology $\mathcal{T}$ on X, which possesses (9) and contains the topology $\mathcal{T}'_X$.

4. Some particular cases

First we show that path-connectedness can be used instead of the connectedness whenever all sets X_i (see Definition 3.1) are convex in $\mathcal{G}$.

Proposition 4.1. *Let X be equipped with a topology $\mathcal{T}$ and $\mathcal{G}$ be a convexity on X. Assume that N sets $X_1, \dots, X_N \in \mathcal{G}$ exist such that $X = X_1 \cup \dots \cup X_N$ and for each $i = 1, \dots, N$ the following condition holds: for any two points $x, y \in X_i$ a continuous mapping $\omega : [0,1] \to [x,y]_{\mathcal{G}}$ exists such that $\omega(0) = x$ and $\omega(1) = y$. Then the space $(X, \mathcal{T})$ is N-connected with respect to $\mathcal{G}$.*

Proof. We need to prove that all intervals of sets X_i are connected in $\mathcal{T}$. So let $x, y \in X_i$. Let $[x,y]_{\mathcal{G}}$ be represented as the union of two nonempty sets U_1 and U_2. Since $X_i \in \mathcal{G}$ then $U_1 \cup U_2 = [x,y]_{\mathcal{G}} \subset X_i$. Take arbitrary $u_1 \in U_1$ and $u_2 \in U_2$. We have $u_1, u_2 \in X_i$. Hence there is a continuous mapping $\omega : [0,1] \to [u_1,u_2]_{\mathcal{G}}$ with $\omega(0) = u_1$ and $\omega(1) = u_2$. Inclusion $u_1, u_2 \in [x,y]_{\mathcal{G}} = U_1 \cup U_2$ implies that $\omega(t) \in U_1 \cup U_2$ for all $t \in [0,1]$. Thus, we have a continuous mapping $\omega : [0,1] \to U_1 \cup U_2$, where $\omega(0) \in U_1$ and $\omega(1) \in U_2$. This means that either U_1 has a limit point of U_2 or U_2 has a limit point of U_1. Therefore the interval $[x,y]_{\mathcal{G}}$ is connected. $\square$

Let Y be a topological space. Consider the following interpretation of continuity of a mapping $\omega : Y \to \mathcal{H}$ in cases, when $\mathcal{H}$ is equipped with one of the topologies: $\mathcal{T}_{\mathcal{H}}$ or $\mathcal{T}'_{\mathcal{H}}$.

Proposition 4.2. *Let $y_0 \in Y$ and $\omega : Y \to \mathcal{H}$ be a mapping. If $\mathcal{H}$ is equipped with the topology $\mathcal{T}_{\mathcal{H}}$ then ω is continuous at y_0 if and only if for each $x \in \omega(y_0)$ a neighbourhood U of y_0 exists such that $x \in \omega(y)$ for all $y \in U$. If $\mathcal{H}$ is equipped with the topology $\mathcal{T}'_{\mathcal{H}}$ then ω is continuous at y_0 if and only if for each $x \notin \omega(y_0)$ a neighbourhood U of y_0 exists such that $x \notin \omega(y)$ for all $y \in U$.*

Proof. Let $\mathcal{H}$ be equipped with the topology $\mathcal{T}_{\mathcal{H}}$, and assume that ω is continuous at the point y_0. Take a point $x \in \omega(y_0)$. Then $\omega(y_0) \in \mathcal{H}_x \in \mathcal{H}^* \subset \mathcal{T}_{\mathcal{H}}$. Hence the set $\mathcal{H}_x$ is a neighbourhood of $\omega(y_0)$. Since ω is continuous at y_0 then we can find a neighbourhood U of y_0 such that $\omega(y) \in \mathcal{H}_x$ for all $y \in U$. In other words, $x \in \omega(y)$ for all $y \in U$.

Conversely, assume that for each $x \in \omega(y_0)$ a neighbourhood U of y_0 exists such that $x \in \omega(y)$ for all $y \in U$. Let $\mathcal{S}$ be a neighbourhood of $\omega(y_0)$. Since the topology $\mathcal{T}_{\mathcal{H}}$ is generated by $\mathcal{H}^*$ then a finite collection $\{\mathcal{H}_{x_1}, \dots, \mathcal{H}_{x_k}\}$ of elements of $\mathcal{H}^*$ exists such that $\omega(y_0) \in \bigcap_{i=1}^k \mathcal{H}_{x_i} \subset \mathcal{S}$. This implies $x_i \in \omega(y_0)$ for all $i = 1, \dots, k$. By our assumption, there exist a neighbourhoods $U_1, \dots, U_k$ of the point y_0 such that $x_i \in \omega(y)$ for all $y \in U_i$. Then the set $U = \bigcap_{i=1}^k U_i$ is a neighbourhood of y_0 and $\omega(y) \in \bigcap_{i=1}^k \mathcal{H}_{x_i} \subset \mathcal{S}$ for all $y \in U$. So the mapping $\omega : Y \to \mathcal{H}$ is continuous at y_0.

We omit the second part of the proof since all arguments are the same as in the first one. □

A similar interpretation of continuity of a mapping $\omega : Y \to X$ is valid for the topologies $\mathcal{T}_X$ and $\mathcal{T}'_X$.

Proposition 4.3. *Let $y_0 \in Y$ and $\omega : Y \to X$ be a mapping. If X is equipped with the topology $\mathcal{T}_X$ ($\mathcal{T}'_X$) then ω is continuous at y_0 if and only if for each $H \in \mathcal{H}$ such that $\omega(y_0) \in H$ ($\omega(y_0) \notin H$) a neighbourhood U of y_0 exists such that $\omega(y) \in H$ ($\omega(y) \notin H$) for all $y \in U$.*

Proof. The proof is straightforward. □

Remark 4.4. Let X be equipped with a topology $\mathcal{T}$. Then all sets $H \in \mathcal{H}$ are open (closed) in the topology $\mathcal{T}$ if and only if $\mathcal{T}_X \subset \mathcal{T}$ ($\mathcal{T}'_X \subset \mathcal{T}$). As one can see from Proposition 4.2, it is natural to apply the topology $\mathcal{T}_{\mathcal{H}}$ in the case, when all sets $H \in \mathcal{H}$ are open. At the same time, the topology $\mathcal{T}'_{\mathcal{H}}$ on $\mathcal{H}$ is suitable when all sets $H \in \mathcal{H}$ are closed.

In order to check the connectedness of a topological space with respect to a convexity on this space we need to describe convex hulls of each pair of elements of this space, or at least to indicate some points of these convex hulls. Here we consider two particular cases, where the sets $H \in \mathcal{H}$ are expressed via real-valued functions, and get some formulas for the convex hulls in terms of these functions.

Subbases of level sets $S_0(l) = \{x \in X : l(x) \le 0\}$

Let L be a family of real-valued functions defined on a set X. Consider the collection $\mathcal{H}$ of all sets $S_0(l) = \{x \in X : l(x) \le 0\}$, where $l \in L$.

Let $x_1, x_2 \in X$. Then, by Proposition 2.1, the set $[x_1, x_2]_{\bar{\mathcal{G}}}$ consists of all points $x \in X$ such that for any $l \in L$ the following implications hold

$$\begin{aligned} \max\{l(x_1), l(x_2)\} \le 0 &\implies l(x) \le 0, \\ l(x) \le 0 &\implies \min\{l(x_1), l(x_2)\} \le 0. \end{aligned}$$

In particular, $[x_1, x_2]_{\bar{\mathcal{G}}}$ contains all points $x \in X$ such that

$$\min\{l(x_1), l(x_2)\} \le l(x) \le \max\{l(x_1), l(x_2)\} \quad \forall\, l \in L.$$

Let $l_1, l_2 \in L$. Due to Proposition 2.2, we have

$$[S_0(l_1), S_0(l_2)]_{\bar{\mathcal{G}}^*} = \{S_0(l) \in \mathcal{H} : S_0(l_1) \cap S_0(l_2) \subset S_0(l) \subset S_0(l_1) \cup S_0(l_2)\}.$$

In other words, the set $[S_0(l_1), S_0(l_2)]_{\bar{\mathcal{G}}^*}$ consists of all $S_0(l)$ such that for any $x \in X$ the following implications hold

$$\begin{aligned} \max\{l_1(x), l_2(x)\} \le 0 &\implies l(x) \le 0, \\ l(x) \le 0 &\implies \min\{l_1(x), l_2(x)\} \le 0. \end{aligned}$$

In particular, $[S_0(l_1), S_0(l_2)]_{\bar{\mathcal{G}}^*}$ contains all $S_0(l)$ such that

$$\min\{l_1(x), l_2(x)\} \le l(x) \le \max\{l_1(x), l_2(x)\} \quad \forall\, x \in X. \tag{15}$$

Proposition 4.5. *Assume that L is closed under vertical shifts (this means that for each $l \in L$ and $c \in \mathbb{R}$ the function $h(x) = l(x) + c$ belongs to L). Let $x_1, x_2 \in X$. Then*

$$[x_1, x_2]_{\bar{\mathcal{G}}} = \{x \in X : \min\{l(x_1), l(x_2)\} \leq l(x) \leq \max\{l(x_1), l(x_2)\} \;\forall\, l \in L\}. \tag{16}$$

If, moreover, X is equipped with a topology $\mathcal{T}$ such that

$$\{x \in X : l(x) < 0\} \subset \operatorname{int} S_0(l) \quad \forall\, l \in L \tag{17}$$

then condition (9) is valid for $\mathcal{T}$.

Proof. First we check the equality (16). Let $x \in [x_1, x_2]_{\bar{\mathcal{G}}}$. Take an arbitrary $l \in L$ and consider the number $c = \max\{l(x_1), l(x_2)\}$. Since L is closed under vertical shifts then the function $h(z) = l(z) - c$ belongs to L. It is easy to see that $x_1, x_2 \in S_0(h)$. Since $x \in [x_1, x_2]_{\bar{\mathcal{G}}}$ then $x \in S_0(h)$, therefore $h(x) = l(x) - \max\{l(x_1), l(x_2)\} \leq 0$.

In order to check the inequality $\min\{l(x_1), l(x_2)\} \leq l(x)$ consider the function $h(z) = l(z) - l(x)$. Since $S_0(h) \in \mathcal{H}$ and $x \in S_0(h)$ then, by Proposition 2.1, $\{x_1, x_2\} \cap S_0(h) \neq \emptyset$. Hence either $h(x_1) \leq 0$ or $h(x_2) \leq 0$. This means that $\min\{l(x_1), l(x_2)\} \leq l(x)$.

Let $\mathcal{T}$ be a topology on X, which enjoys (17). We will show that (14) holds for $\mathcal{T}$. Then, by Proposition 3.7, condition (9) holds as well. Let F be a finite subset of X and $a \notin \operatorname{conv}_{\mathcal{G}} F$. Then there is a set $S_0(l) \in \mathcal{H}$ with $l \in L$ such that $F \subset S_0(l)$ and $a \notin S_0(l)$. Take a positive number ε and consider the function $h_\varepsilon(x) = l(x) - \varepsilon$. Since L is closed under vertical shifts then $h_\varepsilon \in L$ and, by (17),

$$\{x \in X : l(x) < \varepsilon\} = \{x \in X : h_\varepsilon(x) < 0\} \subset \operatorname{int} S_0(h_\varepsilon).$$

Hence $S_0(l) \subset \operatorname{int} S_0(h_\varepsilon)$ for any positive ε. In particular, we can take $\varepsilon = l(a)/2 > 0$ because $a \notin S_0(l)$. We have: $F \subset S_0(l) \subset \operatorname{int} S_0(h_\varepsilon)$ and $a \notin S_0(h_\varepsilon)$. Thus, condition (14) is valid. $\square$

Remark 4.6. Note that the continuity of all functions in L is sufficient for (17) to hold.

Consider the classical convex case.

Proposition 4.7. *Let $(X, \mathcal{T})$ be a topological linear space and L be the set of all continuous affine functions $l : X \to \mathbb{R}$. Let $\mathcal{H}$ be the collection of all level sets $S_0(l) = \{x \in X : l(x) \leq 0\}$, where $l \in L$ (in other words, $\mathcal{H}$ consists of the empty set, whole X and all closed half-spaces of X). Then the convexity $\mathcal{G}$ generated by $\mathcal{H}$ is of arity 2.*

Proof. We will prove that the space $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to the convexity $\bar{\mathcal{G}}^*$, and $(X, \mathcal{T})$ is connected with respect to $\bar{\mathcal{G}}$.

Let $l_1, l_2 \in L$. Since l_1 and l_2 are continuous and affine then for every $\alpha \in [0,1]$ the function $l(x) = (1-\alpha)l_1(x) + \alpha l_2(x)$ is also continuous and affine. Consider the mapping $\omega : [0,1] \to \mathcal{H}$ defined by

$$\omega(\alpha) = \{x \in X : \; (1-\alpha)l_1(x) + \alpha l_2(x) \leq 0\}. \tag{18}$$

Then $\omega(0) = S_0(l_1)$ and $\omega(1) = S_0(l_2)$. Moreover, since for any $\alpha \in [0,1]$ and $x \in X$

$$\min\{l_1(x), l_2(x)\} \leq (1-\alpha)l_1(x) + \alpha l_2(x) \leq \max\{l_1(x), l_2(x)\}$$

then, due to (15), $\omega(\alpha) \in [S_0(l_1), S_0(l_2)]_{\bar{\mathcal{G}}^*}$ for all $\alpha \in [0,1]$.

Assume that $\mathcal{H}$ is equipped with the topology $\mathcal{T}'_{\mathcal{H}}$. We need to check that ω is continuous on $[0,1]$. Take an arbitrary $\alpha_0 \in [0,1]$ and $x \notin \omega(\alpha_0)$. Then $(1-\alpha_0)l_1(x) + \alpha_0 l_2(x) > 0$ and we can find a sufficiently small number $\varepsilon > 0$ such that $(1-\alpha)l_1(x) + \alpha l_2(x) > 0$ for all $\alpha \in [0,1] \cap (\alpha_0 - \varepsilon, \alpha_0 + \varepsilon)$. This implies continuity of ω (see Proposition 4.2). Thus, by Proposition 4.1, the space $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to the convexity $\bar{\mathcal{G}}^*$.

Let $x_1, x_2 \in X$. Consider the mapping $\omega : [0,1] \to X$ defined by

$$\omega(\alpha) = (1-\alpha)x_1 + \alpha x_2. \tag{19}$$

Then $\omega(0) = x_1$ and $\omega(1) = x_2$. Since $l((1-\alpha)x_1 + \alpha x_2) = (1-\alpha)l(x_1) + \alpha l(x_2)$ whenever l is affine then

$$\min\{l(x_1), l(x_2)\} \le l(\omega(\alpha)) \le \max\{l(x_1), l(x_2)\} \quad \forall\, l \in L.$$

Hence, by (16), $\omega(\alpha) \in [x_1, x_2]_{\bar{\mathcal{G}}}$ for any $\alpha \in [0,1]$. Since $(X, \mathcal{T})$ is a topological linear space then ω is continuous on $[0,1]$. Due to Proposition 4.1, the space $(X, \mathcal{T})$ is connected with respect to the convexity $\bar{\mathcal{G}}$.

Note that L is closed under vertical shifts, and all functions $l \in L$ are continuous in topology $\mathcal{T}$ (in particular, they enjoy (17)). Then, by Proposition 4.5, condition (9) is valid for $\mathcal{T}$. At last, it follows from Theorem 3.6 that the convexity $\mathcal{G}$ generated by the collection of all closed half-spaces of X is of arity 2. □

Now consider the case of affine functions defined on an arbitrary linear space.

Example 4.8. Let X be a linear space and L be the set of all affine functions $l : X \to \mathbb{R}$. As in Proposition 4.7, let $\mathcal{H}$ be the collection of all level sets $S_0(l) = \{x \in X : \ l(x) \le 0\}$ $(l \in L)$ and $\mathcal{G}$ the convexity on X generated by $\mathcal{H}$. Then the space $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to the convexity $\bar{\mathcal{G}}^*$ and $(X, \mathcal{T}'_X)$ is connected with respect to $\bar{\mathcal{G}}$. Indeed, for any $l_1, l_2 \in L$ the function (18) enjoys all required properties. For $x_1, x_2 \in X$ we only need to check that the function (19) is continuous on $[0,1]$ if X is equipped with the topology $\mathcal{T}'_X$. Let $\alpha_0 \in [0,1]$ and $l \in L$ be such that $l(\omega(\alpha_0)) > 0$. Since l is affine then $(1-\alpha_0)l(x_1) + \alpha_0 l(x_2) = l(\omega(\alpha_0)) > 0$, hence a positive number ε exists such that $l(\omega(\alpha)) = (1-\alpha)l(x_1) + \alpha l(x_2) > 0$ for all $\alpha \in [0,1] \cap (\alpha_0 - \varepsilon, \alpha_0 + \varepsilon)$. This implies continuity of ω (see Proposition 4.3).

Since for any affine function l the function $-l$ is also affine then the set

$$\{x \in X : \ l(x) < 0\} = \{x \in X : \ -l(x) > 0\} = X \backslash S_0(-l) \in \mathcal{H}'$$

is open in the topology $\mathcal{T}'_X$ for all $l \in L$. Hence (17) holds true for $\mathcal{T} = \mathcal{T}'_X$, and, by Proposition 4.5, condition (9) is also valid. Thus, due to Theorem 3.6, the convexity $\mathcal{G}$ on X is of arity 2.

The following example demonstrates that the estimate of arity number in Theorem 3.6 is sharp.

Example 4.9. Let $N \ge 2$. Choose an arbitrary vectors $e^1, \dots, e^N \in \mathbb{R}^{N-1}$ such that every $(N-1)$ of them are linearly independent and zero is a convex combination of all

e^i (for example, we can take usual orthogonal base of $\mathbb{R}^{N-1}$ and vector $(-1,\dots,-1)$). Let $X = X_1 \cup \dots \cup X_N$, where $X_i = \{ae^i : a \geq 0\}$ for any $i = 1,\dots,N$. Let L be the set of all affine functions defined on $\mathbb{R}^{N-1}$ and $\mathcal{H}$ the collection of all level sets $S_0(l) = \{x \in X : l(x) \leq 0\}$, $l \in L$. Then all sets X_i are convex in convexity $\mathcal{G}$ generated by $\mathcal{H}$. Proposition 4.1 allows to prove that $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to $\bar{\mathcal{G}}^*$ and $(X, \mathcal{T}'_X)$ is N-connected with respect to $\bar{\mathcal{G}}$. Indeed, we can use the same functions $\omega : [0,1] \to \mathcal{H}$ and $\omega : [0,1] \to X$ as in the proof of Proposition 4.7 and Example 4.8. For each $i = 1,\dots,N$ we have: $\omega(\alpha) = (1-\alpha)x_1 + \alpha x_2$ belongs to $[x_1, x_2]_{\bar{\mathcal{G}}}$ for any $x_1, x_2 \in X_i$ and $\alpha \in [0,1]$. Condition (17) is also valid for the topology $\mathcal{T} = \mathcal{T}'_X$. Then, by Proposition 4.5 and Theorem 3.6, the convexity $\mathcal{G}$ on $X = X_1 \cup \dots \cup X_N$ is of arity N. Now we show that $\mathcal{G}$ is not of arity $N-1$. Consider the set $A = \{e^1,\dots,e^N\}$. Then, due to our choice of vectors e^i, $\mathrm{conv}_{\mathcal{G}} F = X \cap \mathrm{conv} F = F \subset A$ for any $F \in [A]^{\leq N-1}$ (here $\mathrm{conv} F$ is the classical convex hull of F in $\mathbb{R}^{N-1}$). However, the set A does not belong to $\mathcal{G}$ since $0 \in \mathrm{conv}_{\mathcal{G}} A$.

Subbases of epigraphs $\mathrm{epi}\, l = \{(y,c) \in Y \times \mathbb{R} : l(y) \leq c\}$

Let L be a set of real-valued functions defined on a set Y. Let $X = Y \times \mathbb{R}$. Consider the collection $\mathcal{H}$ of all epigraphs $\mathrm{epi}\, l = \{(y,c) \in Y \times \mathbb{R} : l(y) \leq c\}$, where $l \in L$.

Let $(y_1,c_1),(y_2,c_2) \in Y \times \mathbb{R}$. Then the set $[(y_1,c_1),(y_2,c_2)]_{\bar{\mathcal{G}}}$ consists of all points $(y,c) \in Y \times \mathbb{R}$ such that for any $l \in L$ the following implications hold

$$\begin{aligned} \max\{l(y_1)-c_1, l(y_2)-c_2\} \leq 0 &\implies l(y) \leq c, \\ l(y) \leq c &\implies \min\{l(y_1)-c_1, l(y_2)-c_2\} \leq 0. \end{aligned}$$

In particular, $[(y_1,c_1),(y_2,c_2)]_{\bar{\mathcal{G}}}$ contains all (y,c) such that

$$\min\{l(y_1)-c_1, l(y_2)-c_2\} \leq l(y) - c \leq \max\{l(y_1)-c_1, l(y_2)-c_2\} \quad \forall\, l \in L. \tag{20}$$

At the same time, we have a very easy description of the set $[\mathrm{epi}\, l_1, \mathrm{epi}\, l_2]_{\bar{\mathcal{G}}^*}$ for every $l_1, l_2 \in L$:

$$\begin{aligned} [\mathrm{epi}\, l_1, \mathrm{epi}\, l_2]_{\bar{\mathcal{G}}^*} &= \{\mathrm{epi}\, l : l \in L,\ (\mathrm{epi}\, l_1 \cap \mathrm{epi}\, l_2) \subset \mathrm{epi}\, l \subset (\mathrm{epi}\, l_1 \cup \mathrm{epi}\, l_2)\} \\ &= \{\mathrm{epi}\, l : l \in L,\ \min\{l_1(y), l_2(y)\} \leq l(y) \leq \max\{l_1(y), l_2(y)\}\ \ \forall\, y \in Y\}. \end{aligned}$$

Proposition 4.10. *Assume that L is closed under vertical shifts. Let $(y_1,c_1),(y_2,c_2) \in Y \times \mathbb{R}$. Then a point $(y,c) \in Y \times \mathbb{R}$ belongs to $[(y_1,c_1),(y_2,c_2)]_{\bar{\mathcal{G}}}$ if and only if (20) holds.*

In other words,

$$[(y_1,c_1),(y_2,c_2)]_{\bar{\mathcal{G}}} = \{(y,c) : f(y) \leq c \leq g(y)\},$$

where the functions f and g are defined by

$$\begin{aligned} f(x) &= \sup_{l \in L}(l(x) - \max\{l(y_1)-c_1, l(y_2)-c_2\}), \\ g(x) &= \inf_{l \in L}(l(x) - \min\{l(y_1)-c_1, l(y_2)-c_2\}). \end{aligned}$$

Let, moreover, $Y \times \mathbb{R}$ be equipped with a topology $\mathcal{T}$ such that

$$\{(y,c) : l(y) < c\} \subset \mathrm{int}\, \mathrm{epi}\, l \quad \forall\, l \in L.$$

Then condition (9) is valid for $\mathcal{T}$.

Proof. Let $(y,c) \in [(y_1,c_1),(y_2,c_2)]_{\bar{\mathcal{G}}}$. Take an arbitrary $l \in L$ and consider the following functions defined on Y:

$$h(z) = l(z) - \max\{l(y_1) - c_1, l(y_2) - c_2\}, \qquad h'(z) = l(z) - l(y) + c.$$

Since L is closed under vertical shifts then $h, h' \in L$. We have

$$h(y_1) \le c_1, \qquad h(y_2) \le c_2, \qquad h'(y) = c.$$

Since $(y,c) \in [(y_1,c_1),(y_2,c_2)]_{\bar{\mathcal{G}}}$ then $h(y) \le c$ and $\min\{h'(y_1)-c_1, h'(y_2)-c_2\} \le 0$. This means that $l(y)-c \le \max\{l(y_1)-c_1, l(y_2)-c_2\}$ and $\min\{l(y_1)-c_1, l(y_2)-c_2\} \le l(y)-c$.

Let $\mathcal{T}$ be a topology on $Y \times \mathbb{R}$ such as in the statement of proposition. Let F be a finite subset of $Y \times \mathbb{R}$ and $(y,c) \notin \mathrm{conv}_{\mathcal{G}} F$. Then there is a function $l \in L$ such that $F \subset \mathrm{epi}\, l$ and $(y',c') \notin \mathrm{epi}\, l$. Consider the function $h_\varepsilon(z) = l(z) - \varepsilon$, where $\varepsilon = (l(y') - c')/2 > 0$. Since L is closed under vertical shifts then $\mathrm{epi}\, h_\varepsilon \in \mathcal{H}$. We have: $(y',c') \notin \mathrm{epi}\, h_\varepsilon$ and

$$F \subset \mathrm{epi}\, l \subset \{(y,c) :\, h_\varepsilon(y) < c\} \subset \mathrm{int}\,\mathrm{epi}\, h_\varepsilon.$$

Proposition 3.7 implies that condition (9) is valid for $\mathcal{T}$. □

Let L be a set of real valued functions defined on a set Y and let Z be a subset of Y. Recall (see [1]) that a function $f : Y \to \mathbb{R}_{+\infty} = \mathbb{R} \cup \{+\infty\}$ is called L-convex on Z if a subfamily $T \subset L$ exists such that $f(z) = \sup_{l \in T} l(z)$ for all $z \in Z$.

Proposition 4.11. *Let $N \ge 2$ and $\mathcal{T}$ be a topology on X, which enjoys* (9). *Assume that $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to $\bar{\mathcal{G}}^*$ and $(X,\mathcal{T})$ is N-connected with respect to $\bar{\mathcal{G}}$. Then for any function $f : Y \to \mathbb{R}_{+\infty}$ the following conditions are equivalent:*

(*i*) *For every $y, y_1, \dots, y_N \in Y$*

$$f(y) \le \sup\{l(y) :\, l \in L,\, l(y_i) \le f(y_i)\ \ \forall\, i = 1, \dots, N\}. \tag{21}$$

(*ii*) *f is L-convex on every finite subset of Y.*

Proof. Theorem 3.6 implies that the convexity $\mathcal{G}$ generated by $\mathcal{H}$ is of arity N.

(*i*) $\Longrightarrow$ (*ii*) Let a function f enjoy (21) for all $y, y_1, \dots, y_N \in Y$. Then its epigraph $\mathrm{epi}\, f$ belongs to the convexity $\mathcal{G}$. Indeed, since $\mathcal{G}$ is N-ary then $\mathrm{epi}\, f$ belongs to $\mathcal{G}$ if and only if $\mathrm{conv}_{\mathcal{G}}\{(y_1,c_1), \dots, (y_N,c_N)\} \subset \mathrm{epi}\, f$ for any $(y_1,c_1), \dots, (y_N,c_N) \in \mathrm{epi}\, f$. So let $(y_1,c_1), \dots, (y_N,c_N) \in \mathrm{epi}\, f$. Then we have

$$\begin{aligned} \mathrm{conv}_{\mathcal{G}}\{(y_1,c_1), \dots, (y_N,c_N)\} &= \{(y,c) :\, \sup\{l(y) :\, l(y_i) \le c_i\ \ \forall\, i \le N\} \le c\} \\ &\subset \{(y,c) :\, \sup\{l(y) :\, l(y_i) \le f(y_i)\ \ \forall\, i \le N\} \le c\} \\ &\subset \mathrm{epi}\, f. \end{aligned}$$

Let Z be a finite subset of Y. If $f(z) = +\infty$ for all $z \in Z$ then also $\sup_{l \in L} l(z) = +\infty$ for all $z \in Z$, and therefore f is L-convex on Z. Indeed, if $f(z) \equiv +\infty$ on Z then it follows from (21) that for any $z, y_1, \dots, y_N \in Z$

$$\sup_{l \in L} l(z) = \sup\{l(z) :\, l \in L,\, l(y_i) \le +\infty\ \ \forall\, i = 1, \dots, N\} \ge f(z) = +\infty.$$

Now assume that the set $F = \{(z, f(z)) : \; z \in Z, \; f(z) < +\infty\}$ is not empty. Since F is a finite subset of $Y \times \mathrm{I\!R}$ then, due to (2),

$$\mathrm{conv}_{\mathcal{G}} F = \bigcap \{H \in \mathcal{H} : \; F \subset H\} = \bigcap \{\mathrm{epi}\, l : \; l \in L, \, l(z) \leq f(z) \;\; \forall\, z \in Z\}. \tag{22}$$

Let T be the collection of all functions $l \in L$ such that $l(z) \leq f(z)$ for any $z \in Z$. Since $\mathrm{epi}\, f \in \mathcal{G}$ and $F \subset \mathrm{epi}\, f$ then $\mathrm{conv}_{\mathcal{G}} F \subset \mathrm{epi}\, f$. This means, in view of (22), that T is nonempty (otherwise $\mathrm{conv}_{\mathcal{G}} F = Y \times \mathrm{I\!R} \not\subset \mathrm{epi}\, f$) and $f(y) \leq \sup_{l \in T} l(y)$ for all $y \in Y$. On the other hand, $\sup_{l \in T} l(z) \leq f(z)$ for any $z \in Z$ by definition of T. Hence $f(z) = \sup_{l \in T} l(z) \;\; \forall\, z \in Z$. In other words, f is L-convex on Z.

$(ii) \Longrightarrow (i)$ Let f be L-convex on every finite subset of Y. Let $y, y_1, \ldots, y_N \in Y$. Since f is L-convex on $\{y, y_1, \ldots, y_N\}$ then

$$\begin{aligned} f(y) &= \sup\{l(y) : \; l \in L, \, l(y) \leq f(y), \, l(y_i) \leq f(y_i) \;\; \forall\, i = 1, \ldots, N\} \\ &\leq \sup\{l(y) : \; l \in L, \, l(y_i) \leq f(y_i) \;\; \forall\, i = 1, \ldots, N\}. \end{aligned}$$

□

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