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Convex Decompositions

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We consider decompositions S of a closed, convex set P into smaller, closed and convex regions. The *thin convex* decompositions are those having a certain strong convexity property as a set of sets. Thin convexity is directly connected to our intended application in voting theory (see [8, 9]), via the *consistency* property for abstract voting systems. The *facial* decompositions are those for which each intersecting pair of regions meet at a common face. The class of *neat* decompositions is defined by a separation property, *neat separability* by a hyperplane, applied to the regions. The *regular* decompositions are those whose regions, when we take cross sections by lines, reduce to closed intervals, any two of which are equal, or are disjoint, or overlap only at their endpoints. Our main result is that for polytopes P these four classes of decompositions are the same. The *Voronoi* decompositions of P are those whose regions are determined by the point (chosen from a designated finite subset Y of P) to which they are closest. These form a fifth class of decompositions, which is strictly contained in any of the first four classes.

Keywords: Convex set, decomposition, consistent voting system, hyperplane separation properties, Voronoi regions, regular decomposition, neat decomposition

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1. Introduction

We divide a closed and convex set P into smaller closed and convex regions, and consider properties of the resulting decomposition. Our goal is to provide useful characterizations of the class of thin convex decompositions, which plays a role in geometric representations of voting systems (see [8, 9]). We begin with a few key definitions.

Definition 1.1. If $X = \{x_1, x_2, \dots, x_m\}$ is a finite set of points of $\mathbf{R}^n$, then an *affine combination* of X is a sum of the form $\sum_{i=1}^m \alpha_i x_i$, where the α_j are real numbers, not necessarily positive, that sum to 1. The *affine span* $\mathrm{Aff}(r)$ of a set $r \subseteq \mathbf{R}^n$ is the set of all affine combinations of finite sets $X \subseteq r$. A set $Y \subseteq \mathbf{R}^n$ is *affinely independent* if no member y of Y lies in the affine span $\mathrm{Aff}(Y - \{y\})$ of the other points. The *affine dimension* of a set $r \subseteq \mathbf{R}^n$ is $m - 1$, where m is the maximal size of an affinely

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independent subset $Y \subseteq r$. The *relative interior* $\mathrm{r.i.}(r)$ of a set $r \subseteq \mathbf{R}^n$ is the interior of r as taken according to the induced or relative topology on $\mathrm{Aff}(r)$, taken as a subset of $\mathbf{R}^n$ with the standard topology. A *convex polyhedron* is an intersection of finitely many closed half spaces of $\mathbf{R}^n$. A *polytope* is a bounded convex polyhedron; equivalently, it is the convex hull of some finite set V of points of $\mathbf{R}^n$.

Definition 1.2. Let P be a convex, closed subset of $\mathbf{R}^n$. A *closed decomposition of* P is a finite set S of non-empty closed subsets of P, called *regions*, whose union is P. A non-empty intersection of one or more regions of S will be called a *subregion*.

Definition 1.3. A point q of a subset P of $\mathbf{R}^n$ is *strictly between* two other points q_1 and q_2 of P if q lies on the open line segment from q_1 to q_2 (equivalently, $q = \alpha q_1 + (1-\alpha)q_2$ for some α in $(0, 1)$).

The first class of decompositions we consider is the one most directly suggested by the intended application to voting systems that are based on the mean.

Definition 1.4. A closed decomposition S of P is *thin convex* if whenever q_1 and q_2 lie in at least one common region of S, and q is strictly between q_1 and q_2, the regions of S containing q as a member are precisely those regions that contain both q_1 and q_2 as members.

Notice that thin convexity implies, but is strictly stronger than, the requirement that every region (and subregion) of S be convex. For example, the closed decomposition of Figure 1.1a, in which the members of S are the three planar regions r, u, and v, is thin convex. In particular, for the labeled points q_1, q_2, and q, we see that q is strictly between q_1 and q_2, that the set of regions containing q_1 as a member is $\{r, u\}$, the set containing q_2 as a member is $\{u, v\}$ and the set containing q as a member is $\{u\} = \{r, u\} \cap \{u, v\}$. This decomposition S would still be thin convex if we added, as additional members of S, any or all of the lower dimensional subregions $r \cap u$, $r \cap v$, $u \cap v$, or $r \cap u \cap v$. The decompositions of Figure 1.1b and 1.1c fail to be thin convex, however, as witnessed by the labeled points p_1, p_2, and p; and t_1, t_2, and t, respectively.

The definition of thin convexity is inspired by the voting-theoretic property of consistency, as introduced by Smith [4] and Young [6, 7]. Our first characterization of thin convexity is based on the strong similarity between this definition and the standard notion of a *face* in convexity theory (see Weber [5] or Rockafellar [3], for example).

Definition 1.5. A *face* of a set u of points of $\mathbf{R}^n$ is a subset f of u with the property that whenever two points p and q of u have a point of f lying strictly between them, both p and q belong to f.

Definition 1.6. A closed decomposition S of P is *facial* if every region of S is convex, and every intersection of two regions is a face of each of the regions.

Observation 1.7. Let S be a closed decomposition of a closed, convex set P. Then S is thin convex if and only if it is facial.

We leave the proof of Observation 1.7 as an exercise for the reader. While this observation already begins to give us some insight into what thin convex decompositions look like, the

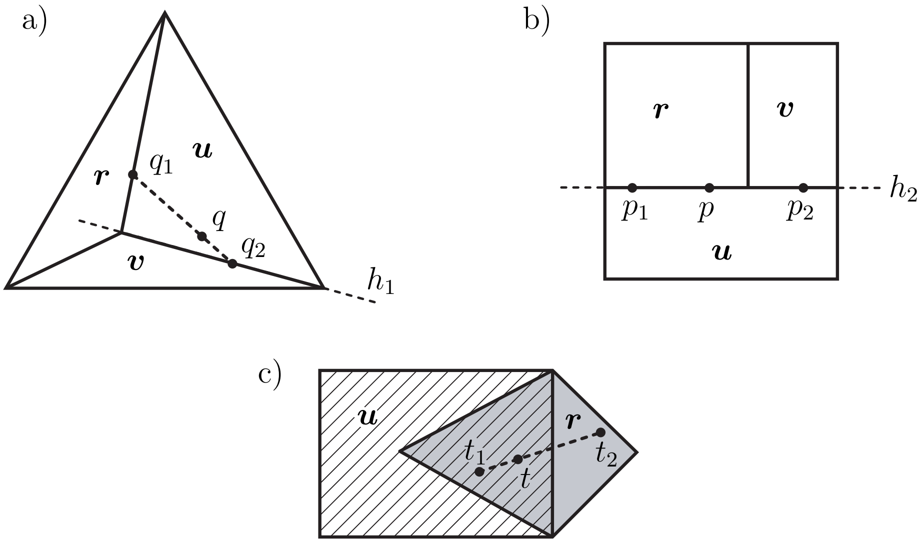

Figure 1.1: Three closed decompositions.

examples in Figure 1.1 suggest that there may be a more constructive characterization, posed in terms of the hyperplanes that slice up P.

Definition 1.8. Two sets u and v of points of $\mathbf{R}^n$ are *weakly separated by a hyperplane h of* $\mathbf{R}^n$ if every point of u lies either on h or to one side of h, and every point of v lies either on h or to the other side of h. These sets are *properly separated by h* if they are weakly separated and they are not both contained as subsets of h, and are *neatly separated by h* if they are weakly separated by h and satisfy the additional requirement that $u \cap v = h \cap u = h \cap v$. They are *strictly separated by h* if they are weakly separated and each is disjoint from h.

The definition mentions these four properties in the order of (strictly) increasing strength, if we limit ourselves to distinct sets $u \neq v$. To get a feel for the meaning of neat separation, the one property that is new, note that in the closed decomposition of Figure 1.1a, P is a 2-simplex Δ_2 in $\mathbf{R}^2$ and there are three slicing hyperplanes, which are lines. Each pair of regions is neatly separated by one of these lines. In the closed decomposition of Figure 1.1b, however, the line labeled h_2 is the only hyperplane weakly (and properly) separating regions r and u, and h_2 does not separate neatly. In the decomposition of Figure 1.1c, regions r and u cannot be weakly separated.

Definition 1.9. A closed decomposition of P is *neat* if each pair of distinct regions of S is neatly separated by some hyperplane.

Before turning to the proof that neat is equivalent to thin convex, we consider an additional condition on closed decompositions that facilitates this argument. This *regularity* condition also provides some intuition for these decompositions. The main idea arises from considering cross sections of decompositions.

Definition 1.10. If S is a closed decomposition of the closed, convex set $P \subseteq \mathbf{R}^n$, and A is a closed and convex subset of $\mathbf{R}^n$ that meets P, then the *restriction* of S to A is given by $S|_A = \{\, r \cap A \mid r \in S \text{ and } r \cap A \neq \emptyset \,\}$.

It is immediate that $S|_A$ in the above definition will always be a closed decomposition of the closed, convex set $P \cap A \subseteq \mathbf{R}^n$. Note that the restriction of a decomposition to a set A may have fewer regions than does S, even when A meets every region of S, because two distinct regions u and v of S may intersect A in the same region. In Figure 1.1a, for example, when we restrict S to the hyperplane h_1 the resulting decomposition $S|_{h_1}$ has only two regions, one of which is equal both to $u \cap h_1$ and to $v \cap h_1$.

Proposition 1.11. *Let S be a closed decomposition of the closed, convex set $P \subseteq \mathbf{R}^n$, and A be a closed, convex subset of $\mathbf{R}^n$ with $P \cap A \neq \emptyset$. Then if S is neat, its restriction $S|_A$ is also neat as a decomposition of $P \cap A$, and if S is thin convex then $S|_A$ is also thin convex.*

Proof. If some hyperplane k neatly separates regions r and u of S, and $r \cap A \neq u \cap A$, then the same hyperplane k neatly separates regions $r \cap A$ and $u \cap A$ of $S|_A$. Thus, if S is neat, $S|_A$ is also neat. The proof for thin convexity is immediate. □

In particular, Proposition 1.11 holds when A is a hyperplane, a line, or a line segment. In fact, the definition of neat can be simplified when applied to decompositions of a line segment. This allows us to state several equivalent reformulations.

Proposition 1.12. *Let S be a closed decomposition of a closed, convex set $P \subseteq \mathbf{R}^n$, and L be any line of $\mathbf{R}^n$ that meets P. Then the following conditions on $S|_L$ are equivalent:*

(a) *$S|_L$ is thin convex.*
(b) *$S|_L$ is neat.*
(c) *Every region of $S|_L$ is a closed interval of L, and every two such regions are either identical or have no common points in the relative interior of either.*
(d) *Every region of $S|_L$ is a closed interval of L, and if any such region r contains a point lying in the relative interior of another such region u, then $u \subseteq r$.*
(e) *Every region of $S|_L$ is a closed interval, and every two such regions are equal, or are disjoint, or overlap only at a single point which is an endpoint of each.*

Proof. Note that for the purposes of this proposition we classify a singleton set $\{a\}$ as a closed interval $[a, a]$ of any line through a. Condition (e) thus requires that any overlap between two regions $[a, a]$ and $[b, c]$ of $S|_L$ implies $a = b$ or $a = c$. It is routine to check that conditions (c), (d), and (e) are equivalent. If $S|_L$ is neat then each neatly separating hyperplane intersects L in a single point. (Observe that neat and proper separation are equivalent in the case of closed subsets of a line L.) It follows easily that condition (e) is satisfied, with some of these single points serving as the endpoints of regions of $S|_L$. Conversely, if condition (e) is satisfied, then each endpoint a of a region of $S|_L$ may be extended to the hyperplane $h_{a,L}$ through a that is normal to L, and these special hyperplanes suffice to obtain all the neat separations required to confirm condition (b). It remains to show that condition (a) is equivalent to the others.

(a) ⇒ (d) Assume $S|_L$ is thin convex. It follows immediately that each region r of $S|_L$ is an interval, which must be closed as the regions of S are closed. Let r and u be regions of $S|_L$, and assume point q lies both in r and in the relative interior of u. To see that $u \subseteq r$, let p be any other point of u. As q is in the relative interior of u we can find a point t in u such that q lies strictly between p and t on L. Thin convexity now implies that as p and t lie in a common region u, and the point q (lying strictly between p and t) is a member of r, then $p \in r$.

$(c) \Rightarrow (a)$ Assume $S|_L$ satisfies condition (c). Let q_1 and q_2 be distinct points lying in some common region r of $S|_L$, and let q be any point in the interior of the line segment $\overline{q_1 q_2}$. Let r' be any region of $S|_L$. If both $q_1 \in r'$ and $q_2 \in r'$ hold, then as r' is an interval, $q \in r'$. Now assume $q \in r'$. Then there exists a point of r' lying in the interior of r, so $r = r'$, and thus $q_1, q_2 \in r'$. Hence $q \in r'$ if and only if both $q_1 \in r'$ and $q_2 \in r'$, as required. □

Note that a set $P \subseteq \mathbf{R}^n$ is convex if *and only if* every line L of $\mathbf{R}^n$ has a convex intersection $P \cap L$ with P. This suggests that perhaps Proposition 1.11 has a converse given in terms of *all* linear cross sections, and inspires the following definition:

Definition 1.13. Let S be a closed decomposition of the closed, convex set $P \subseteq \mathbf{R}^n$. Then S is *regular* if for every line L of $\mathbf{R}^n$ that meets P, $S|_L$ satisfies any or all of the equivalent conditions of Proposition 1.12.

This preservation under cross-section is a key property that distinguishes the classes we have been discussing from a narrower, closely related class that is of direct interest to the voting-theoretic properties considered in [8, 9]. One easy way to generate a neat decomposition of a closed, convex set P is to choose a finite set Y of points of $\mathbf{R}^n$, and for each $y \in Y$ let

$$V_y = \left\{ q \in \mathbf{R}^n \;\middle|\; \|q - y\| \leq \|q - z\| \text{ for each } z \in Y \right\}.$$

These proximity regions V_y of points that are at least as close to y as to any other point of Y are called *Voronoi regions* (see, for example, [2]).

Definition 1.14. Let P be a closed and convex subset of $\mathbf{R}^n$ and Y be a finite set of points of P. Then the *Voronoi decomposition* V_Y of P is given by

$$V_Y = \{ V_y \cap P \mid y \in Y \text{ and } V_y \cap P \neq \emptyset \},$$

and S is said to be a *Voronoi decomposition* of P if $S = V_Y$ for some finite set Y of points in P.

Proposition 1.15. *Let P be a closed and convex subset of $\mathbf{R}^n$ and Y be a finite set of points of P. Then the Voronoi decomposition V_Y of P is neat.*

Proof. Let $V_y \cap P$ and $V_z \cap P$ be distinct regions of V_Y, and $h_{y,z}$ be the hyperplane consisting of all points that are equidistant from y and z. Then it is straightforward to check that $h_{y,z}$ neatly separates $V_y \cap P$ and $V_z \cap P$. □

For the applications to voting in [8, 9], it would in some respects be preferable to characterize Voronoi decompositions, rather than neat decompositions, but we do not at this time see how to do this in a way that establishes a clear link with representations of voting systems. One obstacle seems to be that Voronoi decompositions fail to have the sort of hereditary behavior (as revealed by Proposition 1.11 and Theorem 2.1) exhibited by our other classes. In fact, it is easy to see that if P is a line segment, and S^* is a closed decomposition of P into five intervals whose lengths alternate long-short-long-short-long (where a long interval is more than twice as long as a short one) then it is impossible to position points y_1, y_2, y_3, y_4, y_5 on P (or to position them anywhere on the straight line extending P) in any way that makes the decomposition Voronoi. It *is* possible, however, to position the y_j off to the side of L, as in Figure 2.1, which depicts a decomposition S^* that is a non-Voronoi restriction (to a line) of a Voronoi decomposition of a planar polytope into hexagonal proximity regions.

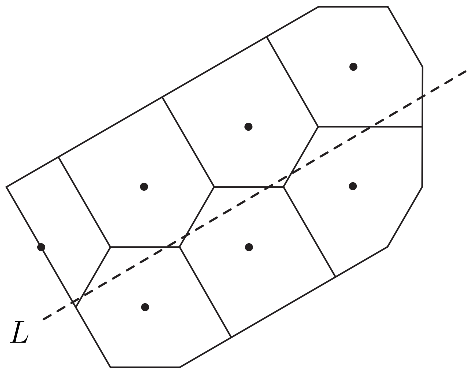

Figure 1.2: A Voronoi decomposition whose restriction to the line L is neat, but not Voronoi.

In one sense, the difficulty may be said to lie with our choice of definition for "Voronoi decomposition." Suppose we define *quasi-Voronoi decomposition* by dropping the requirement, in the definition above, that members of Y must lie inside P, and allowing them to lie anywhere in a Euclidean space E of dimension possibly higher than that of P. Then it is clear that restrictions (to affine subspaces) of quasi-Voronoi decompositions are quasi-Voronoi. The twisted triangle example in [1] provides a neat decomposition that fails to be quasi-Voronoi. With the help of a more recent example of Juan Enrique Martínez-Legaz we have formulated some new axioms – strong forms of thin convexity – that may help characterize the class of quasi-Voronoi decompositions. Appropriate characterizations of these classes would lead to characterizations of some large and natural classes of voting systems – results in the same spirit as the main theorem of [9], which would settle a question left open in [1].

2. Proof of the Main Theorem

Our goal is to prove the following result:

Theorem 2.1. *Let S be a closed decomposition of a polytope $P \subseteq \mathbf{R}^n$. Then the following are equivalent:*

(a) *S is thin convex,*
(b) *S is facial,*
(c) *S is neat,*
(d) *S is regular.*

We begin by pointing out that, with the exception of the following Lemma, the proof of equivalence is clear.

Lemma 2.2. *Let S be a closed decomposition of a polytope $P \subseteq \mathbf{R}^n$. Then if S is regular, it is neat.*

Proof of Theorem 2.1. We've already seen that (a) is equivalent to (b). It is immediate from the definition that S is thin convex if and only if every restriction $S|_L$ to a straight line is thin convex, so the equivalence of (a) and (d) follows. From Propositions 1.11 and 1.12, it is also immediate that if S is neat then S is regular. It remains only to establish the lemma.

Proof of Lemma 2.2.[1] Assume S is regular. As each linear cross sections of each region of S is an interval, it follows that the regions of S are convex. We divide the main argument into following series of four claims:

Claim 2.3. *The relative interiors of the regions of S are pairwise disjoint.*

Claim 2.4. *Every pair of distinct regions of S can be properly separated by a hyperplane.*

Claim 2.5. *Each region r of S is a polytope.*

Claim 2.6. *S is neat.*

Proof of Claim 2.3. Suppose by way of contradiction that x is any element of $\text{r. i.}(r) \cap \text{r. i.}(u)$. As r and u are different regions, without loss of generality assume that the set difference $r - u$ is nonempty, and choose a point y in this difference. Let L be the line containing x and y.

Case 1: Assume that $y \in \text{Aff}(u)$. Then any point on L lying strictly between x and y, and sufficiently close to x, is an element of u. So $S|_L$ is not thin convex, violating our assumption that S is regular.

Case 2: Assume that $y \notin \text{Aff}(u)$. Then any point z on L having the property that x lies strictly between y and z and such that z is sufficiently close to x, will be a member of r and not of u (as $z \notin \text{Aff}(u)$). Again, this violates our assumption that S is regular.

Proof of Claim 2.4. The following separation theorem, which appears as Corollory 2.4.11 on page 71 of [5], now establishes proper separability via some hyperplane h:

Theorem 2.7. *Each pair of nonempty convex sets A and B whose relative interiors are disjoint can be properly separated by a hyperplane.*

Proof of Claim 2.5. Boundedness of r follows from boundedness of P. Assume that r has dimension k, so that $\text{Aff}(r)$ may be identified with $\mathbf{R}^k$. Then $\text{Aff}(r)$ is itself a finite intersection of closed half spaces of $\mathbf{R}^n$. We will prove that r, viewed as a subset of $\text{Aff}(r)$, is a finite intersection of closed half-spaces of $\text{Aff}(r)$. As each half space H of $\text{Aff}(r)$ has an extension to a half-space H^* of $\mathbf{R}^n$ for which $H^* \cap \text{Aff}(r) = H$, it then follows that r is a polytope. Note that as S is regular, so is its restriction $S|_{\text{Aff}(r)}$.

We'll work inside $\text{Aff}(r)$, thinking of $\text{Aff}(r)$ as a "copy" of $\mathbf{R}^k$; all references to "$\text{int}(X)$" in the proof of Claim 2.5 will refer to X's interior according to the relative topology of

[1]Juan Enrique Martínez-Legaz has an alternate proof, which uses Motzkin's Theorem of the Alternative in place of the argument via rotating hyperplanes.

$\mathrm{Aff}(r)$. For each non-empty region t of $S|_{\mathrm{Aff}(r)}$ with $t \neq r$ choose $h_{r,t}$ to be a hyperplane of $\mathrm{Aff}(r)$ weakly separating t and r, and let $H_{r,t}$ be the corresponding closed half space of $\mathrm{Aff}(r)$ containing r. Let w_r be the closed polyhedron of $\mathrm{Aff}(r)$ formed as the intersection of all $H_{r,t}$ for $t \neq r$, and let p_r be the polytope $w_r \cap P$. We'll show that $r = p_r$. Clearly $r \subseteq p_r$, so it suffices to prove the reverse containment.

First, we'll show $\mathrm{int}(p_r) \subseteq r$. If not, choose $z \in \mathrm{int}(p_r) - r$. Then $z \in \mathrm{int}(H_{r,t})$ for each $H_{r,t}$, so $z \notin h_{r,t}$. As $z \in p_r - r \subseteq P - r$, $z \in u$ for some region u of $S|_{\mathrm{Aff}(r)}$ with $u \neq r$. But then $z \in H_{r,u} \cap u$, so $z \in h_{r,u}$, a contradiction. Finally, as $\mathrm{int}(p_r) \subseteq r$, $p_r = \overline{\mathrm{int}(p_r)} \subseteq \overline{r} = r$.[2]

Proof of Claim 2.6. The argument is by induction on j, the dimension of P, with the dimension $n \geq j$ of the ambient space $\mathbf{R}^n$ being held fixed. If $j = 1$, then S is neat by Proposition 1.12 and the definition (1.13) of regularity.

Now assume that every regular closed decomposition of a dimension j closed polytope $P \subseteq \mathbf{R}^n$ is neat. Let P be a dimension $j+1$ closed polytope in $\mathbf{R}^n$, S be a regular closed decomposition of P, and r and u be any two distinct regions of S. We need to neatly separate r and u. Choose any hyperplane h that properly separates r and u; assume that the equation of h is $\alpha \cdot x = \beta$, and that each point x of r satisfies $\alpha \cdot x \geq \beta$. If h neatly separates r and u we are done. If $r \cap h = \emptyset$ then any sufficiently small increase in β yields a parallel hyperplane h' that strictly separates r and u, while if $u \cap h = \emptyset$ a small decrease in β achieves strict separation.

So, assume $\emptyset \neq r \cap h \neq u \cap h \neq \emptyset$ (see Figure 2.1). Consider the restriction $S|_h$, which is a closed decomposition of $P \cap h$, and is regular. As h does not contain $r \cup u$, $\dim(P \cap h) \leq j$, so by our inductive hypothesis, we may choose a hyperplane k' of $\mathbf{R}^n$ that neatly separates the distinct regions $r \cap h$ and $u \cap h$ of $S|_h$. It is impossible for k' to equal h, or else $r \cap h \cap k' = r \cap h \neq u \cap h = u \cap h \cap k'$, and k' would not separate neatly. It is also impossible for k' to be parallel to h, because r and u both include points on h, and these points would all lie strictly on the same side of k'. It follows that if we set $k = k' \cap h$ then k is a codimension 2 affine subspace of $\mathbf{R}^n$.

As h properly separates r and u, let $\vec{v}$ be a normal to h pointing to the u side of h (so that no member of r lies strictly on the $\vec{v}$ side of h). As k' neatly separates $r \cap h$ and $u \cap h$, $r \cap k = u \cap k$. Let $\vec{w}$ be any vector that lies in h, is normal to k, and points to the $u \cap h$ side of k (so that, inside h, no member of $r \cap h$ lies strictly on the $\vec{w}$ side of k). Let h_ε be the hyperplane obtained by rotating h about k, taking $\vec{v}$ toward $\vec{w}$ through an angle of ε.

Subclaim 2.4.1. For any sufficiently small value of $\varepsilon > 0$, h_ε neatly separates r and u.

Proof of Subclaim 2.4.1. As $h \cap h_\varepsilon = k$, clearly $h_\varepsilon \cap r$ contains $k \cap r$ and $h_\varepsilon \cap u$ contains $k \cap u$. Also $k \cap r = k \cap u$. As r and u are polytopes, they are convex hulls of

[2] As p_r is closed, to see that $p_r = \overline{\mathrm{int}(p_r)}$, it suffices to show that each point x_1 of p_r is arbitrarily close to some point of $\mathrm{int}(p_r)$. Expand $\{x_1\}$ to a set $X = \{x_1, x_2, \ldots, x_{k+1}\}$ of $k+1$ affinely independent points of p_r. Consider all convex combinations $\alpha_1 x_1 + \alpha_2 x_2 + \ldots \alpha_{k+1} x_{k+1}$ for which each α_j is strictly positive. These points form the interior (in the induced topology on $\mathrm{Aff}(r)$) of the convex hull of X, which is entirely contained in p_r, as p_r is convex. As such points come arbitrarily close to x_1, we can locate an open ball B of $\mathrm{Aff}(r)$ with $B \subseteq p_r$, and such that all points of B are as close as desired to x_1.

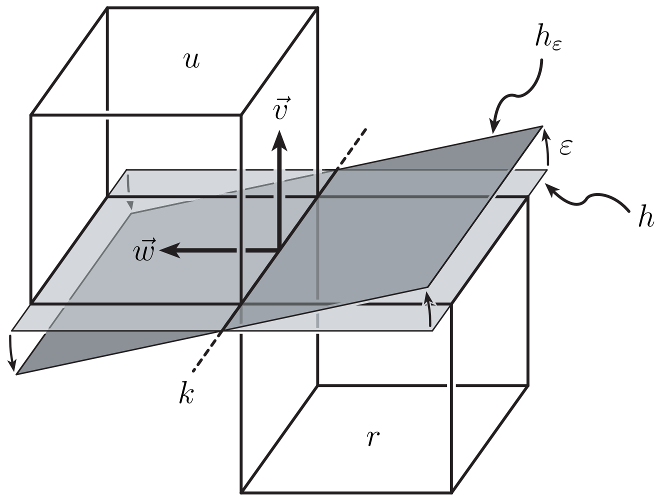

Figure 2.1: Regions u and r are rectangular solids. Plane h separates them weakly, and rotates into h_ε which separates them neatly.

finite sets T_r and T_u, respectively, of vertices.[3] Let $T = (T_r \cup T_u) - k$ consist of those vertices lying off k.

For each vertex s in T there is a unique angle $\theta(s)$ with $0 < \theta(s) \leq \pi$ such that s lies in $h_{\theta(s)}$. Let $\varepsilon < \pi/2$ be any angle satisfying $0 < \varepsilon < \theta(s)$ for each $\theta(s)$. We claim that h_ε neatly separates r and u.

First note that every vertex of r either lies on k or lies strictly on the $-\vec{v}$ side of h_ε, as follows: if $s \in T_r - k$ and $s \in h$, then any rotation of h toward $\vec{v}$ by less than π leaves s strictly on the $-\vec{v}$ side of h_ε, while if $s \notin h$ then, as $\varepsilon < \theta(s)$, s starts strictly on the $-\vec{v}$ side of h and the rotation by ε is too small to change this fact.

It follows that

(i) every point in r lies on h_ε or strictly on the $-\vec{v}$ side of h_ε, and
(ii) $r \cap h_\varepsilon = r \cap k$.

By the same reasoning, we get

(iii) every point in u lies on h_ε or strictly on the $\vec{v}$ side of h_ε, and
(iv) $u \cap h_\varepsilon = u \cap k$.

From points (i) and (iii) we see that h_ε separates weakly, and from (ii) and (iv) it follows that this separation is neat. □

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[3]This application of the Krein-Milman theorem is the only place we use our assumption that P is bounded – i.e., that P is a polytope as well as a convex polyhedron. It seems possible that a suitable extension of Krein-Milman would yield a version of Theorem 2.1 for all convex polyhedra.

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