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The Core of the Infinite Dimensional Generalized Jacobian*

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In this paper, locally Lipschitz functions acting between infinite dimensional normed spaces are considered. When the range is a dual space and satisfies the Radon–Nikodým property, a generalized core-Jacobian, $\Delta f(p)$ is introduced, and its fundamental properties are established. Primarily, it is shown that the β-closure of its convex hull is exactly the generalized Jacobian. Furthermore, the nonemptiness, the β-compactness, the β-upper semicontinuity, and even another representation are obtained. Connections with known notions are derived and chain rules are proved using key results developed. Therefore, the generalized core-Jacobian introduced in this paper is proved to enjoy all the properties that allow this set to be the nucleus of the generalized Jacobian.

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1. Introduction

The field of nonsmooth analysis mainly studies derivative-like objects for nonsmooth functions. The notion of subgradient was introduced for real-valued convex functions in the late fifties by Rockafellar in [23], and in the references therein. Subsequently, the light has been directed on objects for nonconvex, in particular, locally Lipschitz functions acting between two normed spaces X and Y.

When X and Y are *both* finite dimensional and $f : \mathcal{D} \subseteq X \to Y$ is *vector-valued* and locally Lipschitz, Clarke in [2] and [3] introduced the notion of the *generalized Jacobian* $\partial f(p)$ as the convex hull of $\Delta f(p)$ what we call here the *core-Jacobian* of f at p:

$$\begin{gathered}\partial f(p) := \operatorname{co}\Delta f(p),\\ \Delta f(p) := \Big\{\Phi \in \mathcal{L}(X,Y) \mid \exists\,(x_i)_{i\in\mathbb{N}} \text{ in } \Omega(f) : \lim_{i\to\infty} x_i = p \text{ and } \lim_{i\to\infty} Df(x_i) = \Phi\Big\}.\end{gathered} \tag{1}$$

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Here $\Omega(f)$ denotes the set of the points of the open set $\mathcal{D}$ where f is differentiable, it is of full Lebesgue measure. It is shown in [17, Remark 3.4] that this generalized Jacobian is equivalent to the one proposed by Pourciau in [20], namely,

$$\partial^P f(p) := \bigcap_{\delta>0} \overline{\mathrm{co}}\,\{Df(x) : x \in (p+\delta B_X) \cap \Omega(f)\}\,. \tag{2}$$

These two equivalent objects are nonempty, due to Rademacher's theorem on the almost everywhere differentiablity of locally Lipschitz functions. Furthermore, in terms of these Jacobians, results have been derived pertaining optimality conditions, implicit functions theorems, metric regularity, and calculus rules including the sum rule and the chain rule. Thereby, it has already been shown that these generalized Jacobians are successful approximations of f by linear operators.

In [18] and [16] Clarke's generalized Jacobian (1) was extended to the case when X was *any normed* space and Y was a finite dimensional space. In these references the generalized Jacobian $\partial f(p)$ was defined to be a subset of the *set of linear operators* from X to Y, $\mathcal{L}(X,Y)$. There, the difficulty caused by the infinite dimensionality of the domain was handled by introducing an intermediate Jacobian $\partial_L f$ defined on finite dimensional spaces L so that Rademacher theorem remains applicable. More specifically, for a given subspace L of X, we say that $f : \mathcal{D} \to Y$ is *L-(Gâteaux)-differentiable* at a point p if there exists a continuous linear map $D_L f(p) : L \to Y$ (called the *L-Jacobian*) such that, for all $h \in L$,

$$D_L f(p)(h) = f'(p,h) := \lim_{t\to 0} \frac{f(p+th) - f(p)}{t}. \tag{3}$$

Thus, setting $\Lambda(X)$ to be the set of *finite* dimensional linear subspaces of X, and $\Omega_L(f)$ to be the collection of those points p in $\mathcal{D}$ where f is L-differentiable, the generalized Jacobian introduced in [18] and [16] is

$$\partial f(p) := \big\{\Phi \in \mathcal{L}(X,Y) : \Phi|_L \in \partial_L f(p),\ \forall L \in \Lambda(X)\big\}, \tag{4}$$

where the *L-Jacobian* $\partial_L f(p)$ is the convex hull of what we call here the *generalized core-L-Jacobian* $\Delta_L f(p)$, (previously named pre-L-Jacobian), namely,

$$\begin{aligned} \partial_L f(p) &:= \mathrm{co}\,\Delta_L f(p),\\ \Delta_L f(p) &:= \big\{\Psi \in \mathcal{L}(L,Y) \mid \exists\,(x_i)_{i\in\mathbb{N}} \text{ in } \Omega_L(f) : \lim_{i\to\infty} x_i = p \text{ and } \lim_{i\to\infty} D_L f(x_i) = \Psi\big\}. \end{aligned} \tag{5}$$

Note that since L and Y are finite dimensional, the generalized core-L-Jacobian, $\Delta_L f(p)$, is nonempty and *closed*. Also, the sequence (x_i) in (5) is not necessarily contained in the affine subspace $p + L$, and hence, $\partial_L f(p)$ could be larger than Clarke's generalized Jacobian of the restricted function $f|_{p+L}$ at p. However, when the domain is finite dimensional, the generalized Jacobian in (7) coincides with Clarke's definition in (1). On the other hand, when the domain is infinite dimensional and the image space is $\mathbb{R}$, the notion given by (7) coincides with Clarke's generalized gradient $\partial^c f(p)$ which was defined as

$$\partial^c f(p) := \big\{\zeta \in X^* \mid \langle \zeta, h\rangle \le f^\circ(p,h),\ \forall h \in X\big\}, \tag{6}$$

where $f^\circ(p,h)$ stands for Clarke's directional derivative.

In [18] and [16], the nonemptiness, the w^*-compactness, the convexity, and the upper semicontinuity property of the extended generalized Jacobian in (7) were derived in addition to a chain rule for the composition of nonsmooth locally Lipschitz maps with finite dimensional ranges was established.

Recently, in [17], the generalized Jacobian notion given in [18] and [16] was extended to the case when in addition to the domain also *the range is infinite dimensional.* In this case, two extra difficulties manifested. The first is the differentiability issue related to the Rademacher theorem in infinite dimension, which was handled by taking image spaces Y satisfying the Radon–Nikodým property. This implies that the restriction of a Lipschitz function $f : \mathcal{D} \to Y$ to a *finite dimensional* domain is almost everywhere differentiable (cf. [1, Prop. 6.41]). The second difficulty concerned finding a topology in the space of linear operators $\mathcal{L}(X, Y)$, where the generalized Jacobian lives, so that bounded sequences would have cluster points in this topology. This difficulty was overcome by assuming that the image space Y is a dual of a normed space V, and the corresponding topology is denoted by $\beta(X, V)$, see the next section for more details. Thus, when $Y = V^*$ the generalized Jacobian was defined as before, that is,

$$\partial f(p) := \big\{\Phi \in \mathcal{L}(X, Y) : \Phi|_L \in \partial_L f(p),\ \forall L \in \Lambda(X)\big\}, \tag{7}$$

however, the L-Jacobian is now defined by

$$\partial_L f(p) := \overline{\mathrm{co}}^{\beta(L,V)}\, \Gamma_L f(p), \tag{8}$$

where

$$\Gamma_L f(p) := \Big\{\Phi \in \mathcal{L}(L, Y) : \exists\, (x_i)_{i\in\mathbb{N}} \text{ in } \Omega_L(f) \text{ such that } \lim_{i\to\infty} x_i = p \text{ and } \Phi \in \beta(L, V)\text{-}\operatorname*{clus}_{i\to\infty} D_L f(x_i)\Big\}. \tag{9}$$

It is worth noting that in [17], $\partial_L f(p)$ was defined via an extended form of the Pourciau Jacobian, (2), which is shown to be equivalent to (8)–(9).

Note that the nonempty set $\Gamma_L f(p)$ is not in general $\beta(L, V)$-closed. However, if Y is finite dimensional, so is the space $\mathcal{L}(L, Y)$, and thus the cluster points in (9) can be replaced by ordinary limits and hence, in this case, $\Gamma_L f(p)$ is closed and

$$\Delta_L f(p) = \Gamma_L f(p). \tag{10}$$

It results that, the generalized Jacobian defined via (8)–(9) extends the one defined via (5) to the case where the image space Y is infinite dimensional.

Under the assumption that Y satisfies also the the Radon–Nikodým property, it is shown in in [17] that the generalized Jacobian $\partial f(p)$ enjoys all the fundamental properties desired from a derivative set, in particular, the nonemptiness; the $\beta(X, V)$-compactness and convexity; the topological (equivalently, sequential) $\beta(X, V)$-upper semicontinuity at p of $\partial f(\cdot)$; and that $\partial f(p)$ is a singleton if and only if f is strictly w^*-Hadamard-differentiable at p, etc. For details see Section 2.

The following question naturally arise: Is it possible to dispose of the convexity in the definition of the generalized L-Jacobians and at the same the corresponding definition of a generalized Jacobian still preserve many of the important properties expected from a derivative-like set?

In other words, let us define the *generalized core-Jacobian* by

$$\Delta f(p) := \big\{\Phi \in \mathcal{L}(X,Y) : \Phi|_L \in \Delta_L f(p),\ \forall L \in \Lambda(X)\big\}, \tag{11}$$

where,

$$\Delta_L f(p) := \overline{\Gamma_L f(p)}^{\beta(L,V)} \tag{12}$$

is called the *generalized core-L-Jacobian*.

Then, it is vital to know (i) what properties $\Delta f(p)$ still retains from those of $\partial f(p)$, (ii) how good these linear operators are in approximating the function f near p, and (iii) what is the relationship between $\Delta f(p)$ and $\partial f(p)$.

From (8) we have that

$$\partial_L f(p) = \overline{\mathrm{co}}^{\beta(L,V)}\, \Delta_L f(p), \tag{13}$$

and hence, the generalized L-Jacobian has $\partial_L f(p)$ is the smallest $\beta(L,V)$-closed convex subset containing all the $\beta(L,V)$-cluster points of sequences of the form $(D_L f(x_i))$ with $x_i \to p$. For this reason we call the set $\Delta_L f(p)$ as the *generalized core-L-Jacobian.*

Equation (13) pronounces the fundamental role played by the generalized core-L-Jacobian, $\Delta_L f(p)$. It is the nucleus of the $\partial_L f(p)$; any result for $\partial_L f(p)$ would be derived for $\Delta_L f(p)$, first, and then for the $\beta(L,V)$-closure of its convex hull. Given then the importance of the generalized core-L-Jacobians, it is central to learn their basic properties, which are not known.

The main goal of this paper is to derive fundamental properties of $\Delta f(p)$ defined by (11). In order to accomplish this goal, we develop first the important properties for $\Delta_L f(p)$ which form a stepping stone in reaching our goal. No blindness with respect to L-nullset can be derived, however, we are still able to develop some chain rules in terms of both $\Delta_L f(p)$ and $\Delta f(p)$. Another important result in this paper lies in showing that the set $\Delta f(p)$ earns its name: it is the *core* of the generalized Jacobian; as we shall show that the $\beta(X,V)$-closure of its convex hull is exactly the generalized Jacobian $\partial f(p)$. As a consequence, the generalized Jacobian will have a new representation which is simpler than the one defining it in (7).

The paper is divided as follows. In Section 2 we recall from [17] the β-topology on the space of linear operators $\mathcal{L}(X,Y)$, abstract results and properties of the generalized Jacobian. New abstracts results centered on an extendibility theorem needed for the rest of the paper will be derived in Section 3. The main results of the paper lie in Section 4, where we derive first, several rich properties including a characterization and a useful new representation of $\Delta_L f(p)$. The next half of this section is the climax of the paper, as it focuses on the properties of $\Delta f(p)$. It is shown that the $\beta(X,V)$-closure of its convex hull is exactly $\partial f(p)$. In addition, other important properties are obtained, such as, the nonemptiness, the boundedness near p, and $\beta(X,V)$-compactness and the $\beta(X,V)$-upper semicontinuity at p. Furthermore, a useful representation is obtained as well as a scheme to find an upper estimate for the generalized core-Jacobian. In Section 5 different chain rules are derived in terms of both the generalized core-L-Jacobian, $\Delta_L f(p)$, and the generalized core-Jacobian itself, $\Delta f(p)$. Furthermore, a relationship is developed between $\Delta f(p)$ and known notions, such as, Ioffe's approximate subdifferential, Mordukhovich's coderivative and generalized Thibault limit set. An application is given in Section 6. Namely, we derive formulae for both $\Delta_L f(p)$ and $\Delta f(p)$ when the function f is a continuous selection of differentiable functions.

2. Preliminary results

Let X and Y denote a normed and a Banach space, respectively, and let X^* and Y^* stand for their topological dual spaces. The open and closed unit balls in a normed space Z will be denoted by B_Z and $\overline{B}_Z$, respectively.

We say that a normed space V is a *predual* of Y if $V^* = Y$ holds. If there exists a predual for Y, then Y is said to be a *dual space*. When Y is reflexive then it is a dual space and all its predual spaces are isometrically isomorphic to Y^*. A predual space V of Y induces a weak*-topology on Y denoted by $\sigma(Y, V)$. If Y has two different preduals then the weak*-topologies obtained via the different preduals are either identical or incomparable. By the Banach–Alaoglu theorem, the closed unit ball of the Banach space Y is compact in any of these weak*-topologies. For an account on these facts we refer to the book by R. B. Holmes [5].

Now consider $\mathcal{L}(X, Y)$ the Banach space of bounded linear operators from X to Y equipped with the standard operator norm. In order to obtain in $\mathcal{L}(X, Y)$ a topology in which the closed unit ball of $\mathcal{L}(X, Y)$ is compact, we assume that Y is a dual of a normed space V. Then, for all $x \in X$ and $v \in V$, the map $x \otimes v : \mathcal{L}(X, Y) \to \mathbb{R}$ defined by

$$(x \otimes v)(\Phi) := \langle \Phi(x), v \rangle \quad (\Phi \in \mathcal{L}(X, Y)) \tag{14}$$

is linear on $\mathcal{L}(X, Y)$. Denote

$$X \otimes V := \operatorname{span}\{x \otimes v : x \in X,\ v \in V\}. \tag{15}$$

The weak topology induced by $X \otimes V$ on $\mathcal{L}(X, Y)$, i.e., the topology $\sigma(\mathcal{L}(X, Y), X \otimes V)$ will be called the *weak*-operator-topology* and will be denoted by $\beta = \beta(X, V)$ throughout this paper. This notation indicates that the topology is described in terms of the elements of X and V.

Obviously, the following sets form a neighborhood subbase for the origin in the $\beta(X, V)$-topology:

$$\left\{\Phi \in \mathcal{L}(X, Y) : |\langle \Phi(x), v \rangle| < \varepsilon\right\} \quad (x \in X,\ v \in V,\ \varepsilon > 0). \tag{16}$$

that is, the $\beta(X, V)$-topology is the topology of the pointwise convergence for the real valued bilinear functions defined on $X \times V$ by $(x, v) \mapsto \langle \Phi(x), v \rangle$. Trivially, the $\beta(X, V)$-topology is weaker than the norm-topology, hence $\beta(X, V)$-closed sets are automatically norm-closed, and norm-compact sets are automatically $\beta(X, V)$-compact.

If X is finite dimensional with $\dim X = n$, then $\mathcal{L}(X, Y)$ is isomorphic to $Y^n = (V^*)^n$. Then the $\beta(X, V)$-topology is the same as the product of the weak*-topologies of Y. If Y is also reflexive then this topology coincides with the product of the weak topologies of Y. Therefore, if X is finite dimensional and Y is reflexive then norm-closed convex sets of $\mathcal{L}(X, Y)$ are also $\beta(X, V)$-closed.

All the results displayed in this section are proved in [17] and are needed throughout this paper.

The following theorem offers an analog of the Banach–Alaoglu theorem in the space $\mathcal{L}(X,Y)$.

Theorem 2.1. *Let Y be the dual of a normed space V. Then the closed unit ball of the Banach space $\mathcal{L}(X,Y)$ is compact in the $\beta(X,V)$-topology.*

Continuity properties of maps induced by left and right compositions by linear operators are described in the next result.

Lemma 2.2. *Let Q and X be normed spaces and let Y and Z be dual spaces with predual spaces V and W, respectively. Then,*

(*i*) *for all $A \in \mathcal{L}(Q,X)$, the map $\Phi \mapsto \Phi \circ A$ is a $(\beta(X,V), \beta(Q,V))$-continuous linear operator from $\mathcal{L}(X,Y)$ into $\mathcal{L}(Q,Y)$.*

(*ii*) *for all $A \in \mathcal{L}(Y,Z)$, the map $\Phi \mapsto A \circ \Phi$ is a $(\beta(X,V), \beta(X,W))$-continuous linear operator from $\mathcal{L}(X,Y)$ into $\mathcal{L}(X,Z)$ provided that $W \circ A \subseteq V$ holds.*

The set $W \circ A$ is a subset of Y^*, therefore, the condition $W \circ A \subseteq V$ always holds if Y is reflexive.

Recall that $\Lambda(X)$ is the family of finite dimensional linear subspaces of X. For $\Phi \in \mathcal{L}(X,Y)$ and $L \in \Lambda(X)$, define the maps $\Phi|_L \in \mathcal{L}(L,Y)$ and $\mathrm{rest}_L : \mathcal{L}(X,Y) \to \mathcal{L}(L,Y)$ by

$$\Phi|_L(x) := \Phi(x) \quad (x \in L), \qquad \mathrm{rest}_L(\Psi) := \Psi|_L \quad (\Psi \in \mathcal{L}(X,Y)). \tag{17}$$

Similarly, for $\Phi \in \mathcal{L}(X,Y)$ and $\boldsymbol{v} = (v_1, \dots, v_n) \in V^n$, define the maps $\boldsymbol{v} \circ \Phi \in \mathcal{L}(X, \mathbb{R}^n)$ and $\mathrm{vect}_{\boldsymbol{v}} : \mathcal{L}(X,Y) \to \mathcal{L}(X, \mathbb{R}^n)$ by

$$\begin{gathered} \boldsymbol{v} \circ \Phi(x) := (\langle \Phi(x), v_1 \rangle, \dots, \langle \Phi(x), v_n \rangle) \quad (x \in X), \\ \mathrm{vect}_{\boldsymbol{v}}(\Psi) := \boldsymbol{v} \circ \Psi \quad (\Psi \in \mathcal{L}(X,Y)). \end{gathered} \tag{18}$$

If $\mathcal{F} \subseteq \mathcal{L}(X,Y)$, then we also set

$$\mathcal{F}|_L := \big\{\Phi|_L : \Phi \in \mathcal{F}\big\} \qquad \text{and} \qquad \boldsymbol{v} \circ \mathcal{F} := \big\{\boldsymbol{v} \circ \Phi : \Phi \in \mathcal{F}\big\}. \tag{19}$$

The next result states the β-continuity properties of the restriction and vectorization maps introduced in (17) and (18).

Lemma 2.3. *Let Y be the dual of a normed space V. Then,*

(*i*) *for all $L \in \Lambda(X)$, the map $\mathrm{rest}_L : \mathcal{L}(X,Y) \to \mathcal{L}(L,Y)$ is a $(\beta(X,V), \beta(L,V))$-continuous linear operator;*

(*ii*) *for all $n \in \mathbb{N}$ and $\boldsymbol{v} \in V^n$, the map $\mathrm{vect}_{\boldsymbol{v}} : \mathcal{L}(X,Y) \to \mathcal{L}(X,\mathbb{R}^n)$ is a $(\beta(X,V), \beta(X,\mathbb{R}^n))$-continuous linear operator.*

The following result is instrumental when proving inclusions between subsets of the space $\mathcal{L}(X,Y)$.

Theorem 2.4. *Let Y be the dual of a normed space V and let $\mathcal{F}, \mathcal{G} \subseteq \mathcal{L}(X,Y)$ such that $\mathcal{G}$ is $\beta(X,V)$-compact. Then $\mathcal{F} \subseteq \mathcal{G}$ holds if and only if, for all $n \in \mathbb{N}$ and $\boldsymbol{v} \in V^n$, for all $L \in \Lambda(X)$,*

$$\boldsymbol{v} \circ \mathcal{F}|_L \subseteq \boldsymbol{v} \circ \mathcal{G}|_L. \tag{20}$$

Given a family $\{\mathcal{F}_L \subseteq \mathcal{L}(L,Y) : L \in \Lambda(X)\}$, the following result furnishes a sufficiency criterion for the extendibility of every element $\Phi \in \mathcal{F}_K \subseteq \mathcal{L}(K,Y)$ to $\Psi \in \mathcal{L}(X,Y)$ such that $\Psi|_L \in \mathcal{F}_L$ for all $L \in \Lambda(X)$ with $K \subseteq L$.

Theorem 2.5. *Let Y be the dual of a normed space V. Let, for all $L \in \Lambda(X)$, $\mathcal{F}_L$ be a given nonempty $\beta(L,V)$-closed subset of the space $\mathcal{L}(L,Y)$. Then the set $\mathcal{F} \subseteq \mathcal{L}(X,Y)$ defined by*

$$\mathcal{F} := \{\Phi \in \mathcal{L}(X,Y) : \Phi|_L \in \mathcal{F}_L, \ \forall L \in \Lambda(X)\} \tag{21}$$

is $\beta(X,V)$-closed. If, in addition, for $L \in \Lambda(X)$, $\mathcal{F}_L$ is $\beta(L,V)$-compact and, for all $K \in \Lambda(X)$ with $K \subseteq L$, we have

$$\mathcal{F}_L|_K = \mathcal{F}_K, \tag{22}$$

then

$$\mathcal{F}_K = \mathcal{F}|_K \quad (K \in \Lambda(X)). \tag{23}$$

The following fact will be repeatedly used in the sequel.

Lemma 2.6. *If T and R are topological spaces, with Hausdorff topologies τ and ρ, respectively, $f : T \to R$ is (τ,ρ)-continuous and $S \subseteq T$ is τ-precompact (i.e., $\overline{S}^\tau$ is τ-compact), then*

$$\overline{f(S)}^\rho = f(\overline{S}^\tau). \tag{24}$$

The next result will be used when computing the support function of intersecting families of sets. Recall that a family of sets $\{S_\alpha : \alpha \in A\}$ is directed downward if, for any indices $\alpha_1, \alpha_2 \in A$, there is $\alpha \in A$ such that $S_\alpha \subseteq S_{\alpha_1} \cap S_{\alpha_2}$.

Lemma 2.7. *Let T and R be topological spaces with Hausdorff topologies τ and ρ, respectively, and let $\{S_\alpha : \alpha \in A\}$ be a family of τ-compact subsets of T that is directed downward and let $f : T \to R$ be continuous. Then*

$$f\left(\bigcap_{\alpha \in A} S_\alpha\right) = \bigcap_{\alpha \in A} f(S_\alpha). \tag{25}$$

If $(x_i)_{i\in\mathbb{N}}$ is a sequence in T such that $\{x_i : i \in \mathbb{N}\}$ is τ-precompact then

$$f\left(\tau\text{-}\operatorname*{clus}_{i\to\infty} x_i\right) = \rho\text{-}\operatorname*{clus}_{i\to\infty} f(x_i) \tag{26}$$

(where θ-$\mathrm{clus}_{i\to\infty} z_i$ denotes the set of θ-cluster points of the sequence $(z_i)_{i\in\mathbb{N}}$).
If f is real-valued then

$$\sup f\left(\bigcap_{\alpha \in A} S_\alpha\right) = \inf_{\alpha \in A} \sup f(S_\alpha). \tag{27}$$

Let Z be a normed space, (E,d) be a metric space, $D \subseteq E$ be a nonempty subset, and $q \in D$. We say that a set-valued function $\mathcal{F} : D \to 2^Z$ is *bounded near* q if there exist $\delta > 0$ and $r \geq 0$ such that $\mathcal{F}(x) \subseteq rB_Z$ for all $x \in D$ with $d(x,q) < \delta$. $\mathcal{F}$ is said to be locally bounded on D if it is bounded near each point of D. For a function

$F : D \to Z$, the boundedness of F near q means the boundedness of the set-valued map $\mathcal{F}(x) := \{F(x)\}$ near q.

Given a Hausdorff topology τ on Z, the map $\mathcal{F} : D \to 2^Z$ is said to be *topologically τ-usc at q* if, for all τ-open sets U containing $\mathcal{F}(q)$, there exists $\delta > 0$ such that, for all $x \in D$ with $d(x, q) < \delta$, we have $\mathcal{F}(x) \subseteq U$. On the other hand, $\mathcal{F} : D \to 2^Z$ is said to be *sequentially τ-usc at q* if, whenever (x_i, z_i) in $D \times Z$ such that $z_i \in \mathcal{F}(x_i)$ for all i, and $x_i \to q$, then $\tau\text{-clus}_{i\to\infty} z_i \subseteq \mathcal{F}(q)$. Assuming that $\mathcal{F}(q)$ is τ-compact and $\mathcal{F}$ is bounded near q, the next lemma establishes the equivalence of topological and sequential τ-upper semicontinuity properties.

Lemma 2.8. *Let Z be a normed space and τ be a Hausdorff topology on Z such that B_Z is τ-precompact. Let $q \in D$, and let $\mathcal{F} : D \to 2^Z$ be a set-valued mapping such that $\mathcal{F}(q)$ is τ-compact and $\mathcal{F}$ is bounded near q. Then $\mathcal{F}$ is topologically τ-usc at q if and only if it is sequentially τ-usc at q.*

In the rest of this section we first list some properties of the L-Jacobian, $D_L f(p)$, and the generalized L-Jacobian, $\partial_L f(p)$, introduced respectively by (3) and (8)–(9). Then we present properties of the generalized Jacobian, $\partial f(p)$, defined by (7), where $\partial_L f(p)$ is given by (8)–(9).

Let $\mathcal{D}$ be a nonempty open subset of a normed space X, p be an arbitrary point in $\mathcal{D}$, Y be a *dual* of a normed space V and $f : \mathcal{D} \to Y$. The following lemma is immediate.

Lemma 2.9. *Let $K, L \in \Lambda(X)$ with $K \subseteq L$. Then, whenever f is L-differentiable at p, we have*

$$D_L f(p)|_K = D_K f(p). \tag{28}$$

If Y is reflexive, then norm-closed convex sets of $\mathcal{L}(L, Y)$ are also $\beta(L, V)$-closed, therefore, in the definition of $\partial_L f(p) := \overline{\text{co}}^{\beta(L,V)}\, \Gamma_L f(p)$, the operation $\overline{\text{co}}^{\beta(L,V)}$ can be replaced by $\overline{\text{co}} := \overline{\text{co}}^{\|\cdot\|}$. Hence, in this case, the notions of the generalized L-Jacobian and Jacobian do not explicitely involve the β-topology.

Clearly, $\partial_L f(p)$ is a $\beta(L, V)$-closed convex set of the space $\mathcal{L}(L, Y)$ for all $L \in \Lambda(X)$ and, by Theorem 2.5, $\partial f(p)$ is $\beta(X, V)$-closed and also convex in $\mathcal{L}(X, Y)$.

In the rest of this section, assume that f is Lipschitz near p, and let $\ell_f(p)$ to be the Lipschitz modulus of f at p, that is,

$$\ell_f(p) := \inf_{\delta>0} \sup \left\{ \frac{\|f(x) - f(y)\|}{\|x - y\|} \;\middle|\; x, y \in p + \delta B_X,\ x \neq y \right\}. \tag{29}$$

It is obvious that ℓ_f is an usc function on $\mathcal{D}$ and that f is Lipschitz near p if and only if $\ell_f(p) < +\infty$.

Some properties of the generalized L-Jacobian are given in the next lemma in which the Lipschitz modulus of f at p is involved.

Lemma 2.10. *Let $Y = V^*$ and assume that $f : \mathcal{D} \to Y$ is Lipschitz near p. Then, for all $L \in \Lambda(X)$, $\partial_L f(p)$ is a $\beta(L, V)$-compact subset of $\ell_f(p)\overline{B}_{\mathcal{L}(L,Y)}$. Furthermore, the set-valued map $\partial_L f(\cdot)$ is bounded near p and is topologically (and also sequentially) $\beta(L, V)$-usc at p.*

The next result is a characterization of the generalized L-Jacobian in terms of an extension of Pourciau's notion in [20] to infinite dimension. This means that the extensions of each of Clarke's and Pourciau's generalized Jacobians in (1) and (2) to the case of infinite dimensional image spaces remain equivalent.

Lemma 2.11. *Let $Y = V^*$ and assume that $f : \mathcal{D} \to Y$ is Lipschitz near p. Then, for $L \in \Lambda(X)$,*

$$\partial_L f(p) = \bigcap_{\delta>0} \overline{\mathrm{co}}^{\beta(L,V)} \big\{ D_L f(x) : x \in (p + \delta B_X) \cap \Omega_L(f) \big\}. \tag{30}$$

To ensure the nonemptiness of $\partial_L f(p)$, we assume that the image space Y satisfies the well-known Radon–Nikodým property ([1, Def. 5.4]). In the case when Y is reflexive, or when Y is a separable dual space then this property holds automatically for Y ([1, Cor. 5.12]). The next result generalizes the results of Warga [28] and Fabian & Preiss [4] and [16, Thm. 3.1] to functions whose range is a dual space with the Radon–Nikodým property.

Lemma 2.12. *Assume that $Y = V^*$ is a Radon–Nikodým space and $f : \mathcal{D} \to Y$ is Lipschitz near p. Then, for $L \in \Lambda(X)$,*

(i) $\partial_L f(p)$ is nonempty, and
(ii) if $K \in \Lambda(X)$ with $K \subseteq L$, then

$$\partial_L f(p)|_K = \partial_K f(p). \tag{31}$$

The following rule, named as the *restriction and vectorization theorem* of the generalized Jacobian, gives an explicit formula for the restriction of $\partial f(p)$ to finite dimensional subspaces and for the image of $\partial f(p)$ by a finite dimensional linear map. As a consequence, the nonemptiness of the generalized Jacobian $\partial f(p)$ is obtained.

Theorem 2.13. *Let $Y = V^*$ be a Radon–Nikodým space and let $f : \mathcal{D} \to Y$ be a locally Lipschitz near p. Then, for all $K \in \Lambda(X)$,*

$$\partial f(p)|_K = \partial_K f(p) \tag{32}$$

and hence $\partial f(p)$ is nonempty. Furthermore, for all $n \in \mathbb{N}$ and $\boldsymbol{v} \in V^n$,

$$\boldsymbol{v} \circ \partial f(p) = \partial(\boldsymbol{v} \circ f)(p). \tag{33}$$

The following result summarizes the essential properties of the generalized Jacobian.

Theorem 2.14. *Let $Y = V^*$ be a Radon–Nikodým space and let $f : \mathcal{D} \to Y$ be Lipschitz near p. Then the generalized Jacobian has the following properties.*

(i) $\partial f(p)$ is nonempty, $\beta(X,V)$-compact, convex, and $\partial f(p) \subseteq \ell_f(p)\overline{B}_{\mathcal{L}(X,Y)}$.
(ii) The set-valued map $\partial f(\cdot)$ is locally bounded and topologically (equivalently, sequentially) $\beta(X,V)$-usc at p.
(iii) If f is L-differentiable at $p \in \mathcal{D}$ for some $L \in \Lambda(X)$, then $D_L f(p) \in \partial f(p)|_L$.
(iv) $\partial f(p)$ is a singleton if and only if f is strictly w^*-Hadamard-differentiable at p.

A characterization for the generalized Jacobian is displayed in the next result.

Theorem 2.15. *Let $Y = V^*$ be a Radon–Nikodým space and let $f : \mathcal{D} \to Y$ be a locally Lipschitz function. Then $\mathcal{F} = \partial f$ is the smallest set-valued map $\mathcal{F} : \mathcal{D} \to 2^{\mathcal{L}(X,Y)}$ with the following properties:*

(i) $\mathcal{F}(p)$ *is $\beta(X,V)$-compact and convex for all $p \in \mathcal{D}$.*
(ii) $\mathcal{F}$ *is locally bounded on $\mathcal{D}$ and sequentially $\beta(X,V)$-usc on $\mathcal{D}$.*
(iii) *For all $L \in \Lambda(X)$ and for all $x \in \Omega_L(f)$,*

$$D_L f(x) \in \mathcal{F}(x)|_L. \tag{34}$$

In the rest of this section we present the connection between the generalized Jacobian $\partial f(p)$ and other known differentiability notions of f at p, such as the ones defined (35) and (36).

The limit points of directional difference quotients were introduced and investigated by Thibault [26], [27]. A notion more general than Thibault's limit set is defined as follows

$$\delta f(p,h) := \Big\{ y \in Y : \exists\, (x_i,t_i)_{i\in\mathbb{N}} \text{ in } \mathcal{D} \times \mathbb{R}_+ \text{ such that} \\ \lim_{i\to\infty}(x_i,t_i) = (p,0),\ y \in w^*\text{-}\operatorname*{clus}_{i\to\infty} \frac{f(x_i+t_ih)-f(x_i)}{t_i} \Big\}. \tag{35}$$

We introduce the following concept, which is related to Ioffe's fan derivative (see [6]),

$$D^\circ f(p)(h) := \big\{ y \in Y : \langle y, v\rangle \le (v \circ f)^\circ(p,h)\ \forall v \in V \big\}. \tag{36}$$

Note that if Y is reflexive (i.e., when $V = Y^*$) then $D^\circ f(p)$ coincides with Ioffe's fan derivative.

Theorem 2.16. *Let $Y = V^*$ be a Radon–Nikodým space. Let $f : \mathcal{D} \to Y$ be a locally Lipschitz function. Then, for all $p \in \mathcal{D}$ and $h \in X$,*

$$\partial f(p)(h) = \overline{\mathrm{co}}^{w^*} \delta f(p,h) = D^\circ f(p)(h). \tag{37}$$

Another known notions for the differentiation of a nonsmooth function are the *normal and mixed coderivatives* introduced by Mordukhovich in [12], [13] and in [14], respectively. (See also [15, Definition 1.32] for their detailed definition.)

Lemma 2.17. *Let X be a normed space and $Y = V^*$ be a Radon–Nikodým space. Let $f : \mathcal{D} \to Y$ be a Lipschitz function near $p \in \mathcal{D}$. Then, for all $v \in V$,*

$$\overline{\mathrm{co}}^{w^*} D_M^* f(p)(v) = v \circ \partial f(p). \tag{38}$$

If, in addition, X is an Asplund space, then, for all $v \in V$,

$$\overline{\mathrm{co}}^{w^*} D_N^* f(p)(v) = v \circ \partial f(p). \tag{39}$$

3. New Abstract Results

In this section we present new abstract results which form the foundation for proving the main results of this paper.

The next lemma shows that for a special family of sets, the convex-closure operator commutes with the intersection.

Lemma 3.1. *Let Z be a locally convex linear space with a Hausdorff topology τ and let $\{S_\alpha : \alpha \in A\}$ be a family of τ-compact subsets of Z that is directed downward. Then*

$$\overline{\mathrm{co}}^\tau \left(\bigcap_{\alpha \in A} S_\alpha \right) = \bigcap_{\alpha \in A} \overline{\mathrm{co}}^\tau\, S_\alpha. \tag{40}$$

Proof. The inclusion "$\subseteq$" is obvious because the right hand side is a convex and τ-closed set containing $\bigcap_{\alpha \in A} S_\alpha$.

The other inclusion is shown via contrapositive. If $z \in \bigcap_{\alpha \in A} \overline{\mathrm{co}}^\tau(S_\alpha)$ and $z \notin \overline{\mathrm{co}}^\tau \left(\bigcap_{\alpha \in A} S_\alpha\right)$, then by the strict form of the Hahn-Banach separation theorem, there exists $\zeta \in Z^*$ such that

$$\begin{aligned}\langle \zeta, z\rangle > \sup \left\{ \langle \zeta, u\rangle \mid u \in \overline{\mathrm{co}}^\tau \left(\bigcap_{\alpha \in A} S_\alpha \right) \right\} &= \sup \left\{ \langle \zeta, u\rangle \mid u \in \bigcap_{\alpha \in A} S_\alpha \right\} \\ \overset{(1)}{=} \inf_{\alpha \in A} \sup \big\{ \langle \zeta, u\rangle \mid u \in S_\alpha \big\} &= \inf_{\alpha \in A} \sup \big\{ \langle \zeta, u\rangle \mid u \in \overline{\mathrm{co}}^\tau(S_\alpha) \big\},\end{aligned} \tag{41}$$

where $\overset{(1)}{=}$ is a consequence of (27) of Lemma 2.7. Thus, there exists an element $\alpha \in A$ such that

$$\langle \zeta, z\rangle > \sup \big\{ \langle \zeta, u\rangle \mid u \in \overline{\mathrm{co}}^\tau(S_\alpha) \big\}, \tag{42}$$

proving that $z \notin \overline{\mathrm{co}}^\tau(S_\alpha)$, whence the contradiction $z \notin \bigcap_{\alpha \in A} \overline{\mathrm{co}}^\tau(S_\alpha)$ follows. $\square$

The following result is a backbone for establishing the new representation of the generalized Jacobian, $\partial f(p)$, which was introduced in [17] and is defined in (7) via (8).

Theorem 3.2. *Let Y be the dual of a normed space V. Let, for all $L \in \Lambda(X)$, $\mathcal{G}_L$ be a given nonempty $\beta(L, V)$-compact subset of the space $\mathcal{L}(L, Y)$. Assume that there exists a positive number ρ such that, for all $L \in \Lambda(X)$,*

$$\mathcal{G}_L \subseteq \rho \overline{B}_{\mathcal{L}(L,Y)} \tag{43}$$

and, for all $K \in \Lambda(X)$ with $K \subseteq L$,

$$\mathcal{G}_L|_K \subseteq \mathcal{G}_K \subseteq \left(\overline{\mathrm{co}}^{\beta(L,V)}\, \mathcal{G}_L \right)|_K. \tag{44}$$

Then the set $\mathcal{G} \subseteq \mathcal{L}(X, Y)$ defined by

$$\mathcal{G} := \big\{ \Phi \in \mathcal{L}(X, Y) : \Phi|_L \in \mathcal{G}_L,\ \forall L \in \Lambda(X) \big\} \tag{45}$$

is $\beta(X, V)$-compact, nonempty and satisfies

$$\mathcal{G} \subseteq \rho \overline{B}_{\mathcal{L}(X,Y)} \tag{46}$$

and

$$\mathcal{G}_K \subseteq \left(\overline{\mathrm{co}}^{\beta(X,V)}\, \mathcal{G} \right)|_K \quad (K \in \Lambda(X)). \tag{47}$$

Proof. By the first assertion of Theorem 2.5, the set $\mathcal{G}$ is $\beta(X,V)$-closed. To prove its $\beta(X,V)$-compactness, by Theorem 2.1, we show that (46) holds. Let $\Phi \in \mathcal{G}$ and $x \in X$. Now let L be the linear span of the vector x. The inclusion $\Phi|_L \in \mathcal{G}_L$ yields that $\|\Phi(x)\| \le \rho\|x\|$, hence $\|\Phi\| \le \rho$. Thus, (46) holds, and hence

$$\mathcal{G} = \bigcap_{L\in\Lambda(X)} S_L, \tag{48}$$

where the set S_L is defined by

$$S_L := \big\{\Phi \in \mathcal{L}(X,Y) : \Phi|_L \in \mathcal{G}_L,\ \|\Phi\| \le \rho\big\}. \tag{49}$$

In order to complete the proof of the theorem, we will need to establish that (47) holds. For, we shall show the following properties of S_L:

(i) For all $L \in \Lambda(X)$, the set S_L is $\beta(X,V)$-compact.
(ii) For all $L \in \Lambda(X)$, the inclusion $\mathcal{G}_L \subseteq S_L|_L$ holds. In particular, S_L is nonempty.
(iii) For all $L, K \in \Lambda(X)$ with $K \subseteq L$, the inclusion $S_L \subseteq S_K$ holds.
(iv) The family $\{S_L : L \in \Lambda(X)\}$ is directed downward.

For (i), it suffices to show that the set $\hat{S}_L := \big\{\Phi \in \mathcal{L}(X,Y) : \Phi|_L \in \mathcal{G}_L\big\}$ is $\beta(X,V)$-closed. Let Φ be a $\beta(X,V)$-cluster point of a net Φ_α in $\mathcal{L}(X,Y)$ satisfying $\Phi_\alpha|_L \in \mathcal{G}_L$. Then, by Lemma 2.3, $\Phi|_L$ is also a $\beta(L,V)$-cluster point of $\Phi_\alpha|_L$. The $\beta(L,V)$-closedness of $\mathcal{G}_L$ will then yield that $\Phi|_L \in \mathcal{G}_L$, therefore $\hat{S}_L$ is $\beta(X,V)$-closed.

To verify (ii), define $\mathcal{F}_K := \overline{\mathrm{co}}^{\beta(K,V)}\, \mathcal{G}_K$ for $K \in \Lambda(X)$. Because $\mathcal{G}_K$ is bounded, hence $\mathrm{co}\,\mathcal{G}_K$ is also bounded. Thus, by Theorem 2.1, $\mathcal{F}_K := \overline{\mathrm{co}}^{\beta(K,V)}\, \mathcal{G}_K$ is $\beta(K,V)$-compact convex. Taking the $\beta(K,V)$-closed convex hulls of each of the sets in (44), we get

$$\overline{\mathrm{co}}^{\beta(K,V)}\left(\mathcal{G}_L|_K\right) \subseteq \overline{\mathrm{co}}^{\beta(K,V)}\, \mathcal{G}_K = \mathcal{F}_K \subseteq \overline{\mathrm{co}}^{\beta(K,V)}\left(\left(\overline{\mathrm{co}}^{\beta(L,V)}\, \mathcal{G}_L\right)|_K\right) = \overline{\mathrm{co}}^{\beta(K,V)}\left(\mathcal{F}_L|_K\right). \tag{50}$$

By the $\beta(L,V)$-compactness of the set $\mathcal{F}_L$ and the $(\beta(L,V),\beta(K,V))$-continuity of the restriction map rest_K, we get that $\mathcal{F}_L|_K$ is $\beta(K,V)$-compact and convex, hence $\overline{\mathrm{co}}^{\beta(K,V)}\left(\mathcal{F}_L|_K\right) = \mathcal{F}_L|_K$. From (50) it also follows that

$$\overline{\mathrm{co}}^{\beta(K,V)}\left(\mathcal{G}_L|_K\right) \subseteq \mathcal{F}_L|_K. \tag{51}$$

On the other hand,

$$\left(\mathrm{co}\,\mathcal{G}_L\right)|_K \subseteq \mathrm{co}\left(\mathcal{G}_L|_K\right). \tag{52}$$

Upon taking the $\beta(K,V)$-closure of this inclusion and by using that $\mathrm{co}\,\mathcal{G}_L$ is $\beta(L,V)$-precompact, it results that

$$\mathcal{F}_L|_K = \left(\overline{\mathrm{co}}^{\beta(L,V)}\, \mathcal{G}_L\right)|_K \subseteq \overline{\left(\mathrm{co}\,\mathcal{G}_L\right)|_K}^{\beta(K,V)} \subseteq \overline{\mathrm{co}}^{\beta(K,V)}\left(\mathcal{G}_L|_K\right). \tag{53}$$

Hence (51), and therefore, (50) hold with equalities. Thus, $\mathcal{F}_L|_K = \mathcal{F}_K$ is valid for all subspaces $K, L \in \Lambda(X)$ with $K \subseteq L$.

Now, we are in the position of applying Theorem 2.5. If $\mathcal{F}$ is defined by (21), then we obtain that $\mathcal{F}_L = \mathcal{F}|_L$ for all L. We have that $\|F_L\| \le \rho$ for all L, therefore $\|\mathcal{F}\| \le \rho$ holds. It then follows that, for all $L \in \Lambda(X)$,

$$\mathcal{G}_L \subseteq \mathcal{F}|_L \subseteq \overline{B}_{\mathcal{L}(X,Y)}(\rho)|_L. \tag{54}$$

Hence, for every element $\Psi \in \mathcal{G}_L$, there exists an element Φ in S_L such that $\Phi|_L = \Psi$. Therefore, (ii) holds.

To prove (iii), let $L, K \in \Lambda(X)$ with $K \subseteq L$, and let $\Phi \in S_L$. Then $\Phi|_L \in \mathcal{G}_L$ and $\|\Phi\| \le \rho$. By condition (44),

$$\Phi|_K = (\Phi|_L)\,|_K \in \mathcal{G}_L|_K \subseteq \mathcal{G}_K \tag{55}$$

hence $\Phi \in S_K$.

For (iv), observe that, by (iii), we have $S_{L+K} \subseteq S_L \cap S_K$ for all $L, K \in \Lambda(X)$.

Now we prove (47). Let $K \in \Lambda(X)$ be fixed and let $L \supseteq K$ be arbitrary. From (ii), we have that

$$\mathcal{G}_L \subseteq S_L|_L \subseteq \left(\overline{\mathrm{co}}^{\beta(X,V)}\, S_L\right)|_L. \tag{56}$$

By the $\beta(X,V)$-compactness of S_L and by the $(\beta(X,V), \beta(L,V))$-continuity of rest_L, the right hand side is $\beta(L,V)$-compact and convex. Therefore, it follows that

$$\overline{\mathrm{co}}^{\beta(L,V)}\, \mathcal{G}_L \subseteq \left(\overline{\mathrm{co}}^{\beta(X,V)}\, S_L\right)|_L, \tag{57}$$

which, in view of (44), implies

$$\mathcal{G}_K \subseteq \left(\overline{\mathrm{co}}^{\beta(L,V)}\, \mathcal{G}_L\right)\big|_K \subseteq \left(\left(\overline{\mathrm{co}}^{\beta(X,V)}\, S_L\right)\big|_L\right)\Big|_K = \left(\overline{\mathrm{co}}^{\beta(X,V)}\, S_L\right)\Big|_K, \tag{58}$$

and thus,

$$\mathcal{G}_K \subseteq \bigcap_{\substack{L \in \Lambda(X) \\ K \subseteq L}} \left(\overline{\mathrm{co}}^{\beta(X,V)}\, S_L\right)\Big|_K. \tag{59}$$

Since, by (iv), the family $\{S_L : L \in \Lambda(X)\}$ is directed downward, so does the family $\{\overline{\mathrm{co}}^{\beta(X,V)}\, S_L : L \in \Lambda(X)\}$ of $\beta(X,V)$-compact sets. Thus, by the $(\beta(X,V), \beta(K,V))$-continuity of rest_K, and Lemma 2.7 it follows that the right-hand-side of (59) can be written as

$$\bigcap_{\substack{L \in \Lambda(X) \\ K \subseteq L}} \left(\overline{\mathrm{co}}^{\beta(X,V)}\, S_L\right)\Big|_K = \left(\bigcap_{\substack{L \in \Lambda(X) \\ K \subseteq L}} \overline{\mathrm{co}}^{\beta(X,V)}\, S_L\right)\Bigg|_K. \tag{60}$$

On the other hand, by the downward directedness of the family S_L,

$$\bigcap_{\substack{L \in \Lambda(X) \\ K \subseteq L}} \overline{\mathrm{co}}^{\beta(X,V)}\, S_L = \bigcap_{L \in \Lambda(X)} \overline{\mathrm{co}}^{\beta(X,V)}\, S_L, \tag{61}$$

since if x belongs to the left hand side of (61), then, for any $L_0 \in \Lambda(X)$, we have $L_0 + K \supseteq K$ and hence $x \in \overline{\mathrm{co}}^{\beta(X,V)}\, S_{L_0+K}$. On the other hand, the downward directedness of S_L yields that $S_{L_0+K} \subseteq S_{L_0} \cap S_K$, and thus,

$$x \in \overline{\mathrm{co}}^{\beta(X,V)}\, S_{L_0+K} \subseteq \overline{\mathrm{co}}^{\beta(X,V)}\, (S_{L_0} \cap S_K) \subseteq \overline{\mathrm{co}}^{\beta(X,V)}\, S_{L_0}. \tag{62}$$

Thus, using (59)–(61) together with Lemma 3.1 applied to the downward directed family S_L of $\beta(X,V)$-compact sets, it results that

$$\mathcal{G}_K \subseteq \left(\bigcap_{L \in \Lambda(X)} \overline{\mathrm{co}}^{\beta(X,V)}\, S_L\right)\Bigg|_K = \left(\overline{\mathrm{co}}^{\beta(X,V)} \bigcap_{L \in \Lambda(X)} S_L\right)\Bigg|_K = \left(\overline{\mathrm{co}}^{\beta(X,V)}\, \mathcal{G}\right)\Big|_K. \tag{63}$$

Since K is arbitrary in $\Lambda(X)$, this proves that (47) holds. From this inclusion and from the nonemptiness of $\mathcal{G}_K$ it results that $\mathcal{G}$ is nonempty. $\square$

4. Generalized Core-Jacobian

Let $\mathcal{D}$ be a nonempty open subset of a normed space X, p be an arbitrary point in $\mathcal{D}$, Y be a *dual* of a normed space V and $f : \mathcal{D} \to Y$.

Consider the *generalized core-Jacobian* $\Delta f(p) \subseteq \mathcal{L}(X, Y)$ of the function f at the point p defined in (11), in which $\Delta_L f(p)$ is the *generalized core-L-Jacobian* introduced in (12) and (9).

The next result establishes the most basic properties of the generalized core-L-Jacobian.

Proposition 4.1. *Let $Y = V^*$ and assume that $f : \mathcal{D} \to Y$ is Lipschitz near p. Then, for $L \in \Lambda(X)$, we have the following properties:*

(*i*) *$\Delta_L f(p)$ is $\beta(L, V)$-compact, $\Delta_L f(p) \subseteq \ell_f(p)\overline{B}_{\mathcal{L}(L,Y)}$, and $D_L f(p) \in \Delta_L f(p)$ whenever f is L-differentiable at p. If, in addition, Y has the Radon–Nikodým property, then $\Delta_L f(p)$ is nonempty.*

(*ii*) *For all $K \in \Lambda(X)$ with $K \subseteq L$,*
$$(\Delta_L f(p))|_K \subseteq \Delta_K f(p). \tag{64}$$

(*iii*) *For all $K \in \Lambda(X)$ with $K \subseteq L$,*
$$\Delta_K f(p) \subseteq \left(\overline{\mathrm{co}}^{\beta(L,V)}\, \Delta_L f(p)\right)\big|_K. \tag{65}$$

(*iv*) *The relation*
$$\overline{\mathrm{co}}^{\beta(L,V)}\, \Delta_L f(p) = \partial_L f(p) \tag{66}$$
holds.

Proof. Property (*iv*) is exactly (13), which holds due to the definitions of $\partial_L f(p)$ and $\Delta_L f(p)$ given by (9), (8) and (12).

From (*iv*) and Lemma 2.10, it follows that $\Delta_L f(p) \subseteq \partial_L f(p) \subseteq \ell_f(p)\overline{B}_{\mathcal{L}(L,Y)}$. The $\beta(L, V)$-closedness of $\Delta_L f(p)$ and its boundedness implies by Theorem 2.1 its $\beta(L, V)$-compactness. If f is L-differentiable at p then $D_L f(p) \in \Gamma_L f(p) \subseteq \Delta_L f(p)$.

If Y has the Radon–Nikodým property then, by Lemma 2.12, the L-Jacobian $\partial_L f(p) \neq \emptyset$ and thus, by property (*iv*), $\Delta_L f(p)$ is also nonempty. Hence, (*i*) holds.

To prove (*ii*), let $K \in \Lambda(X)$ with $K \subseteq L$. By definition of $\Gamma_L f(p)$ and Lemma 2.9, it follows that $\Gamma_L f(p)|_K \subseteq \Gamma_K f(p)$. Using the $(\beta(L, V), \beta(K, V))$-continuity of the restriction map rest_K, the $\beta(L, V)$-precompactness of $\Gamma_L f(p)$ and Lemma 2.6, we get
$$\Delta_L f(p)|_K = \overline{\Gamma_L f(p)}^{\beta(L,V)}\Big|_K = \overline{(\Gamma_L f(p)|_K)}^{\beta(K,V)} \subseteq \overline{\Gamma_K f(p)}^{\beta(K,V)} = \Delta_K f(p). \tag{67}$$

To prove (*iii*), we use Lemma 2.12 and property (*iv*) to obtain
$$\Delta_K f(p) \subseteq \partial_K f(p) = (\partial_L f(p))|_K = \left(\overline{\mathrm{co}}^{\beta(L,V)}\, \Delta_L f(p)\right)\big|_K. \tag{68}$$
□

The following result establishes a useful representation for the generalized core-L-Jacobian. It is parallel to the result given for the generalized L-Jacobian in Lemma 2.11.

Proposition 4.2. *Let $Y = V^*$ and assume that $f : \mathcal{D} \to Y$ is Lipschitz near p. Then, for $L \in \Lambda(X)$,*

$$\Delta_L f(p) = \bigcap_{\delta>0} \beta(L,V)\text{-}c\ell \left\{ D_L f(x) : x \in (p+\delta B_X) \cap \Omega_L(f) \right\}. \tag{69}$$

Proof. To prove the inclusion "$\subseteq$", by the $\beta(L,V)$-closedness of the right-hand side, it suffices to show that, for all $\delta > 0$,

$$\Gamma_L f(p) \subseteq \beta(L,V)\text{-}c\ell \left\{ D_L f(x) : x \in (p+\delta B_X) \cap \Omega_L(f) \right\}, \tag{70}$$

where $\Gamma_L f(p)$ is defined by (9). Fix $\delta > 0$, and let $\Phi \in \Gamma_L f(p)$. Then there exists $(x_i)_{i\in\mathbb{N}}$ in $\Omega_L(f)$ such that $\lim_{i\to\infty} x_i = p$ and $\Phi \in \beta(L,V)\text{-clus}_{i\to\infty} D_L f(x_i)$. Choose i large enough so that all the members of the sequence (x_i) belong to $p + \delta B_X$. Thus,

$$\Phi \in \beta(L,V)\text{-}\operatorname*{clus}_{i\to\infty} D_L f(x_i) \subseteq \beta(L,V)\text{-}c\ell \left\{ D_L f(x) : x \in (p+\delta B_X) \cap \Omega_L(f) \right\}, \tag{71}$$

proving the first inclusion.

For the inclusion "$\supseteq$", the $\beta(L,V)$-compactness of $\Delta_L f(p)$ (Proposition 4.1) and Theorem 2.4 imply that it suffices to show that, for all $n \in \mathbb{N}$ and for all $\boldsymbol{v} \in V^n$ we have

$$\boldsymbol{v} \circ \left(\bigcap_{\delta>0} \beta(L,V)\text{-}c\ell \left\{ D_L f(x) : x \in (p+\delta B_X) \cap \Omega_L(f) \right\} \right) \subseteq \boldsymbol{v} \circ \Delta_L f(p). \tag{72}$$

Set $S_\delta := \left\{ D_L f(x) : x \in (p+\delta B_X) \cap \Omega_L(f) \right\}$. Then, $\beta(L,V)\text{-}c\ell\, S_\delta$ is $\beta(L,V)$-compact and $\boldsymbol{v}$ is $(\beta(L,V), \beta(L,\mathbb{R}^n))$-continuous. From Lemma 2.7 and Lemma 2.6 we obtain that

$$\boldsymbol{v} \circ \left(\bigcap_{\delta>0} \beta(L,V)\text{-}c\ell\, S_\delta \right) = \bigcap_{\delta>0} \boldsymbol{v} \circ (\beta(L,V)\text{-}c\ell\, S_\delta) = \bigcap_{\delta>0} c\ell\, (\boldsymbol{v} \circ S_\delta)\,. \tag{73}$$

Then, (72) will be satisfied when we prove that

$$\bigcap_{\delta>0} c\ell \left\{ \boldsymbol{v} \circ D_L f(x) : x \in (p+\delta B_X) \cap \Omega_L(f) \right\} \subseteq \boldsymbol{v} \circ \Delta_L f(p). \tag{74}$$

For, let Ψ be an element of the left-hand-side of (74). Then, for each $i \in \mathbb{N}$, there exists $x_i \in (p + (1/i)B_X) \cap \Omega_L(f)$ such that $\|\Psi - \boldsymbol{v} \circ D_L f(x_i)\| < 1/i$, that is, $\lim_{i\to\infty} x_i = p$ and

$$\Psi = \lim_{i\to\infty} \boldsymbol{v} \circ D_L f(x_i) \in \boldsymbol{v} \circ \beta(L,V)\text{-}\operatorname*{clus}_{i\to\infty} D_L f(x_i) \subseteq \boldsymbol{v} \circ \Gamma_L f(p) \subseteq \boldsymbol{v} \circ \Delta_L f(p), \tag{75}$$

which shows the second inclusion. $\square$

Using the representation in the previous proposition, the local boundedness and the $\beta(L,V)$-upper semicontinuity of the generalized core-L-Jacobian are shown in the following.

Proposition 4.3. *Let $Y = V^*$ and assume that $f : \mathcal{D} \to Y$ is Lipschitz near p. Then, for $L \in \Lambda(X)$, the map $\Delta_L f(\cdot)$ is bounded near p and topologically (and also sequentially) $\beta(L, V)$-usc at p.*

Proof. Let $r > \ell_f(p)$ be arbitrary. Then, there exists a positive number δ_0 such that f is Lipschitz in the ball $p + \delta_0 B_X$ with Lipschitz modulus r.

Let $L \in \Lambda(X)$ be fixed. By the property (i) of Proposition 4.1, we get $\Delta_L f(x) \subseteq r\overline{B}_{\mathcal{L}(L,Y)}$ for all $x \in p + \delta_0 B_X$, proving the boundedness of $\Delta_L f(\cdot)$.

For $\delta > 0$ and u near p, set

$$S_\delta(u) := \beta(L, V)\text{-}c\ell\left\{D_L f(x) : x \in (u + \delta B_X) \cap \Omega_L(f)\right\} \tag{76}$$

Then, $S_\delta(u)$ is $\beta(L, V)$-compact and

$$S_\gamma(u) \subseteq S_\delta(p) \tag{77}$$

whenever $u + \gamma B_X \subseteq p + \delta B_X$, i.e., when $\gamma \leq \delta - \|p - u\|$.

To prove the topological $\beta(L, V)$-usc at p, let $U \subseteq \mathcal{L}(L, Y)$ be a $\beta(L, V)$-open set such that $\Delta_L f(p) \subseteq U$. By Proposition 4.2, we have

$$\Delta_L f(p) = \bigcap_{\delta>0} S_\delta(p) \subseteq U. \tag{78}$$

In view of the $\beta(L, V)$-compactness and of the downward directedness of the sets S_δ, there exist $\delta_0 > 0$, such that $S_{\delta_0}(p) \subseteq U$. Hence, using Proposition 4.2 again,

$$\Delta_L f(u) = \bigcap_{\gamma>0} S_\gamma(u) \subseteq S_{\gamma_0}(u) \subseteq S_{\delta_0}(p) \subseteq U \tag{79}$$

holds whenever $\|u - p\| < \delta_0$ and $\gamma_0 := \delta_0 - \|u - p\|$. This proves the topological $\beta(L, V)$-usc of the map $\Delta_L f(\cdot)$ at p.

The sequential $\beta(L, V)$-usc of the map $\Delta_L f(\cdot)$ at p is now a consequence of Lemma 2.8. □

The next proposition is a key for the characterization of the generalized core-L-Jacobian and it is also instrumental when establishing upper estimates for $\Delta_L f(p)$.

Proposition 4.4. *Let $Y = V^*$ be a Radon–Nikodým space and $L \in \Lambda(X)$. Let $f : \mathcal{D} \to Y$ be Lipschitz near $p \in \mathcal{D}$, and let $\mathcal{F}_L : \mathcal{D} \to 2^{\mathcal{L}(L,Y)}$ be a set-valued map with the following properties:*

(i) *$\mathcal{F}_L$ is sequentially $\beta(L, V)$-usc at p.*

(ii) *There exists $\delta > 0$ such that, for all $x \in \Omega_L(f) \cap (p + \delta B_X)$, we have*

$$D_L f(x) \in \mathcal{F}_L(x). \tag{80}$$

Then,

$$\Delta_L f(p) \subseteq \overline{\mathcal{F}_L(p)}^{\beta(L,V)}. \tag{81}$$

Proof. By the $\beta(L,V)$-closedness of the right hand side in (81), it is enough to prove that

$$\Gamma_L f(p) \subseteq \mathcal{F}_L(p). \tag{82}$$

If $\Phi \in \Gamma_L f(p)$, then there exists a sequence $x_i \in (p+\delta B_X)\cap\Omega_L(f)$ converging to p such that $\Phi \in \beta(L,V)\text{-clus}_{i\to\infty} D_L f(x_i)$.

By property (ii), we have that $D_L f(x_i) \in \mathcal{F}_L(x_i)$, therefore the sequential $\beta(L,V)$-usc of $\mathcal{F}_L$ at p yields $\Phi \in \mathcal{F}_L(p)$. □

The next result is a characterization of the generalized core-L-Jacobian.

Corollary 4.5. *Let $Y = V^*$ be a Radon–Nikodým space and $L \in \Lambda(X)$. Let $f : \mathcal{D} \to Y$ be a locally Lipschitz function. Then $\mathcal{F}_L = \Delta_L f(\cdot)$ is the smallest set-valued mapping $\mathcal{F}_L : \mathcal{D} \to 2^{\mathcal{L}(L,Y)}$ with the following properties:*

(i) *For all $p \in \mathcal{D}$, $\mathcal{F}_L(p)$ is $\beta(L,V)$-closed.*

(ii) *$\mathcal{F}_L$ is sequentially $\beta(L,V)$-usc at p.*

(iii) *For all $x \in \Omega_L(f)$, we have*

$$D_L f(x) \in \mathcal{F}_L(x). \tag{83}$$

Proof. Indeed, if a set-valued map $\mathcal{F}_L : \mathcal{D} \to 2^{\mathcal{L}(L,Y)}$ satisfies properties (ii) and (iii), then, it also fulfills the conditions (i) and (ii) of Proposition 4.4. Thus, by Proposition 4.4, the inclusion (81) follows for all $p \in \mathcal{D}$. Due to assumption (i), (81) is equivalent to $\Delta_L f(p) \subseteq \mathcal{F}_L(p)$. Hence, by Proposition 4.4, all the set-valued maps $\mathcal{F}$ with properties (i)–(iii) contain the generalized core-L-Jacobian map $\Delta_L f(\cdot)$.

On the other hand, by property (i) of Proposition 4.1 and Proposition 4.3, $\mathcal{F}_L(\cdot) = \Delta_L f(\cdot)$ satisfies (i)–(iii). □

Now we shift our attention to the generalized core-Jacobian, $\Delta f(p)$. We derive for this set important properties that remain valid despite the removal of the convexity in the definition of $\Delta_L f(p)$.

The next theorem forms one of the main results of the paper. It shows fundamental properties of the set $\Delta f(p)$, among which condition (iv) shows that it is the core of the generalized Jacobian.

Theorem 4.6. *Let $Y = V^*$ be Radon–Nikodým space and let that $f : \mathcal{D} \to Y$ be Lipschitz near p.*

(i) *The generalized core-Jacobian $\Delta f(p)$ is $\beta(X,V)$-compact, nonempty, $\Delta f(p) \subseteq \ell_f(p)\overline{B}_{\mathcal{L}(X,Y)}$, and $Df(p) \in \Delta f(p)$ whenever f is Gâteaux-differentiable at p.*

(ii) *For all $K \in \Lambda(X)$,*

$$(\Delta f(p))\big|_K \subseteq \Delta_K f(p). \tag{84}$$

(iii) *For $K \in \Lambda(X)$,*

$$\Delta_K f(p) \subseteq \left(\overline{\text{co}}^{\beta(X,V)}\,\Delta f(p)\right)\Big|_K . \tag{85}$$

(iv) *The relation*

$$\overline{\text{co}}^{\beta(X,V)}\,\Delta f(p) = \partial f(p) \tag{86}$$

holds.

(*v*) *The generalized core-Jacobian $\Delta f(p)$ is a singleton if and only if f is strictly w^*-Hadamard-differentiable at p.*

Proof. (*ii*) directly follows from the definition of $\Delta f(p)$.

We shall apply Theorem 3.2 to the family $\mathcal{G}_L := \Delta_L f(p)$ $(L \in \Lambda(X))$. Note that property (*i*) of Proposition 4.1 yields that (43) is satisfied with $\rho = \ell_f(p)$. Also, properties (*ii*) and (*iii*) of the same proposition imply that (44) holds. Hence, by Theorem 3.2, the set $\mathcal{G}$ which is now equal to $\Delta f(p)$ satisfies (46) and (47), which are exactly conditions (*i*) and (*iii*) of the theorem. If f is Gâteaux differentiable at p, then it is also L-differentiable and $Df(p)|_L = D_L f(p) \in \Delta_L f(p)$ for all $L \in \Lambda(X)$. Hence $Df(p) \in \Delta f(p)$.

For the inclusion "$\subseteq$" in (*iv*), it suffices to show that $\Delta f(p) \subseteq \partial f(p)$, because the latter is $\beta(X, V)$-closed and convex by Lemma 2.10. Let $\Phi \in \Delta f(p)$. Then, by the definition of $\Delta f(p)$ and property (*iv*) of Proposition 4.1, we get, for all $L \in \Lambda(X)$,

$$\Phi|_L \in \Delta_L f(p) \subseteq \partial_L f(p). \tag{87}$$

Hence, $\Phi \in \partial f(p)$.

Conversely, let $\Phi \in \partial f(p)$. Then, for all $L \in \Lambda(X)$, we have that $\Phi|_L \in \partial_L f(p)$. Then, property (*iv*) of Proposition 4.1 implies

$$\Phi|_L \in \overline{\operatorname{co}}^{\beta(L,V)}\, \Delta_L f(p). \tag{88}$$

Now, using property (*iii*) of this theorem in conjunction with the convexity and the $\beta(X, V)$-compactness of the set $\overline{\operatorname{co}}^{\beta(X,V)}\, \Delta f(p)$, it results, for all $L \in \Lambda(X)$,

$$\Phi|_L \in \left(\overline{\operatorname{co}}^{\beta(X,V)}\, \Delta f(p)\right)\Big|_L. \tag{89}$$

By Theorem 2.4, the inclusion $\Phi \in \overline{\operatorname{co}}^{\beta(X,V)}\, \Delta f(p)$ follows. Hence (*iv*) is proved.

Obviously, $\Delta f(p)$ is a singleton if and only if $\partial f(p)$ is a singleton, which, by Theorem 2.14, is equivalent to f being w^*-Hadamard-differentiable at p. □

The topological properties of the set-valued map $\Delta f(\cdot)$ are given in this theorem.

Theorem 4.7. *Let $Y = V^*$ and assume that $f : \mathcal{D} \to Y$ is Lipschitz near p. Then, the map $\Delta f(\cdot)$ is bounded near p and is topologically (and also sequentially) $\beta(X, V)$-usc at p.*

Proof. The proof of the boundedness of $\Delta f(\cdot)$ is analogous to that of $\Delta_L f(\cdot)$.

We prove the sequential $\beta(X, V)$-usc which, by Lemma 2.8, yields also the topological $\beta(X, V)$-usc.

Let (Ψ_i, x_i) be a sequence such that, for all i, $\Psi_i \in \Delta f(x_i)$, and $\lim_{i\to\infty} x_i = p$. We need to prove that

$$\underset{i\to\infty}{\beta(X,V)\text{-clus}}\, \Psi_i \subseteq \Delta f(p). \tag{90}$$

By the definition of $\Delta f(p)$, it suffices to show that, for all $L \in \Lambda(X)$,

$$\left(\underset{i\to\infty}{\beta(X,V)\text{-clus}}\, \Psi_i\right)\Bigg|_L \subseteq \Delta_L f(p). \tag{91}$$

By the boundedness of the map $\Delta f(\cdot)$, the sequence Ψ_i is $\beta(X,V)$-precompact. Thus, the continuity of the restriction map rest_L and the identity (26) of Lemma 2.7 imply that the above inclusion is equivalent to

$$\beta(L,V)\text{-}\underset{i\to\infty}{\mathrm{clus}}\,(\Psi_i|_L) \subseteq \Delta_L f(p). \tag{92}$$

Since $\Psi_i \in \Delta f(x_i)$, we have that $\Psi_i|_L \in \Delta_L f(x_i)$ for all $L \in \Lambda(X)$. Therefore (92) holds by the sequential $\beta(L,V)$-usc of the map $\Delta_L f(\cdot)$ at p, □

To establish a useful representation for the generalized core-Jacobian, we need the following result.

Lemma 4.8. *Let X, Y, and Z be normed spaces such that Z is finite dimensional, and let $K \in \Lambda(X)$. Let $C \in \mathcal{L}(Y,Z)$. Then there exists a constant $\gamma \geq 0$ such that, for all $A \in \mathcal{L}(X,Y)$, there exists an extension $\overline{A} \in \mathcal{L}(K,Y)$ with*

$$\|C \circ \overline{A}\| \leq \gamma \|C \circ A\|. \tag{93}$$

Proof. Without loss of generality, we may assume that C is surjective. Since Z is finite dimensional, there exists a continuous linear right inverse $C^+ \in \mathcal{L}(Z,Y)$ of C.

Since Z is finite dimensional, by applying the Hahn-Banach extension theorem to the componentwise, we can see that there exists a constant $\gamma \geq 0$ such that, every $B \in \mathcal{L}(K,Z)$ admits an extension $\overline{B} \in \mathcal{L}(X,Z)$ with $\|\overline{B}\| \leq \gamma\|B\|$. In particular, let $\overline{B}$ the extension of $B := C \circ A$ and consider $D := A - C^+ \circ B$. Then $D \in \mathcal{L}(K,Y)$ and $C \circ D = 0$ holds. The space K is finite dimensional, hence the range of D is also finite dimensional. Therefore, D can be extended to a continuous linear map $\overline{D} \in \mathcal{L}(X,Y)$ such that the ranges of D and $\overline{D}$ coincide. Hence, $C \circ \overline{D} = 0$ is also valid. Now set $\overline{A} := C^+ \circ \overline{B} + \overline{D}$. Then, $\overline{A}$ is an extension of A, because $\overline{A}|_K = C^+ \circ \overline{B}|_K + \overline{D}|_K = C^+ \circ B + D = A$. On the other hand, $C \circ \overline{A} = \overline{B}$, which yields (93). □

Theorem 4.9. *Let $Y = V^*$ and assume that $f : \mathcal{D} \to Y$ is Lipschitz near p. Then,*

$$\begin{aligned}\Delta f(p) = \bigcap_{\substack{\delta > 0 \\ L \in \Lambda(X)}} \; &\beta(X,V)\text{-}c\ell\{\Phi \in \mathcal{L}(X,Y) : \\ &\exists\, x \in (p + \delta B_X) \cap \Omega_L(f) : \Phi|_L = D_L f(x)\}.\end{aligned} \tag{94}$$

Proof. To prove the inclusion "$\supseteq$", let Ψ be a member of the right hand side of (94). To show that $\Psi \in \Delta f(p)$, we need to verify, for all $L \in \Lambda(X)$, that $\Psi|_L \in \Delta_L f(p)$, which, by Proposition 4.2, is equivalent to proving

$$\Psi|_L \in \bigcap_{\delta>0} \beta(L,V)\text{-}c\ell\,\big\{D_L f(x) : x \in (p + \delta B_X) \cap \Omega_L(f)\big\}. \tag{95}$$

Fix $L \in \Lambda(X)$. Since Ψ is a member of the right hand side of (94), it follows that, for all $\delta > 0$,

$$\Psi \in \beta(X,V)\text{-}c\ell\,\big\{\Phi \in \mathcal{L}(X,Y) : \exists\, x \in (p + \delta B_X) \cap \Omega_L(f) : \Phi|_L = D_L f(x)\big\}. \tag{96}$$

We show that this inclusion implies, for all $\delta > 0$,

$$\Psi|_L \in \beta(L,V)\text{-}c\ell \big\{D_L f(x) : x \in (p + \delta B_X) \cap \Omega_L(f)\big\}. \tag{97}$$

For, let U be an arbitrary $\beta(L,V)$-neighborhood of $\Psi|_L$. Then, there exist $\varepsilon > 0$, $h_1, \ldots, h_n \in L$ and $v_1, \ldots, v_n \in V$ such that

$$\big\{A \in \mathcal{L}(L,Y) : \max_{1\le i\le n} |\langle (A - \Psi|_L)(h_i), v_i\rangle| < \varepsilon\big\} \subseteq U. \tag{98}$$

Now the set

$$W := \big\{\Pi \in \mathcal{L}(X,Y) : \max_{1\le i\le n} |\langle (\Pi - \Psi)(h_i), v_i\rangle| < \varepsilon\big\} \tag{99}$$

is a $\beta(X,V)$-neighborhood of Ψ, therefore, by (96), there exist an element $\Phi \in \mathcal{L}(X,Y)$ and a point $x \in (p + \delta B_X) \cap \Omega_L(f)$ such that $\Phi|_L = D_L f(x)$ and $\Phi \in W$. The last inclusion yields $\Phi|_L \in U$, whence $D_L f(x) \in U$ follows. Thus,

$$U \cap \beta(X,V)\text{-}c\ell \big\{D_L f(x) : x \in (p + \delta B_X) \cap \Omega_L(f)\big\} \ne \emptyset \tag{100}$$

showing that (97) holds.

Upon taking the intersection of the right hand side of (97) with respect to $\delta > 0$, we obtain (95) and hence the proof of the inclusion (94) is complete.

For the inclusion "$\subseteq$", let $\Psi \in \Delta f(p)$. It suffices to show that, for all $\delta > 0$ and $L \in \Lambda(X)$,

$$\Psi \in \overline{S_{\delta,L}}^{\beta(X,V)}, \tag{101}$$

where

$$S_{\delta,L} := \big\{\Phi \in \mathcal{L}(X,Y) : \exists x \in (p + \delta B_X) \cap \Omega_L(f) : \Phi|_L = D_L f(x)\big\}. \tag{102}$$

Let $\delta > 0$ and $L \in \Lambda(X)$ be fixed and W be a $\beta(X,V)$-neighborhood of Ψ. Then, there exist $\varepsilon > 0$, $h_1, \ldots, h_m \in X$, and $v_1, \ldots, v_n \in V$ such that $\|h_i\| = 1$ for all i, and

$$\big\{\Phi \in \mathcal{L}(X,Y) : \max_{1\le i\le m,\, 1\le j\le n} |\langle (\Phi - \Psi)(h_i), v_j\rangle| < \varepsilon\big\} \subseteq W. \tag{103}$$

Let K be the smallest linear subspace containing L and $h_1, \ldots, h_m$. Define

$$C(y) := \boldsymbol{v} \circ y = (\langle y, v_1\rangle, \ldots, \langle y, v_n\rangle) \quad (y \in Y). \tag{104}$$

Then $C \in \mathcal{L}(Y, \mathbb{R}^n)$, and hence, the conclusion of Lemma 4.8 holds for some constant $\gamma > 0$. We equip the space $\mathbb{R}^n$ with the norm $\|(t_1, \ldots, t_n)\| := \max_{1\le i\le n} |t_i|$ and define the set U by

$$U := \left\{M \in \mathcal{L}(K,Y) : \|C \circ (M - \Psi|_K)\| < \frac{\varepsilon}{\gamma}\right\}. \tag{105}$$

Let $u_1, \ldots, u_k$ is a basis for the subspaces K, then the norm on $\mathcal{L}(K, \mathbb{R}^n)$ defined by $|||B||| := \max_{1\le i\le k} \|B(u_i)\|$, is equivalent to standard operator norm $\|\cdot\|$, that is, there exists a constant $\alpha > 0$ such that $\|B\| \le \alpha|||B|||$. Thus, we get

$$\left\{M \in \mathcal{L}(K,Y) : \max_{1\le i\le m,\, 1\le j\le n} |\langle (M - \Psi|_K)(u_i), v_j\rangle| < \frac{\varepsilon}{\alpha\gamma}\right\} \subseteq U. \tag{106}$$

Therefore, U is a $\beta(K,V)$-neighborhood of $\Psi|_K$.

The element Ψ is in $\Delta f(p)$, hence $\Psi|_K \in \Delta_K f(p)$. In view of Proposition 4.2, we get that

$$\Psi|_K \in \beta(K,V)\text{-}c\ell\,\big\{D_K f(x) : x \in (p+\delta B_X)\cap\Omega_K(f)\big\}. \tag{107}$$

Thus, since U is a neighborhood of $\Psi|_K$, there exists $x \in (p+\delta B_X)\cap\Omega_K(f)$ such that $D_K f(x) \in U$, that is,

$$\|C\circ(D_K f(x) - \Psi|_K)\| < \frac{\varepsilon}{\gamma}. \tag{108}$$

Now, apply Lemma 4.8 to $A := D_K f(x) - \Psi|_K$, and C defined in (104), it follows that there exists an extension $\overline{A} \in \mathcal{L}(X,Y)$ of A with $\|C\circ\overline{A}\| \le \gamma\|C\circ A\|$. Define

$$\Phi := \overline{A} + \Psi, \tag{109}$$

it results that

$$\Phi|_K := \big(\overline{A}+\Psi\big)\,|_K = A + \Psi|_K = D_K f(x), \tag{110}$$

and thus, using $L \subseteq K$, it results that $\Phi|_L = D_L f(x)$, proving that $\Phi \in S_{\delta,L}$.

On the other hand, using (108) and that h_i is of norm 1, we obtain

$$\max_{1\le i\le m,\,1\le j\le n} |\langle(\Phi-\Psi)(h_i), v_j\rangle| = \max_{1\le i\le m} \big\|C\circ\overline{A}(h_i)\big\| \le \|C\circ\overline{A}\| \le \gamma\|C\circ A\| \le \gamma\frac{\varepsilon}{\gamma}, \tag{111}$$

which shows that Φ is also in W. Therefore, $W\cap S_{\delta,L} \ne \emptyset$, proving the inclusion (101), and hence, the theorem. □

The next result is useful when finding upper estimates for the generalized core-Jacobian.

Theorem 4.10. *Let $Y = V^*$ be a Radon–Nikodým space, $f : \mathcal{D} \to Y$ be Lipschitz function near $p \in \mathcal{D}$, and $\mathcal{F} : \mathcal{D} \to 2^{\mathcal{L}(X,Y)}$ be a set-valued map with the following properties:*

- *(i) $\mathcal{F}$ is bounded near p and is sequentially $\beta(X,V)$-usc at p.*
- *(ii) There exists $\delta > 0$ such that, for all $L \in \Lambda(X)$ and $x \in \Omega_L(f)\cap(p+\delta B_X)$, we have*

$$D_L f(x) \in \mathcal{F}(x)|_L. \tag{112}$$

Then,

$$\Delta f(p) \subseteq \overline{\mathcal{F}(p)}^{\beta(X,V)}. \tag{113}$$

Proof. We show first that the $\beta(X,V)$-usc of $\mathcal{F}$ at p implies the $\beta(L,V)$-usc of the map $\mathcal{F}_L := \mathcal{F}|_L$ at p for all $L \in \Lambda(X)$.

Indeed, let (Ψ_i, x_i) be a sequence such that $\Psi_i \in \mathcal{F}(x_i)|_L$ and (x_i) tends to p. Then there exist elements $\Phi_i \in \mathcal{F}(x_i)$ satisfying $\Psi_i = \Phi_i|_L$. By the $\beta(X,V)$-usc of $\mathcal{F}$, we have $\beta(X,V)\text{-clus}_{i\to\infty}\Phi_i \subseteq \mathcal{F}(p)$. On the other hand, by the boundedness of $\mathcal{F}$ near p, the sequence (Φ_i) is $\beta(X,V)$-precompact. Thus, using the continuity of the restriction map rest_L and the second assertion of Lemma 2.7, we get

$$\beta(L,V)\text{-}\underset{i\to\infty}{\mathrm{clus}}\,\Psi_i = \beta(L,V)\text{-}\underset{i\to\infty}{\mathrm{clus}}\,(\Phi_i|_L) = \left(\beta(X,V)\text{-}\underset{i\to\infty}{\mathrm{clus}}\,\Phi_i\right)\Bigg|_L \subseteq \mathcal{F}(p)|_L. \tag{114}$$

To prove (113), by applying Theorem 2.4, it suffices to show that, for $L \in \Lambda(X)$,

$$\Delta f(p)|_L \subseteq \left(\overline{\mathcal{F}(p)}^{\beta(X,V)}\right)\Big|_L, \tag{115}$$

which is satisfied when we show that

$$\Delta_L f(p) \subseteq \left(\overline{\mathcal{F}(p)}^{\beta(X,V)}\right)\Big|_L = \overline{(\mathcal{F}(p)|_L)}^{\beta(L,V)} = \overline{\mathcal{F}_L(p)}^{\beta(L,V)}, \tag{116}$$

where the second last equality is a consequence of applying Lemma 2.6 to the $\beta(X, V)$-precompact set $\mathcal{F}(p)$.

Since $\mathcal{F}_L$ is $\beta(L, V)$-usc and satisfies condition (ii) of this theorem, it follows that $\mathcal{F}_L$ satisfies the assumptions of Proposition 4.4. Hence, by the same proposition (116) holds. □

Remark 4.11. The lack of convexity in the sets $\Delta_L f(p)$ defining $\Delta f(p)$, is behind the fact that the blindness with respect to L-nullset of $\Delta_L f(p)$ is not necessarily valid, and hence, a Fabian–Preiss type result is no longer possible. As a result, a characterization of $\Delta f(p)$ similar to the one in Theorem 2.15 for $\partial f(p)$ is not obtained.

5. Chain Rules and Relation to Known Notions

The first chain rule for the generalized core-Jacobian is for the case when one composes a nonsmooth function with a continuous linear one. For this we need the following result which is a special case of the one established in [16, Lemma 4.1].

Lemma 5.1. *Let Y and Z be arbitrary normed spaces, and $L \in \Lambda(X)$. Assume that $A \in \mathcal{L}(X, Y)$ and $g : \mathcal{O} \to Z$ is Lipschitz near $A(p)$, where $\mathcal{O} \subseteq Y$ is an open set containing $A(p)$. Then $g \circ A$ is L-differentiable at $p \in \mathcal{D}$, if and only if g is K-differentiable at $A(p)$ and*

$$D_L(g \circ A)(p) = D_K g\,(A(p)) \circ A|_L, \tag{117}$$

where $K \subseteq Y$ denotes the image space of A.

Theorem 5.2. *Let X, Y be arbitrary normed spaces and $Z = W^*$ be a Radon–Nikodým space. Let $A : X \to Y$ be a continuous linear operator and let $g : \mathcal{O} \to Z$ be Lipschitz near $A(p)$, where $\mathcal{O} \subseteq Y$ is an open set containing $A(p)$. Then, for all $L \in \Lambda(X)$,*

$$\Delta_L(g \circ A)(p) \subseteq \Delta_K g\,(A(p)) \circ A|_L, \tag{118}$$

where $K = A(L)$. If, in addition X and Y are Banach spaces and A is surjective then (118) *holds with equality and the following inclusion holds*

$$\Delta g\,(A(p)) \circ A \subseteq \Delta(g \circ A)(p). \tag{119}$$

If $g \circ A$ is strictly w^*-Hadamard-differentiable at p, then (119) *holds with equality.*

Proof. By shrinking $\mathcal{O}$, we may assume that g is Lipschitz on $\mathcal{O}$. Now choose a neighborhood $U \subseteq X$ of p such that $A(U) \subseteq \mathcal{O}$.

To prove (118), we first show the following inclusion

$$\Gamma_L(g \circ A)(p) \subseteq \Gamma_K g\,(A(p)) \circ A|_L. \tag{120}$$

For, let $\Psi \in \beta(L,W)\text{-clus}_{i\to\infty} D_L(g\circ A)(x_i)$, where $\lim_{i\to\infty} x_i = p$. By Lemma 5.1, it results that, for each i, the function g is K-differentiable at $A(x_i)$ with

$$D_L(g\circ A)(x_i) = D_K g(A(x_i))\circ A|_L. \tag{121}$$

Hence, from (121) and Lemma 2.2(i) applied to $A|_L \in \mathfrak{L}(L,Y)$, we get that the inclusion $\Psi \in \beta(L,W)\text{-clus}_{i\to\infty} D_L(g\circ A)(x_i)$ is equivalent to

$$\Psi \in \underset{i\to\infty}{\beta(L,W)\text{-clus}}\,(D_K g\,(A(x_i))\circ A|_L) = \left(\underset{i\to\infty}{\beta(K,W)\text{-clus}}\, D_K g\,(A(x_i))\right)\circ A|_L, \tag{122}$$

this implies that $\Psi \in \Gamma_K g\,(A(p))\circ A|_L$, proving (120).

Upon taking the $\beta(L,W)$-closure of the inclusion (120) and applying Lemma 2.6, we obtain (118).

In the case when X and Y are Banach spaces and A is surjective, by Banach's theorem, A is an open mapping, and hence it has a right inverse which is continuous at 0, i.e., there exist a mapping $M : Y \to X$ and a constant $C \geq 0$ such that, for all $y \in Y$, $A\circ M(y) = y$ and $\|M(y)\| \leq C\|y\|$.

To show that (118) holds with equality, it suffices to prove that (120) is satisfied with the reversed inclusion. For, let $\Psi \in \Gamma_K g\,(A(p))\circ A|_L$. Then, there exists a sequence $y_i \in \Omega_K(g)$ converging to $A(p)$ such that

$$\Psi \in \left(\underset{i\to\infty}{\beta(K,W)\text{-clus}}\, D_K g(y_i)\right)\circ A|_L. \tag{123}$$

Set $x_i := p + M\,(y_i - A(p))$. Then, by the properties of the right inverse map M, we have that $A(x_i) = y_i$ for all i, and x_i converges to p. Hence, using Lemma 2.6 and Lemma 5.1, we obtain

$$\begin{aligned}\Psi \in &\left(\underset{i\to\infty}{\beta(K,W)\text{-clus}}\, D_K g\,(A(x_i))\right)\circ A|_L = \left(\underset{i\to\infty}{\beta(K,W)\text{-clus}}\, D_K g\,(A(x_i))\right)\circ A|_L \\ &= \underset{i\to\infty}{\beta(L,W)\text{-clus}}\,(D_K g\,(A(x_i))\circ A|_L) \\ &= \underset{i\to\infty}{\beta(L,W)\text{-clus}}\,(D_L(g\circ A)(x_i)) \in \Gamma_L(g\circ A)(p).\end{aligned} \tag{124}$$

Thus, the equality in (118) is proved.

For (119), let $\Psi \in \Delta g\,(A(p))\circ A$. Then, for all $L \in \Lambda(X)$, $\Psi|_L \in \Delta_{A(L)} g\,(A(p))\circ A|_L$. Using the equality in (118), this is equivalent to

$$\Psi|_L \in \Delta_L(g\circ A)(p) \quad (L \in \Lambda(X)) \tag{125}$$

saying that $\Psi \in \Delta(g\circ A)(p)$.

If $g\circ A$ is strictly w^*-Hadamard differentiable, then, by Theorem 4.6, $\Delta(g\circ A)(p)$ is a singleton and hence inclusion (120) yields that the left hand side is a singleton and thus, being nonempty, $\Delta g\,(A(p))\circ A$ is a singleton, from which the equality in (120) follows. $\square$

The result of the next theorem is a *smooth-nonsmooth chain rule.*

Theorem 5.3. *Let $Y = V^*$ and $Z = W^*$ be Radon–Nikodým spaces. Let $f : \mathcal{D} \to Y$ be a locally Lipschitz function near $p \in \mathcal{D}$ and $g : \mathcal{O} \to Z$ be continuously differentiable at $f(p)$, where $\mathcal{O} \subseteq Y$ is an open set containing $f(p)$. Assume that $W \circ Dg\,(f(p)) \subseteq V$. Then, for all $L \in \Lambda(X)$,*

$$Dg\,(f(p)) \circ \Delta_L f(p) \subseteq \Delta_L(g \circ f)(p) \tag{126}$$

and

$$Dg\,(f(p)) \circ \Delta f(p) \subseteq \Delta(g \circ f)(p). \tag{127}$$

Proof. First we show that, for all $L \in \Lambda(X)$,

$$Dg\,(f(p)) \circ \Gamma_L f(p) \subseteq \Gamma_L(g \circ f)(p). \tag{128}$$

Let $\Psi \in Dg\,(f(p)) \circ \Gamma_L f(p)$. Then there exists a sequence (x_i) converging to p such that $x_{\in}\Omega_L(f)$ and

$$\Psi \in Dg\,(f(p)) \circ \left(\beta(L,V)\text{-}\underset{i\to\infty}{\text{clus}}\, D_L f(x_i) \right). \tag{129}$$

By the condition $W \circ Dg\,(f(p)) \subseteq V$ and by Lemma 2.2, we have that the operator $A : \mathcal{L}(L,Y) \to \mathcal{L}(L,Z)$ defined by $A(\Phi) := Dg\,(f(p))$ is $(\beta(L,V), \beta(L,W))$-continuous. Hence,

$$Dg\,(f(p)) \circ \left(\beta(L,V)\text{-}\underset{i\to\infty}{\text{clus}}\, D_L f(x_i) \right) = \beta(L,W)\text{-}\underset{i\to\infty}{\text{clus}} \left(Dg\,(f(p)) \circ D_L f(x_i) \right). \tag{130}$$

On the other hand, by the estimate,

$$\begin{aligned} &\|Dg\,(f(p)) \circ D_L f(x_i) - Dg\,(f(x_i)) \circ D_L f(x_i)\| \\ \le\ &\|Dg\,(f(p)) - Dg\,(f(x_i))\|\, \|D_L f(x_i)\| \end{aligned} \tag{131}$$

and by the continuous differentiability of g at $f(p)$, it follows that

$$\begin{aligned} \Psi \in\ &\beta(L,W)\text{-}\underset{i\to\infty}{\text{clus}} \left(Dg\,(f(p)) \circ D_L f(x_i) \right) \\ =\ &\beta(L,W)\text{-}\underset{i\to\infty}{\text{clus}} \left(Dg\,(f(x_i)) \circ D_L f(x_i) \right) \\ =\ &\beta(L,W)\text{-}\underset{i\to\infty}{\text{clus}} \left(D_L(g \circ f)(x_i) \right) \subseteq \Gamma_L(g \circ f)(p), \end{aligned} \tag{132}$$

proving (128).

Upon taking the $\beta(L,W)$-closure of the two sides of the inclusion in (128), and by using Lemma 2.6, the inclusion (126) follows.

The statement of (127) is an immediate consequence of (126) and the definition of the generalized core-(L-)Jacobians $\Delta f(p)$ and $\Delta(g \circ f)(p)$. $\square$

In order to establish a relationship between our generalized core-Jacobian and Ioffe's approximate subdifferential, we recall the appropriate notions from [7].

Assume that $\varphi : \mathcal{D} \to \mathbb{R}$ is a locally Lipschitz function near p. The *lower Dini-directional derivative* of φ at x near p is defined by

$$d^-\varphi(x,h) := \liminf_{t\to 0} \frac{\varphi(x+th)-\varphi(x)}{t} \quad (h \in X). \tag{133}$$

Define Ioffe's *approximate subdifferential* at p by

$$\partial_a\varphi(p) := \bigcap_{L\in\Lambda(X)} \limsup_{x\to p} \partial_L^-\varphi(x), \tag{134}$$

where

$$\partial_L^-\varphi(x) := \big\{\xi \in X^* : \langle \xi, h\rangle \le d^-\varphi(x,h)\, \forall h \in L\big\} \tag{135}$$

and the limsup is, as usual, taken in the norm-to-weak-star topology.

Theorem 5.4. *Let $Y = V^*$ be a Radon–Nikodým space. Let $f : \mathcal{D} \to Y$ be a locally Lipschitz function near $p \in \mathcal{D}$. Then, for all $v \in V$,*

$$v \circ \Delta f(p) \subseteq \partial_a(v\circ f)(p). \tag{136}$$

Proof. Let $v \in V$ be fixed and set $\varphi := v \circ f$. We show that

$$\Delta\varphi(p) \subseteq \partial_a\varphi(p). \tag{137}$$

Let $\Phi \in \Delta\varphi(p)$. Then, for all $L \in \Lambda(X)$ there exists a sequence (x_i) in $\Omega_L(f)$ converging to p such that, we have that

$$\Phi|_L = \lim_{i\to\infty} D_L\varphi(x_i). \tag{138}$$

By the Hahn–Banach extension theorem, there exist continuous linear functional ξ_i such that

$$\xi_i|_L = \Phi|_L - D_L\varphi(x_i), \qquad \|\xi_i\| = \|\Phi|_L - D_L\varphi(x_i)\|. \tag{139}$$

Set $\Phi_i = \Phi - \xi_i$ for all i. Then, we have

$$\Phi_i|_L = D_L\varphi(x_i), \qquad \lim_{i\to\infty} \Phi_i = \Phi. \tag{140}$$

The function φ is L-differentiable at x_i, hence $\Phi_i \in \partial_L^-\varphi(x_i)$. Therefore, $\Phi \in \limsup_{x\to p} \partial_L^-\varphi(x)$ for every $L \in \Lambda(X)$, proving that $\Phi \in \partial_a\varphi(p)$.

To complete proof, note that by the chain rule Theorem 5.3,

$$v\circ\Delta f(p) \subseteq \Delta(v\circ f)(p) = \Delta\varphi(p) \subseteq \partial_a\varphi(p) \subseteq \partial_a(v\circ f)(p), \tag{141}$$

which completes the proof (136). □

Remark 5.5. In terms of the approximate subdifferential, a coderivative notion $D^*f(p)Y^* \to X^*$ for locally Lipschitz vector-valued functions f can be defined (see [7] for more details). Using [7, Theorem 4] and the inclusion in the above theorem, it follows that, for all $v \in V$,

$$v\circ\Delta f(p) \subseteq D^*f(p)(v). \tag{142}$$

The next result establishes a connection between the generalized core-Jacobian and the notion in (35).

Theorem 5.6. *Let $Y = V^*$ be a Radon–Nikodým space. Let $f : \mathcal{D} \to Y$ be a locally Lipschitz function near $p \in \mathcal{D}$. Then, for all $h \in X$,*

$$(\Delta f(p))(h) \subseteq \overline{\delta f(p,h)}^{w^*}. \tag{143}$$

Proof. For $h \in X$ with $h \neq 0$, denote by $\langle h \rangle$ the linear subspace generated by h. First we prove, for $\Gamma_{\langle h \rangle} := \Gamma_{\langle h \rangle} f(p)$, that

$$\Gamma_{\langle h \rangle}(h) \subseteq \delta f(p,h). \tag{144}$$

Let $\Psi \in \Gamma_{\langle h \rangle}(h)$, then there exists a sequence (x_i) in $\Omega_{\langle h \rangle}(f)$ such that $\lim_{i\to\infty} x_i = p$, and $\Psi \in \beta(\langle h \rangle, V)\text{-clus}_{i\to\infty} D_{\langle h \rangle} f(x_i)$. Since by definition, $D_{\langle h \rangle} f(x_i)(h) = f'(x_i, h)$, it follows that

$$\Psi(h) \in \Big(\beta(\langle h \rangle, V)\underset{i\to\infty}{\text{-clus}}\, D_{\langle h \rangle} f(x_i)\Big)(h) = w^*\underset{i\to\infty}{\text{-clus}}\, D_{\langle h \rangle} f(x_i)(h) = w^*\underset{i\to\infty}{\text{-clus}}\, f'(x_i, h). \tag{145}$$

Using the definition of $f'(x_i, h)$ it follows that there exists $0 < t_i < 1/i$ such that

$$\left\| \frac{f(x_i + t_i h) - f(x_i)}{t_i} - f'(x_i, h) \right\| < \frac{1}{i}, \tag{146}$$

hence, using (145) we obtain

$$\Psi \in w^*\underset{i\to\infty}{\text{-clus}}\, f'(x_i, h) = w^*\underset{i\to\infty}{\text{-clus}}\, \frac{f(x_i + t_i h) - f(x_i)}{t_i} \subseteq \delta f(p,h), \tag{147}$$

proving that (144) holds.

Upon taking the w^*-closure in (144) and by using the definition of $\Delta_{\langle h \rangle} f(p)$ and the $\beta(\langle h \rangle, V)$-precompactness of $\Gamma_{\langle h \rangle}$, Lemma 2.6, it follows that

$$\big(\Delta_{\langle h \rangle} f(p)\big)(h) = \Big(\overline{\Gamma}_{\langle h \rangle}^{\beta(\langle h \rangle, V)}\Big)(h) = \overline{\Gamma_{\langle h \rangle}(h)}^{w^*} \subseteq \overline{\delta f(p,h)}^{w^*}. \tag{148}$$

Now we show that (143) is satisfied. Let $y \in \Delta f(p)(h)$, then, $y = \Phi(h)$, where $\Phi \in \Delta f(p)$. Thus, $\Phi|_{\langle h \rangle} \in \Delta_{\langle h \rangle} f(p)$. Hence, using (148), we get

$$y = \Phi(h) \in \big(\Delta_{\langle h \rangle} f(p)\big)(h) \subseteq \overline{\delta f(p,h)}^{w^*}, \tag{149}$$

which completes the proof. $\square$

6. Generalized Jacobian for Continuous Selections

"Piecewise smooth" functions have received considerable attention in the last few years because of applications to solution methodology in optimization. See [8], [9], [10], [11], [19], [18], [16], [21], [22], [24], [25]. Given a finite system of some locally Lipschitz functions $g_1, \ldots, g_k : \mathcal{D} \to Y$, a continuous function $f : \mathcal{D} \to Y$ is called a *continuous* *selection of* $\{g_1, \dots, g_k\}$ if, for all $x \in \mathcal{D}$, there exists an index $j \in \{1, \dots, k\}$ such that $f(x) = g_j(x)$, that is, for all $x \in \mathcal{D}$,

$$f(x) \in \{g_1(x), \dots, g_k(x)\}. \tag{150}$$

When the functions $g_1, \dots, g_k$ are differentiable (resp. C^1) on $\mathcal{D}$, then we say that f is *piecewise differentiable* (resp. *piecewise smooth*).

The main result of this section offers formulae for the generalized core-L-Jacobian and for the generalized core-Jacobian of a piecewise smooth function. For its proof, we need the following lemma (which was stated as Step 1 in the proof of Theorem 5.5 in [18]).

Lemma 6.1. *Let $Y = V^*$ be a Radon–Nikodým space and let $f : \mathcal{D} \to Y$ be a continuous selection of $\{g_1, \dots, g_k\}$, where $g_1, \dots, g_k : \mathcal{D} \to Y$ are continuously Gâteaux-differentiable at $p \in \mathcal{D}$. Then, for all $L \in \Lambda(X)$ and for all $x \in (p + \delta B_X) \cap \Omega_L(f)$ there exists*

$$j_0 \in I(p) := \bigl\{ j \in \{1, \dots, k\} : \exists U \subseteq \mathcal{D} \text{ open, } p \in \overline{U},\ f|_U = g_j|_U \bigr\}. \tag{151}$$

such that

$$D_L f(x) = Dg_{j_0}(x)|_L \in \bigl\{ Dg_j(x) : j \in I(p) \bigr\}\big|_L . \tag{152}$$

Theorem 6.2. *Let $Y = V^*$ be a Radon–Nikodým space and let $f : \mathcal{D} \to Y$ be a continuous selection of $\{g_1, \dots, g_k\}$, where $g_1, \dots, g_k : \mathcal{D} \to Y$ are continuously Gâteaux-differentiable at $p \in \mathcal{D}$. Then f is Lipschitz near p, for all $L \in \Lambda(X)$,*

$$\Delta_L f(p) = \bigl\{ Dg_j(p)|_L : j \in I(p) \bigr\}, \tag{153}$$

and

$$\Delta f(p) = \bigl\{ Dg_j(p) : j \in I(p) \bigr\}, \tag{154}$$

where the set $I(p)$ is defined in (151).

Proof. By the continuous differentiability of the functions $g_1, \dots, g_k : \mathcal{D} \to Y$ at $p \in \mathcal{D}$, without loss of generality, we can assume that these functions are Gâteaux-differentiable on $\mathcal{D}$.

For the inclusion "$\subseteq$" in (153) and (154), define on $\mathcal{D}$ the set valued maps $\mathcal{F}$ and $\mathcal{F}_L$ ($L \in \Lambda(X)$) by

$$\mathcal{F}(x) := \bigl\{ Dg_j(x) : j \in I(p) \bigr\}, \qquad \mathcal{F}_L(x) := \mathcal{F}(x)|_L. \tag{155}$$

The continuous differentiability of the functions g_j at p yields that $\mathcal{F}$ and hence $\mathcal{F}_L$ are topologically $\beta(X, V)$-usc and $\beta(L, V)$-usc at p, respectively. It is clear that $\mathcal{F}$ is bounded near p and is $\beta(X, V)$-closed valued. By Lemma 6.1, we have that $\mathcal{F}$ and $\mathcal{F}_L$ also satisfy condition (ii) of Theorem 4.10 and Proposition 4.4, respectively. Hence, by the conclusion of these results, we get the inclusion "$\subseteq$" in equations (153) and (154).

For the reversed inclusion, let $j \in I(p)$. Then, there exists an open set $U \subseteq \mathcal{D}$ such that $p \in \overline{U}$ and $f|_U = g_j|_U$. Thus, there exists a sequence $(x_i)_{i \in \mathbb{N}}$ such that $x_i \to p$ as $i \to \infty$ and f and g_j coincide in a neighborhood of x_i. Whence, since g_j is Gâteaux-differentiable at x_i so does f and we have $Df(x_i) = Dg_j(x_i)$ for all i. By the continuity of Dg_j at p, we have that $Df(x_i) \to Dg_j(p)$ as $i \to \infty$, proving that $Dg_j(p) \in \Delta f(p)$, and also $Dg_j(p)|_L \in \Gamma_L f(p) \subseteq \Delta_L f(p)$, for all $L \in \Lambda(X)$. $\square$

As a consequence of this theorem, we obtain the following result derived in [17, Theorem 6.1] and in [18, Theorem 5.5].

Corollary 6.3. *Let $Y = V^*$ be a Radon–Nikodým space and let $f : \mathcal{D} \to Y$ be a continuous selection of $\{g_1, \ldots, g_k\}$, where $g_1, \ldots, g_k : \mathcal{D} \to Y$ are continuously Gâteaux-differentiable at $p \in \mathcal{D}$. Then f is Lipschitz near p, for all $L \in \Lambda(X)$,*

$$\partial_L f(p) = \mathrm{co}\left\{Dg_j(p)|_L : j \in I(p)\right\}, \tag{156}$$

and

$$\partial f(p) = \mathrm{co}\left\{Dg_j(p) : j \in I(p)\right\}. \tag{157}$$

References

[1] Y. Benyamini, J. Lindenstrauss: Geometric Nonlinear Functional Analysis. Vol. 1, Colloquium Publications 48, American Mathematical Society, Providence (2000).

[2] F. H. Clarke: On the inverse function theorem, Pacific J. Math. 64(1) (1976) 97–102.

[3] F. H. Clarke: Optimization and Nonsmooth Analysis, John Wiley & Sons, New York (1983).

[4] M. Fabián, D. Preiss: On the Clarke's generalized Jacobian, Suppl. Rend. Circ. Mat. Palermo, II. Ser. 14 (1987) 305–307.

[5] R. B. Holmes: Geometric Functional Analysis and its Applications, Graduate Texts in Mathematics 24, Springer, Berlin (1975).

[6] A. D. Ioffe: Nonsmooth analysis: differential calculus of nondifferentiable mappings, Trans. Amer. Math. Soc. 266(1) (1981) 1–56.

[7] A. D. Ioffe: Directional compactness, scalarization and nonsmooth semi-Fredholm mappings, Nonlinear Anal., Theory Methods Appl. 29(2) (1997) 201–219.

[8] H. Th. Jongen, D. Pallaschke: On linearization and continuous selections of functions, Optimization 19(3) (1988) 343–353.

[9] D. Klatte, B. Kummer: Nonsmooth Equations in Optimization. Regularity, Calculus, Methods and Applications, Nonconvex Optimization and its Applications 60, Kluwer, Dordrecht (2002).

[10] L. Kuntz, S. Scholtes: Structural analysis of nonsmooth mappings, inverse functions, and metric projections, J. Math. Anal. Appl. 188(2) (1994) 346–386.

[11] L. Kuntz, S. Scholtes: Qualitative aspects of the local approximation of a piecewise differentiable function, Nonlinear Anal., Theory Methods Appl. 25(2) (1995) 197–215.

[12] B. S. Mordukhovich: Metric approximations and necessary conditions for optimality for general classes of nonsmooth extremal problems, Dokl. Akad. Nauk SSSR 254(5) (1980) 1072–1076.

[13] B. S. Mordukhovich: Generalized differential calculus for nonsmooth and set-valued mappings, J. Math. Anal. Appl. 183(1) (1994) 250–288.

[14] B. S. Mordukhovich: Coderivatives of set-valued mappings: calculus and applications, Nonlinear Anal., Theory Methods Appl. 30(5) (1997) 3059–3070.

[15] B. S. Mordukhovich: Variational Analysis and Generalized Differentiation. I: Basic Theory, Grundlehren der Mathematischen Wissenschaften 330, Springer, Berlin (2006).

[16] Zs. Páles, V. Zeidan: Infinite dimensional generalized Jacobian: properties and calculus rules, J. Math. Anal. Appl. 344(1) (2008) 55–75.

[17] Zs. Páles, V. Zeidan: Generalized Jacobian for functions with infinite dimensional range and domain, Set-Valued Anal. 15(4) (2007) 331–375.

[18] Zs. Páles, V. Zeidan: Infinite dimensional Clarke generalized Jacobian, J. Convex Analysis 14(2) (2007) 433–454.

[19] J.-S. Pang, D. Ralph: Piecewise smoothness, local invertibility, and parametric analysis of normal maps, Math. Oper. Res. 21(2) (1996) 401–426.

[20] B. H. Pourciau: Analysis and optimization of Lipschitz continuous mappings, J. Optimization Theory Appl. 22(3) (1977) 311–351.

[21] D. Ralph, S. Scholtes: Sensitivity analysis of composite piecewise smooth equations, Math. Program., Ser. B 76(3) (1997) 593–612.

[22] D. Ralph, H. Xu: Implicit smoothing and its application to optimization with piecewise smooth equality constraints, J. Optim. Theory Appl. 124(3) (2005) 673–699.

[23] R. T. Rockafellar: Convex Analysis, Princeton Mathematical Series 28, Princeton University Press, Princeton (1970).

[24] R. T. Rockafellar: A property of piecewise smooth functions, Comput. Optim. Appl. 25(1-3) (2003) 247–250.

[25] S. Scholtes: Introduction to Piecewise Differentiable Equations, Habilitation Thesis, University of Karlsruhe, Karlsruhe (1994).

[26] L. Thibault: Subdifferentials of compactly Lipschitzian vector-valued functions, Ann. Mat. Pura Appl., IV. Ser. 125 (1980) 157–192.

[27] L. Thibault: On generalized differentials and subdifferentials of Lipschitz vector-valued functions, Nonlinear Anal., Theory Methods Appl. 6(10) (1982) 1037–1053.

[28] J. Warga: Fat homeomorphisms and unbounded derivate containers, J. Math. Anal. Appl. 81(2) (1981) 545–560; Errata: ibid. 82(2) (1982) 582–583.