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Convex Solids with Planar Homothetic Sections Through Given Points

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Extending results of Rogers, Burton and Mani to the case of unbounded convex sets, we prove that line-free closed convex sets K_1 and K_2 of dimension n in $\mathbb{R}^n$, $n \geq 4$, are homothetic provided there are points $p_1 \in \operatorname{int} K_1$ and $p_2 \in \operatorname{int} K_2$ such that for every pair of parallel 2-dimensional planes L_1 and L_2 through p_1 and p_2, respectively, the sections $K_1 \cap L_1$ and $K_2 \cap L_2$ are homothetic. Furthermore, if there is a homothety $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(K_1) = K_2$ and $f(p_1) \neq p_2$, then K_1 and K_2 are convex cones or their boundaries are convex quadric surfaces. Related results on elliptic and centrally symmetric 2-dimensional bounded sections of convex sets are considered.

Keywords: Homothety, convex body, planar section, quadric surface

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1. Introduction and Main Results

A well-known result of convex geometry states that convex bodies K_1 and K_2 in $\mathbb{R}^n$, $n \geq 3$, are homothetic if and only if the orthogonal projections of K_1 and K_2 on every hyperplane in $\mathbb{R}^n$ are homothetic, where similarity ratio may depend on the projection hyperplane (see, e.g., Bonnesen and Fenchel [2] for historical references and Gardner [7] for an overview and further results). This statement was refined by Rogers [16], who proved that the bodies K_1 and K_2 are homothetic if and only if the orthogonal projections of K_1 and K_2 on every 2-dimensional plane are homothetic. (In a standard way, sets F_1 and F_2 in $\mathbb{R}^n$ are *homothetic* provided $F_1 = z + \lambda F_2$ for a suitable point $z \in \mathbb{R}^n$ and a scalar $\lambda > 0$.)

Similar results involving planar sections of convex bodies were obtained by Rogers [16] (for the case when $p_1 \in \operatorname{int} K_1$ and $p_2 \in \operatorname{int} K_2$) and later by Burton [3], who proved that convex bodies K_1 and K_2 in $\mathbb{R}^n$, $n \geq 3$, are homothetic provided there are points p_1 and p_2 in $\mathbb{R}^n$ such that for every pair of parallel 2-dimensional planes L_1 and L_2 through p_1 and p_2, respectively, the sections $K_1 \cap L_1$ and $K_2 \cap L_2$ are homothetic or empty. Furthermore, as shown by Burton and Mani [4], K_1 and K_2 are homothetic ellipsoids provided there is a homothety $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(K_1) = K_2$ and $f(p_1) \neq p_2$. (Let us recall that a mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ of the form $f(x) = z + \lambda x$ is the *homothety* with center $z \in \mathbb{R}^n$ and ratio $\lambda > 0$.)

In this paper we discuss possible extensions of the results above to the case of unbounded

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convex sets in $\mathbb{R}^n$, motivated by the demands of convex analysis in the study of non-compact convex sets in $\mathbb{R}^n$ (see, e.g., [6, 8, 15]). In what follows, by a *convex solid* we mean a closed convex set, possibly unbounded, with nonempty interior in $\mathbb{R}^n$. A convex solid is called *line-free* provided it contains no line. As usual, $\operatorname{bd} K$, $\operatorname{rbd} K$, $\operatorname{int} K$, and $\operatorname{rint} K$ stand, respectively, for the boundary, relative boundary, interior, and relative interior of a convex set $K \subset \mathbb{R}^n$.

In view of the results above, the following two problems were posed in [18] and confirmed for the case of translates in $\mathbb{R}^n$.

Problem 1.1. Is it true that convex solids K_1 and K_2 in $\mathbb{R}^n$, $n \geq 3$, are homothetic provided the orthogonal projections of K_1 and K_2 on each 2-dimensional plane are homothetic?

Problem 1.2. Is it true that convex solids K_1 and K_2 in $\mathbb{R}^n$, $n \geq 3$, are homothetic provided there are points p_1 and p_2 in $\mathbb{R}^n$ such that for every pair of parallel 2-dimensional planes L_1 and L_2 through p_1 and p_2, respectively, the sections $K_1 \cap L_1$ and $K_2 \cap L_2$ are homothetic or empty?

Even for the case of translates, 2-dimensional planes in Problem 1.2 cannot be replaced by lines. Indeed, Larman and Morales-Amaya [13] gave an example of a pair of unbounded line-free convex solids K_1, K_2 in $\mathbb{R}^2$ with the following properties: (i) K_1 is not a translate of K_2 or of $-K_2$, (ii) there are points $p_1 \in \operatorname{int} K_1$ and $p_2 \in \operatorname{int} K_2$ such that any two parallel chords of K_1 and K_2 that contain p_1 and p_2, respectively, are of equal length.

As proved in [18], convex solids K_1 and K_2 in $\mathbb{R}^n$, $n \geq 3$, are homothetic if and only if the orthogonal projections of K_1 and K_2 on each 3-*dimensional* plane are homothetic. Our first statement here is that Problem 1.1 has a surprisingly negative answer, as follows from the example below.

Example 1.3. Let K_1 and K_2 be solid paraboloids in $\mathbb{R}^3$, given, respectively, by

$$K_1 = \{(x, y, z) \mid x^2 + y^2 \leq z\} \quad \text{and} \quad K_2 = \{(x, y, z) \mid 2x^2 + y^2 \leq z\}.$$

Obviously, K_1 and K_2 are not homothetic. At the same time, their orthogonal projections $\pi_L(K_1)$ and $\pi_L(K_2)$ on each 2-dimensional plane $L \subset \mathbb{R}^3$ are homothetic. Indeed, if $L = \{(x, y, z) \mid z = \text{const}\}$, then $\pi_L(K_1) = \pi_L(K_2) = L$. For any other 2-dimensional plane L in $\mathbb{R}^3$, the projections $\pi_L(K_1)$ and $\pi_L(K_2)$ are planar convex solids bounded by parabolas with axis of symmetry parallel to the projection of the z-axis on L. Since any two parabolas in the plane with parallel axes of symmetry are homothetic, the sets $\pi_L(K_1)$ and $\pi_L(K_2)$ are also homothetic.

Theorem 1.4 below partly solves Problem 1.2 and extends the results from [4, 16] to the case of convex solids K_1 and K_2 in $\mathbb{R}^n$, $n \geq 4$, with $p_1 \in \operatorname{int} K_1$ and $p_2 \in \operatorname{int} K_2$. The question whether Theorem 1.4 holds for $n = 3$ remains open.

Theorem 1.4. *Let K_1 and K_2 be line-free convex solids in $\mathbb{R}^n$, $n \geq 4$, and let points $p_1 \in \operatorname{int} K_1$ and $p_2 \in \operatorname{int} K_2$ be such that for every pair of parallel 2-dimensional planes L_1 and L_2 through p_1 and p_2, respectively, the sections $K_1 \cap L_1$ and $K_2 \cap L_2$ are homothetic. Then K_1 and K_2 are homothetic. Furthermore, if there is a homothety $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(K_1) = K_2$ and $f(p_1) \neq p_2$, then K_1 and K_2 are convex cones or their boundaries are convex quadric surfaces.*

In our terminology, convex cones can have nonzero apices: a convex set $C \subset \mathbb{R}^n$ is a cone with apex $p \in \mathbb{R}^n$ provided $p+\lambda(x-p) \in C$ for all $x \in C$ and $\lambda \geq 0$. By a *convex surface* in $\mathbb{R}^n$ we mean the boundary of a convex solid. This definition includes a hyperplane or a pair of parallel hyperplanes. We say that a convex surface in $\mathbb{R}^n$ is a *convex quadric surface* provided it is a connected component of a quadric surface. The classification of quadric surfaces (see, for example, [17]) implies that a convex quadric surface in $\mathbb{R}^n$ that contains no line can be expressed in suitable coordinates by one of the equations:

$$\begin{aligned} &\alpha_1 x_1^2 + \cdots + \alpha_n x_n^2 = 1, && \text{(ellipsoid)} \\ &\alpha_1 x_1^2 - \alpha_2 x_2^2 - \cdots - \alpha_n x_n^2 = 1,\ x_1 \geq 0, && \text{(convex elliptic hyperboloid)} \\ &\alpha_1 x_1^2 - \alpha_2 x_2^2 - \cdots - \alpha_n x_n^2 = 0,\ x_1 \geq 0, && \text{(convex elliptic cone)} \\ &\alpha_1 x_1^2 + \cdots + \alpha_{n-1} x_{n-1}^2 = x_n, && \text{(elliptic paraboloid)} \end{aligned}$$

where all scalars $\alpha_1, \ldots, \alpha_n$ are positive. Convex quadric surfaces containing lines are both-way unbounded cylinders based on convex quadrics of the same type that are situated in proper subspaces of $\mathbb{R}^n$.

Analysis of the proof of Theorem 1.4 reveals the following corollary. The question whether Corollary 1.5 holds for $n = 3$ remains open.

Corollary 1.5. *Let $K \subset \mathbb{R}^n$, $n \geq 4$, be a line-free convex solid and $p_1, p_2 \in \operatorname{int} K$ be distinct points such that for every pair of parallel 2-dimensional planes L_1 and L_2 through p_1 and p_2, respectively, the sections $K \cap L_1$ and $K \cap L_2$ are homothetic. Then $\operatorname{bd} K$ is a convex quadric surface or K is a convex cone whose apex belongs to the line through p_1 and p_2.* □

We note that Corollary 1.5 fails provided K is not line-free. Indeed, if K is a both-way infinite cylinder in $\mathbb{R}^n$, $n \geq 3$, given by

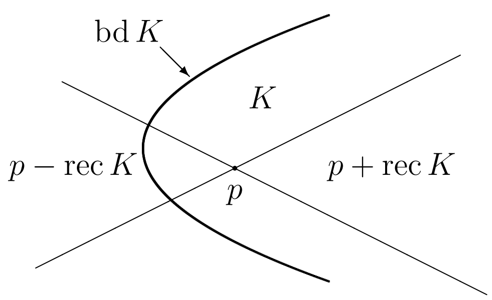
$$K = \{(x_1, \ldots, x_n) \mid 0 \leq x_1, x_2 \leq 1, x_3, \ldots, x_n \in \mathbb{R}\},$$

and if $p_1 = \frac{1}{4}(1, 1, 0, \ldots, 0)$, $p_2 = \frac{1}{2}(1, 1, 0, \ldots, 0)$, then for any 2-dimensional subspace $L \subset \mathbb{R}^n$ the sections $K \cap (p_1 + L)$ and $K \cap (p_2 + L)$ are homothetic, while neither $\operatorname{bd} K$ is a convex quadric surface nor K is a convex cone.

Let us recall that the *recession cone* of a convex set $K \subset \mathbb{R}^n$ is defined by

$$\operatorname{rec} K = \{y \in \mathbb{R}^n \mid x + \alpha y \in K \text{ for all } x \in K \text{ and } \alpha \geq 0\}.$$

We will be using the "double cone" $(p + \operatorname{rec} K) \cup (p - \operatorname{rec} K)$ with apex p, as depicted below.



The proof of Theorem 1.4 is based on Theorem 1.6 below. By an m-dimensional plane in $\mathbb{R}^n$ we mean a translate of an m-dimensional subspace. We say that a plane $F \subset \mathbb{R}^n$ *properly* intersects a convex solid $K \subset \mathbb{R}^n$ (or, equivalently, the boundary of K) provided it intersects both the boundary $\operatorname{bd} K$ and the interior $\operatorname{int} K$ of K.

Theorem 1.6. *For a line-free convex solid $K \subset \mathbb{R}^n$ and a point $p \in \mathbb{R}^n$, $n \geq 3$, the following conditions are equivalent:*

1) *all proper bounded sections of* $\operatorname{bd} K$ *by 2-dimensional planes through* p *are ellipses,*
2) *the set* $\operatorname{bd} K \setminus [(p + \operatorname{rec} K) \cup (p - \operatorname{rec} K)]$ *lies in a convex quadric surface.*

We remark that condition *1*) of Theorem 1.6 implicitly covers the trivial case when no proper section of $\operatorname{bd} K$ by a 2-dimensional plane through p is bounded. For the line-free convex solid K, this happens if and only if $K \subset p + \operatorname{rec} K$, or, equivalently, when the set $\operatorname{bd} K \setminus [(p + \operatorname{rec} K) \cup (p - \operatorname{rec} K)]$ is empty, thus ensuring the equivalence of conditions *1*) and *2*) of the theorem.

Since a convex solid $K \subset \mathbb{R}^n$ with $\operatorname{rec} K = \{0\}$ is compact, Theorem 1.6 is an extension of a well-known result of convex geometry, which states that the boundary of a convex body $K \subset \mathbb{R}^n$, $n \geq 3$, is an ellipsoid if and only if there is a point $p \in \operatorname{int} K$ such that all sections of $\operatorname{bd} K$ by 2-dimensional planes through p are ellipses (see Kubota [11] for $n = 3$ and Busemann [5, pp. 91–92] for $n \geq 3$). Höbinger [9, Theorems 2 and 6] and independently Burton [3] refined Busemann's statement by showing that the point p above can be chosen arbitrarily in $\mathbb{R}^n$. A similar result, which includes unbounded sets into consideration, is proved in [19]: the boundary of a convex solid $K \subset \mathbb{R}^n$, $n \geq 3$, is a convex quadric surface if and only if there is a point $p \in \operatorname{int} K$ such that all sections of $\operatorname{bd} K$ by 2-dimensional planes through p are convex quadric curves.

The following examples show that in Theorem 1.6 the boundary of K can be different from a convex quadric surface.

Example 1.7. Let K be a convex solid bounded by a truncated sheet of a convex circular cone in $\mathbb{R}^n$, given by

$$K = \big\{(x_1, \ldots, x_n) \mid x_n \geq \max\big\{1, (x_1^2 + \cdots + x_{n-1}^2)^{1/2}\big\}\big\}.$$

If $p = (0, \ldots, 0, 2)$, then $p \in \operatorname{int} K$ and all bounded sections of $\operatorname{bd} K$ by 2-dimensional planes through p are ellipses. Clearly, $\operatorname{bd} K \setminus [(p+\operatorname{rec} K) \cup (p-\operatorname{rec} K)]$ is the part of $\operatorname{bd} K$ disjoint from the hyperplane $x_n = 1$. We note that a 2-dimensional plane L through p intersects K along a bounded set if and only if L misses the $(n-1)$-dimensional open ball

$$\{(x_1, \ldots, x_n) \mid x_1^2 + \cdots + x_{n-1}^2 < 1, \quad x_n = 1\}.$$

Example 1.8. Another example gives a convex solid $K \subset \mathbb{R}^n$ bounded by a truncated sheet of a convex elliptic hyperboloid in $\mathbb{R}^n$:

$$K = \big\{(x_1, \ldots, x_n) \mid x_n \geq \max\big\{2, (x_1^2 + \cdots + x_{n-1}^2 + 1)^{1/2}\big\}\big\}.$$

If $p = (0, \ldots, 0, 1)$, then $p \in \mathbb{R}^n \setminus K$ and all proper bounded sections of $\operatorname{bd} K$ by 2-dimensional planes through p are ellipses. The set $\operatorname{bd} K \setminus [(p+\operatorname{rec} K) \cup (p-\operatorname{rec} K)]$ is the part of $\operatorname{bd} K$ disjoint from the hyperplane $x_n = 2$. We note that a 2-dimensional plane L

through p intersects K along a bounded set if and only if L misses the $(n-1)$-dimensional open ball

$$\{(x_1, \ldots, x_n) \mid x_1^2 + \cdots + x_{n-1}^2 < 1, \quad x_n = 2\}.$$

Affirmatively solving Rogers's [16] conjecture, Aitchison, Petty, and Rogers [1] proved that a convex body $K \subset \mathbb{R}^n$, $n \geq 3$, is symmetric about a point $p \in \operatorname{int} K$ or $\operatorname{bd} K$ is an ellipsoid provided all sections of K by 2-dimensional planes through p are centrally symmetric. Larman [12] refined this statement by showing that the point p can be chosen anywhere in $\mathbb{R}^n$. Theorem 1.6 allows us to extend these results to the case of convex solids in $\mathbb{R}^n$, $n \geq 4$.

Theorem 1.9. *For a line-free convex solid $K \subset \mathbb{R}^n$ and a point $p \in \mathbb{R}^n$, $n \geq 4$, the following conditions are equivalent:*

1) *all proper bounded sections of* $\operatorname{bd} K$ *by 2-dimensional planes through p are centrally symmetric,*

2) *K is symmetric about p (and thus is bounded) or the set* $\operatorname{bd} K \setminus [(p + \operatorname{rec} K) \cup (p - \operatorname{rec} K)]$ *lies in a convex quadric surface.*

The question whether Theorem 1.9 holds for $n = 3$ remains open. As above, Examples 1.7 and 1.8 show that in Theorem 1.9 the set $\operatorname{bd} K$ can be different from a convex quadric surface. Similarly to Theorem 1.6, condition *1*) of Theorem 1.9 implicitly covers the trivial case when no proper section of $\operatorname{bd} K$ by a 2-dimensional plane through p is bounded; this happens if and only if $\operatorname{bd} K \setminus [(p + \operatorname{rec} K) \cup (p - \operatorname{rec} K)] = \varnothing$.

For the convenience of the reader we provide a relevant list of standard properties of convex sets in $\mathbb{R}^n$ (their proofs can be found in [6, 8, 15]).

(P1) Any line-free closed convex set $K \subset \mathbb{R}^n$ is the convex hull of its extreme points and extreme rays (Klee [10]).

(P2) A closed convex set $K \subset \mathbb{R}^n$ is unbounded if and only if $\operatorname{rec} K \neq \{0\}$.

(P3) Let $K \subset \mathbb{R}^n$ be a closed convex set and $M \subset \mathbb{R}^n$ be a plane of any dimension such that $K \cap M \neq \varnothing$. The intersection $M \cap K$ is bounded if and only if the subspace $L = M - M$ satisfies the condition $L \cap \operatorname{rec} K = \{0\}$.

(P4) If $C \subset \mathbb{R}^n$ is a line-free closed convex cone with apex 0 and $L \subset \mathbb{R}^n$ is a subspace such that $C \cap L = \{0\}$, then there is a hyperplane $H \subset \mathbb{R}^n$ such that $L \subset H$ and $C \cap H = \{0\}$.

(P5) A closed convex set $K \subset \mathbb{R}^n$ is a cone with apex $p \in K$ if and only if $K - p = \alpha(K - p)$ for any given positive scalar $\alpha \neq 1$.

2. Proof of Theorem 1.6

2) $\Rightarrow$ *1*) Let $L \subset \mathbb{R}^n$ be a 2-dimensional plane through p that properly intersects K along a bounded set. Since $L - p$ is a subspace and the cones $\operatorname{rec} K$ and $-\operatorname{rec} K$ are symmetric about 0, (P3) implies that

$$(L - p) \cap [\operatorname{rec} K \cup (-\operatorname{rec} K)] = \{0\}.$$

Hence

$$L \cap [(p + \operatorname{rec} K) \cup (p - \operatorname{rec} K)] = \{p\}.$$

From here we obtain

$$\begin{aligned}(L \cap \operatorname{bd} K) \setminus \{p\} &= (L \cap \operatorname{bd} K) \setminus \big(L \cap [(p + \operatorname{rec} K) \cup (p - \operatorname{rec} K)]\big) \\ &= L \cap \big(\operatorname{bd} K \setminus [(p + \operatorname{rec} K) \cup (p - \operatorname{rec} K)]\big).\end{aligned}$$

By the hypothesis, the set $\operatorname{bd} K \setminus [(p + \operatorname{rec} K) \cup (p - \operatorname{rec} K)]$ lies in a convex quadric surface, S. Hence the set $(L \cap \operatorname{bd} K) \setminus \{p\}$ lies in the convex quadric curve $L \cap S$. Because $L \cap \operatorname{bd} K$ is the relative boundary of the 2-dimensional compact convex set $L \cap K$, we have $L \cap \operatorname{bd} K = L \cap S$, that is, $L \cap \operatorname{bd} K$ is a convex quadric curve itself. Being bounded, $L \cap \operatorname{bd} K$ should be an ellipse.

To show that *1*) $\Rightarrow$ *2*), we first exclude the trivial case when $K \subset p + \operatorname{rec} K$. In this case,

$$\operatorname{bd} K \setminus [(p + \operatorname{rec} K) \cup (p - \operatorname{rec} K)] = \varnothing.$$

On the other hand, the inclusion $K \subset p + \operatorname{rec} K$ obviously implies that all proper sections of K by 2-dimensional planes through p are unbounded, thus ensuring the trivial equivalence of conditions *1*) and *2*) of the theorem.

Hence we assume, in what follows, that $K \not\subset p + \operatorname{rec} K$. Without loss of generality, we may put $p = 0$. The condition $K \not\subset \operatorname{rec} K$ implies the existence of a point $x \in \operatorname{int} K \setminus \operatorname{rec} K$ such that the line $(x, 0)$ does not meet $\operatorname{rec} K \setminus \{0\}$ (equivalently, $(x, 0) \cap \operatorname{rec} K = \{0\}$). Because K is line-free, the cone $\operatorname{rec} K$ is also line-free. By (P4), we can choose a hyperplane H_1 through the line $(x, 0)$ such that $H_1 \cap \operatorname{rec} K = \{0\}$. According to (P3), H_1 properly intersects K along a bounded set. By the hypothesis, all sections of the set $E_1 = H_1 \cap \operatorname{bd} K$ by 2-dimensional subspaces of H_1 are ellipses. Then E_1 is an $(n-1)$-dimensional ellipsoid (see [3]).

Let G be an $(n-2)$-dimensional subspace of H_1 that contains x. By the continuity argument, we can choose a new hyperplane H_2 through G so close to H_1 that $H_2 \cap \operatorname{rec} K = \{0\}$. Then (P3) implies again that H_2 properly intersects K along a bounded set. As above, $H_2 \cap \operatorname{bd} K$ is an $(n-1)$-dimensional ellipsoid, E_2. Applying a suitable linear transformation, we may assume that both E_1 and E_2 are $(n-1)$-dimensional spheres (possibly, of distinct radii).

We state that $K \neq \operatorname{conv}(E_1 \cup E_2)$. Indeed, if $K = \operatorname{conv}(E_1 \cup E_2)$, one could choose a 2-dimensional plane through 0 properly intersecting $\operatorname{bd} K$ along a bounded curve distinct from an ellipse. Hence there is a 2-dimensional subspace N of $\mathbb{R}^n$ that properly intersects K such that the section $N \cap \operatorname{bd} K$ is an ellipse distinct from $N \cap \operatorname{conv}(E_1 \cup E_2)$. Because N intersects $\operatorname{int} K$, we can slightly vary N to satisfy the condition $N \cap G = \{0\}$.

Choose a point $z \in N \cap (\operatorname{bd} K \setminus \operatorname{conv}(E_1 \cup E_2))$. Clearly, there is a convex quadric surface $Q \subset \mathbb{R}^n$ that contains $\{z\} \cup E_1 \cup E_2$. To finalize the proof of *1*) $\Rightarrow$ *2*) we are going to show that

$$\operatorname{bd} K \setminus [\operatorname{rec} K \cup (-\operatorname{rec} K)] = Q \setminus [\operatorname{rec} K \cup (-\operatorname{rec} K)]. \tag{1}$$

First, we state that $N \cap \operatorname{bd} K = N \cap Q$. For this, we consider the cases $0 \notin \operatorname{bd} K$ and $0 \in \operatorname{bd} K$ separately.

Case 1. $0 \notin \operatorname{bd} K$. Then $N \cap (E_1 \cup E_2)$ consists of four distinct points (all different from 0) and $X = \{z\} \cup (N \cap (E_1 \cup E_2))$ is a planar set of five points with no three on a line.

Then there is a unique quadric curve containing X (see, e.g., [14, pp. 369–371]). Since both F and $N \cap Q$ are quadric curves containing X, we conclude that $N \cap \mathrm{bd}\, K = N \cap Q$.

Case 2. $0 \in \mathrm{bd}\, K$. Then $N \cap (E_1 \cup E_2)$ consists of three distinct points (one of them is 0) and $X = \{z\} \cup (N \cap (E_1 \cup E_2))$ is a planar set of four points with no three on a line. Let H be a hyperplane that supports K at 0. Since H also supports $\mathrm{conv}\,(E_1 \cup E_2)$, and since Q is a convex quadric surface containing $E_1 \cup E_2$, we obtain that H supports Q at 0. There is a unique quadric curve in N containing X and supported by the line $N \cap H$ through 0 (see, e.g., [14, pp. 377]). Because both $N \cap \mathrm{bd}\, K$ and $N \cap Q$ are quadric curves containing X and supported by $N \cap H$ at 0, we obtain that $N \cap \mathrm{bd}\, K = N \cap Q$.

Slightly rotating N about the line $(0, z)$, we obtain a family of ellipses that lie in $Q \cap \mathrm{bd}\, K$ and whose union covers an open piece P of $\mathrm{bd}\, K$ consisting of two open loops with common endpoints 0 and z. It is easy to see that for any point $x \in \mathrm{bd}\, K \setminus [\mathrm{rec}\, K \cup (-\mathrm{rec}\, K)]$ which is not the apex of K (if K is a cone) there is a 2-dimensional subspace M through x such that $M \cap \mathrm{rec}\, K = \{0\}$ and M intersects P along an arc distinct from a line segment. By (P3), the set $M \cap K$ is bounded, whence the section $M \cap \mathrm{bd}\, K$ is an ellipse. Since $M \cap P$ is a nontrivial arc of both quadric curves $M \cap \mathrm{bd}\, K$ and $M \cap Q$, we conclude that $M \cap \mathrm{bd}\, K = M \cap Q$.

Property (P3) implies that the union of 2-dimensional subspaces each intersecting K along a bounded set equals

$$T = \{0\} \cup (\mathbb{R}^n \setminus [\mathrm{rec}\, K \cup (-\mathrm{rec}\, K)]).$$

Our consideration shows that the union of subspaces M above is dense in T. Due to $M \cap \mathrm{bd}\, K = M \cap Q$ for any such subspace M, the convex surfaces $\mathrm{bd}\, K$ and Q have in T a common part which is dense in each of them. By the continuity argument, equality (1) holds.

3. Proof of Theorem 1.4

We precede the proof of the theorem with an auxiliary lemma.

Lemma 3.1. *Let M_1 and M_2 be line-free convex solids in $\mathbb{R}^n$, $n \geq 3$, such that (a) $\mathrm{rec}\, M_1 = \mathrm{rec}\, M_2$, (b) $0 \in \mathrm{int}\, M_1 \cap \mathrm{int}\, M_2$, (c) $M_1 \setminus (-\mathrm{rec}\, M_1) = M_2 \setminus (-\mathrm{rec}\, M_2)$, and (d) for any 2-dimensional subspace $L \subset \mathbb{R}^n$ the curves $L \cap \mathrm{bd}\, M_1$ and $L \cap \mathrm{bd}\, M_2$ are homothetic. Then $M_1 = M_2$.*

Proof. In view of (a) and (c), it is sufficient to prove that

$$M_1 \cap (-\mathrm{int}\,\mathrm{rec}\, M_1) = M_2 \cap (-\mathrm{int}\,\mathrm{rec}\, M_1). \tag{2}$$

This is obvious if $\dim \mathrm{rec}\, M_1 < n$ (because of $\mathrm{int}\,\mathrm{rec}\, M_1 = \varnothing$). Assume that $\dim \mathrm{rec}\, M_1 = n$ and choose in $-\mathrm{int}\,\mathrm{rec}\, M_1$ an open halfline h with apex 0. Since M_1 is line-free, we have $\mathrm{rec}\, M_1 \cap (-\mathrm{rec}\, M_1) = \{0\}$. Hence $h \not\subset \mathrm{rec}\, M_1$, which implies that h intersects both $\mathrm{bd}\, M_1$ and $\mathrm{bd}\, M_2$ at some points v_1 and v_2, respectively. Clearly, the coincidence of points v_1 and v_2 for any choice of the halfline $h \subset -\mathrm{int}\,\mathrm{rec}\, M_1$ implies equality (2). We intend to show that $v_1 = v_2$ by considering various cases separately.

(i) Assume first that M_1 has an extreme point $v \in \mathrm{bd}\, M_1 \setminus (-\mathrm{int}\,\mathrm{rec}\, M_1)$. Choose a 2-dimensional subspace L through h and v (L is uniquely defined provided $v_1 \neq v$) and consider the homothetic curves $L \cap \operatorname{bd} M_1$ and $L \cap \operatorname{bd} M_2$. These curves are identical because they have v as a common extreme point and coincide along both unbounded branches that lie in $\operatorname{bd} M_1 \setminus (-\operatorname{int} \operatorname{rec} M_1)$. In particular, $v_1 = v_2$.

(ii) Now assume that M_1 has no extreme points in $\operatorname{bd} M_1 \setminus (-\operatorname{int} \operatorname{rec} M_1)$. This implies that all extreme points of M_1 are in the bounded set $\operatorname{bd} M_1 \cap (-\operatorname{rec} M_1)$. Because the unbounded set M_1 is the convex hull of its extreme points and extreme rays (see (P1)), there is an extreme ray m of M_1. Clearly, an unbounded part of m lies in $\operatorname{bd} M_1 \setminus (-\operatorname{int} \operatorname{rec} M_1)$.

If $h \cup m$ does not lie in a 2-dimensional subspace, then we choose a 2-dimensional subspace L through h that intersects m at a point $w \notin -\operatorname{int} \operatorname{rec} M_1$ and consider the homothetic curves $L \cap \operatorname{bd} M_1$ and $L \cap \operatorname{bd} M_2$. As above, these curves have w as a common extreme point and coincide along both unbounded branches that lie in $\operatorname{bd} M_1 \setminus (-\operatorname{int} \operatorname{rec} M_1)$. Hence $L \cap \operatorname{bd} M_1 = L \cap \operatorname{bd} M_2$, which implies the equality $v_1 = v_2$.

If $h \cup m$ lies in a 2-dimensional subspace, then we choose in $-\operatorname{int} \operatorname{rec} M_1$ another open halfline h' with apex 0 such that such that $h' \cup m$ does not lie in a 2-dimensional subspace (this is possible because $n \geq 3$). By the argument above, the points of intersection of h' with $\operatorname{bd} M_1$ and $\operatorname{bd} M_2$, respectively, coincide. Since h' can be chosen arbitrarily close to h, we conclude that $v_1 = v_2$. Summing up, equality (2) holds. □

We start the proof of Theorem 1.6 by considering line-free convex solids K_1 and K_2 in $\mathbb{R}^n$, $n \geq 4$, that satisfy the hypothesis of Theorem 1.4. Since Theorem 1.4 is proved in [4] for the case when both K_1 and K_2 are bounded, we may assume that at least one of them, say K_1, is unbounded. By (P2), $\operatorname{rec} K_1 \neq \{0\}$. Without loss of generality, we may put $p_1 = p_2 = 0$.

We claim that $\operatorname{rec} K_1 = \operatorname{rec} K_2$. Indeed, if h is a halfline with apex 0 that lies in $\operatorname{rec} K_1$, and if L is a 2-dimensional subspace through h, then $K_1 \cap L$ contains a translate of h. Since $K_2 \cap L$ is homothetic to $K_1 \cap L$, the set K_2 contains a translate of h. Hence h lies in $\operatorname{rec} K_2$, and $\operatorname{rec} K_1 \subset \operatorname{rec} K_2$. By the symmetry argument, $\operatorname{rec} K_2 \subset \operatorname{rec} K_1$.

Our further consideration is divided into *Cases 1* and *2* below. In what follows, $\mathcal{F}$ stands for the family of hyperplanes $H \subset \mathbb{R}^n$ such that $H \cap \operatorname{rec} K_1 = \{0\}$.

Case 1. Assume the existence of a hyperplane $H_0 \in \mathcal{F}$ such that $H_0 \cap \operatorname{bd} K_1$ is different from an $(n-1)$-dimensional ellipsoid. By the hypothesis, $K_1 \cap L$ and $K_2 \cap L$ are homothetic for every 2-dimensional subspace L of H_0. Since $\dim H_0 = n - 1 \geq 3$, the compact sets $H_0 \cap K_1$ and $H_0 \cap K_2$ are homothetic (see [16]). In other words, there is a homothety $g_0 : H_0 \to H_0$ of the form $g_0(x) = z + \gamma x$, with $z \in H_0$ and $\gamma > 0$, such that

$$H_0 \cap K_2 = z + \gamma(H_0 \cap K_1).$$

The assumption that $H_0 \cap \operatorname{bd} K_1$ is different from an $(n-1)$-dimensional ellipsoid in H_0 implies the equality $z = g_0(0) = 0$ (see [4]). Hence

$$H_0 \cap K_2 = \gamma(H_0 \cap K_1) = H_0 \cap \gamma K_1. \tag{3}$$

We claim that $K_2 = \gamma K_1$. Indeed, put $K_1' = \gamma K_1$. We divide the proof of *Case 1* into subcases *1a–1c*.

1a. First we are going to prove that $H \cap K_2 = H \cap K_1'$ for any hyperplane $H \in \mathcal{F}$. We will do this in two steps: initially assuming that H is sufficiently close to H_0, and then letting H be any member of $\mathcal{F}$.

Let e_0 be a unit normal vector to H_0. Since $H_0 \cap \operatorname{rec} K_1 = \{0\}$, there is an $\varepsilon > 0$ such that for any unit vector $e \in \mathbb{R}^n$ with $\|e - e_0\| < \varepsilon$, the hyperplane H through 0 orthogonal to e satisfies $H \cap \operatorname{rec} K_1 = \{0\}$ and thus belongs to $\mathcal{F}$. Furthermore, because the section $H \cap K_1$ depends continuously on the choice of $H \in \mathcal{F}$ and because $H_0 \cap \operatorname{bd} K_1$ is not an $(n-1)$-dimensional ellipsoid, the scalar ε can be chosen so small that $H \cap K_1$ is also different from an $(n-1)$-dimensional ellipsoid provided $\|e - e_0\| < \varepsilon$. Denote by $\mathcal{F}_\varepsilon$ the family of hyperplanes $H \in \mathcal{F}$ with $\|e - e_0\| < \varepsilon$.

(i) We state that $H \cap K_2 = H \cap K_1'$ for any $H \in \mathcal{F}_\varepsilon$. Indeed, since $H \cap \operatorname{bd} K_1$ is not an $(n-1)$-dimensional ellipsoid, similar to (3) we obtain that $H \cap K_2 = H \cap \gamma_H K_1$, where the ratio $\gamma_H > 0$ depends on H. Since $H \cap \operatorname{bd} K_2$ and $H \cap \operatorname{bd} K_1'$ coincide in $H \cap H_0$, we conclude that $\gamma_H = \gamma$. Hence

$$H \cap K_2 = H \cap \gamma K_1 = H \cap K_1' \text{ for all } H \in \mathcal{F}_\varepsilon. \tag{4}$$

From (4) it follows the existence of a scalar $\delta > 0$ such that $\operatorname{bd} K_2$ and $\operatorname{bd} K_1'$ coincide in the δ-neighborhood $U_\delta(H_0)$ of H_0, which is an open slab of $\mathbb{R}^n$ bounded by a pair of hyperplanes parallel to H_0 each at distance δ from H_0.

(ii) Now choose any hyperplane $H \in \mathcal{F}$. If $H \cap \operatorname{bd} K_1$ is not an $(n-1)$-dimensional ellipsoid, then, as above, $H \cap K_2 = H \cap K_1'$. Let $H \cap \operatorname{bd} K_1$ be an $(n-1)$-dimensional ellipsoid. We state that $H \cap \operatorname{bd} K_2$ is an $(n-1)$-dimensional ellipsoid homothetic to $H \cap \operatorname{bd} K_1$. Indeed, each section of $\operatorname{bd} K_1$ by a 2-dimensional subspace $L \subset H$ is an ellipse. By the hypothesis, the sections $L \cap K_1$ and $L \cap K_2$ are homothetic. Hence $L \cap \operatorname{bd} K_2$ is also an ellipse, and [5, p. 92] implies that $H \cap \operatorname{bd} K_2$ is an $(n-1)$-dimensional ellipsoid homothetic to $H \cap \operatorname{bd} K_1$.

Because the ellipsoids $H \cap \operatorname{bd} K_2$ and $H \cap \operatorname{bd} K_1'$ coincide in the slab $U_\delta(H_0)$, they should be identical: $H \cap \operatorname{bd} K_2 = H \cap \operatorname{bd} K_1'$. Hence $H \cap K_2 = H \cap K_1'$.

1b. Due to

$$\cup\{H \mid H \in \mathcal{F}\} = \{0\} \cup \big(\mathbb{R}^n \setminus [\operatorname{rec} K_1 \cup (-\operatorname{rec} K_1)]\big)$$

and to the inclusion $\operatorname{rec} K_1 \subset \operatorname{int} K_1 \cap \operatorname{int} K_2$, the argument of *1a* shows that

$$K_2 \setminus (-\operatorname{rec} K_1) = K_1' \setminus (-\operatorname{rec} K_1).$$

By the continuity,

$$K_2 \setminus (-\operatorname{int} \operatorname{rec} K_1) = K_1' \setminus (-\operatorname{int} \operatorname{rec} K_1). \tag{5}$$

Now Lemma 3.1 implies that $K_2 = K_1' \,(= \gamma K_1)$.

1c. To finalize *Case 1*, it remains to explore the situation when there is another homothety $f(x) = q + \lambda x$, $\lambda > 0$, distinct from $g(x) = \gamma x$, such that $f(K_1) = K_2$. We state that $\lambda \neq \gamma$, since otherwise $\gamma K_1 = q + \gamma K_1$ implies $q = 0$, which results in $f = g$. Hence $\alpha = \gamma/\lambda \neq 1$.

If $q = 0$, then $\lambda K_1 = \gamma K_1$ or $K_1 = \alpha K_1$, and (P5) gives that K_1 is a convex cone with apex 0. Then $K_2 = \lambda K_1 = K_1$.

If $q \neq 0$, then with $r = q/(\alpha - 1)$, we rewrite $\gamma K_1 = q + \lambda K_1$ as

$$\alpha(\lambda K_1 - r) = \lambda K_1 - r.$$

By (P5), $\lambda K_1 - r$ is a convex cone with apex 0. Hence K_1 and K_2 are homothetic convex cones with apices r and $\gamma r \, (= q + \lambda r)$, respectively.

Case 2. Now assume that $H \cap \operatorname{bd} K_1$ is an $(n-1)$-dimensional ellipsoid for any hyperplane $H \in \mathcal{F}$. As in case (ii) of 1a, $H \cap \operatorname{bd} K_2$ is an $(n-1)$-dimensional ellipsoid homothetic to $H \cap \operatorname{bd} K_1$ for any choice of $H \in \mathcal{F}$. By Theorem 1.6, there are convex quadric surfaces S_1 and S_2 such that

$$\operatorname{bd} K_1 \setminus [\operatorname{rec} K_1 \cup (-\operatorname{rec} K_1)] \subset S_1, \qquad \operatorname{bd} K_2 \setminus [\operatorname{rec} K_1 \cup (-\operatorname{rec} K_1)] \subset S_2.$$

Since $\operatorname{rec} K_1 \subset \operatorname{int} K_1 \cap \operatorname{int} K_2$ (due to $0 \in \operatorname{int} K_1 \cap \operatorname{int} K_2$), we rewrite these inclusions as

$$\operatorname{bd} K_1 \setminus (-\operatorname{rec} K_1) \subset S_1, \qquad \operatorname{bd} K_2 \setminus (-\operatorname{rec} K_1) \subset S_2.$$

By the continuity argument, we can write

$$\operatorname{bd} K_1 \setminus (-\operatorname{int} \operatorname{rec} K_1) = S_1 \setminus (-\operatorname{int} \operatorname{rec} K_1), \tag{6}$$
$$\operatorname{bd} K_2 \setminus (-\operatorname{int} \operatorname{rec} K_1) = S_2 \setminus (-\operatorname{int} \operatorname{rec} K_1). \tag{7}$$

2a. Our next goal is to show that S_1 and S_2 are homothetic. To do this, we consider the cases $\dim \operatorname{rec} K_1 < n$ and $\dim \operatorname{rec} K_1 = n$ separately.

(i) Assume first that $\dim \operatorname{rec} K_1 < n$. Then $\operatorname{int} \operatorname{rec} K_1 = \varnothing$, which immediately implies the equalities $\operatorname{bd} K_1 = S_1$ and $\operatorname{bd} K_2 = S_2$. According to the classification of convex quadric surfaces, solid elliptic paraboloids are the only unbounded line-free convex solids $K \subset \mathbb{R}^n$ with quadric boundary and $\dim \operatorname{rec} K < n$. Thus both K_1 and K_2 are solid elliptic paraboloids and $\operatorname{rec} K_1 = \operatorname{rec} K_2$ is a halfline, h. Let G be the hypersubspace orthogonal to h, and let $E_i = G \cap S_i$, $i = 1, 2$. Since $E_i = G \cap \operatorname{bd} K_i$, $i = 1, 2$, the sections E_1 and E_2 are homothetic. This obviously implies that S_1 and S_2 are homothetic, whence K_1 and K_2 are also homothetic.

(ii) Next we suppose that $\dim \operatorname{rec} K_1 = n$. Then each of the surfaces S_1 and S_2 is either a convex elliptic cone or a convex elliptic hyperboloid. We claim that both S_1 and S_2 are of the same type: they are both either convex cones or convex hyperboloids. Indeed, if S_1 is a convex elliptic cone with apex q_1, then we can choose a 2-dimensional subspace L through q_1 that intersects S_1 along two halflines with common endpoint q_1, whence $L \cap (\operatorname{bd} K_1 \cap S_1)$ contains two line segments along these halflines. Since the curves $L \cap \operatorname{bd} K_1$ and $L \cap \operatorname{bd} K_2$ are homothetic, the section $L \cap (\operatorname{bd} K_2 \cap S_2)$ also contains two line segments. This immediately implies that S_2 should be a convex elliptic cone.

Denote by B_1 and B_2 the convex solids bounded by S_1 and S_2, respectively. Clearly, B_1 and B_2 are uniquely defined because both S_1 and S_2 are line-free. Furthermore, $\operatorname{rec} B_1 = \operatorname{rec} B_2 = \operatorname{rec} K_1$ due to (6) and (7).

If both surfaces S_1 and S_2 are convex elliptic cones, then they are homothetic. Indeed, if q_1 and q_2 are the apices of S_1 and S_2, respectively, then

$$B_1 = q_1 + \operatorname{rec} B_1 = (q_1 - q_2) + (q_2 + \operatorname{rec} B_2) = (q_1 - q_2) + B_2,$$

which implies the equality $S_1 = (q_1 - q_2) + S_2$.

Now assume that both S_1 and S_2 are convex elliptic hyperboloids. We can express S_1 in suitable coordinates by an equation

$$\alpha_1(x_1 - x_1')^2 - \alpha_2(x_2 - x_2')^2 - \cdots - \alpha_n(x_n - x_n')^2 = 1, \quad x_1 \geq x_1',$$

where all scalars $\alpha_1, \ldots, \alpha_n$ are positive and $(x_1', \ldots, x_n') \in \mathbb{R}^n$ is a given point. As easy to see, $\operatorname{rec} B_1$ is given by the inequality

$$\alpha_1 x_1^2 - \alpha_2 x_2^2 - \cdots - \alpha_n x_n^2 \geq 0, \quad x_1 \geq 0.$$

Since $\operatorname{rec} B_1 = \operatorname{rec} B_2$, the surface S_2 has to be described by an equation

$$\beta_1(x_1 - x_1'')^2 - \beta_2(x_2 - x_2'')^2 - \cdots - \beta_n(x_n - x_n'')^2 = 1, \quad x_1 \geq x_1'',$$

where $\beta_1, \ldots, \beta_n > 0$, $\alpha_1/\beta_1 = \cdots = \alpha_n/\beta_n$, and $(x_1'', \ldots, x_n'')$ is a given point in $\mathbb{R}^n$. This obviously implies that S_1 and S_2 are homothetic.

2b. Finally, we state that K_1 and K_2 are homothetic. Since this is already done in *2a* when $\dim \operatorname{rec} K_1 < n$, it remains to consider the case $\dim \operatorname{rec} K_1 = n$.

Let $f(x) = v + \mu x$, $\mu > 0$, be a homothety such that $f(S_1) = S_2$. Put $K_1' = v + \mu K_1$. Applying f to both parts of (6), we obtain

$$\operatorname{bd} K_1' \setminus (v - \operatorname{int} \operatorname{rec} K_1) = S_2 \setminus (v - \operatorname{int} \operatorname{rec} K_1). \tag{8}$$

(i) If $v = 0$, then $S_2 = \mu S_1$ and $K_1' = \mu K_1$. Then (7) and (8) imply that

$$\operatorname{bd} K_2 \setminus (-\operatorname{int} \operatorname{rec} K_1) = \operatorname{bd} K_1' \setminus (-\operatorname{int} \operatorname{rec} K_1).$$

Now by Lemma 3.1, $K_2 = K_1' = \mu K_1$.

(ii) Let $v \neq 0$. We are going to prove that $\operatorname{bd} K_1 = S_1$ and $\operatorname{bd} K_2 = S_2$. Assume, for contradiction, that $\operatorname{bd} K_1 \neq S_1$. Denote by Q_i the part of $\operatorname{bd} K_i$ that does not lie in S_i, and let C_i be the smallest convex cone with apex 0 that contains Q_i, $i = 1, 2$.

We claim that $C_1 = C_2$. Clearly, $\operatorname{int} C_1 \neq \varnothing$ (since otherwise $\operatorname{bd} K_1 = S_1$). Choose any 2-dimensional subspace L that intersects $\operatorname{int} C_1$. Then $L \cap \operatorname{bd} K_1$ is not a convex quadric curve, and the homotheticity of $L \cap \operatorname{bd} K_1$ and $L \cap \operatorname{bd} K_2$ implies that $L \cap \operatorname{bd} K_2$ is also distinct from a convex quadric curve. This shows that L intersects $\operatorname{int} C_2$, implying the inclusion $C_1 \subset C_2$. Similarly, $C_2 \subset C_1$.

Since $\operatorname{rec} K_1$ is line-free, not both v and $-v$ can lie in $\operatorname{rec} K_1$. Let, for example, $v \notin -\operatorname{rec} K_1$. As easily seen, there is a 2-dimensional subspace M such that $M \cap \operatorname{int} C_1 \neq \varnothing$ and $(v+M) \cap Q_2 = \varnothing$. Then $M \cap \operatorname{bd} K_1$ is not a convex quadric curve while $(v+M) \cap \operatorname{bd} K_2$ is a convex quadric curve due to $(v+M) \cap \operatorname{bd} K_2 = (v+M) \cap S_2$. This is in contradiction with the hypothesis that $(v+M) \cap \operatorname{bd} K_2$ is homothetic to $M \cap \operatorname{bd} K_1$. Thus $\operatorname{bd} K_1 = S_1$ and $\operatorname{bd} K_2 = S_2$, implying the homotheticity of K_1 and K_2.

4. Proof of Theorem 1.9

2) $\Rightarrow$ *1)* Let $L \subset \mathbb{R}^n$ be a 2-dimensional plane through p that properly intersects K along a bounded set. If K is symmetric about p, then so is the section $L \cap \operatorname{bd} K$. Assume that K is not symmetric about p. Since $L - p$ is a subspace and the cones $\mathrm{rec}\, K$ and $-\mathrm{rec}\, K$ are symmetric about 0, (P3) implies that

$$(L - p) \cap [\mathrm{rec}\, K \cup (-\mathrm{rec}\, K)] = \{0\}.$$

Hence

$$L \cap [(p + \mathrm{rec}\, K) \cup (p - \mathrm{rec}\, K)] = \{p\}.$$

From here we obtain

$$\begin{aligned}(L \cap \mathrm{bd}\, K) \setminus \{p\} &= (L \cap \mathrm{bd}\, K) \setminus \big(L \cap [(p + \mathrm{rec}\, K) \cup (p - \mathrm{rec}\, K)]\big) \\ &= L \cap \big(\mathrm{bd}\, K \setminus [(p + \mathrm{rec}\, K) \cup (p - \mathrm{rec}\, K)]\big).\end{aligned}$$

By the hypothesis, the set $\mathrm{bd}\, K \setminus [(p + \mathrm{rec}\, K) \cup (p - \mathrm{rec}\, K)]$ lies in a convex quadric surface, S. Because $L \cap \mathrm{bd}\, K$ is the relative boundary of the 2-dimensional compact convex set $L \cap K$, we have $L \cap \mathrm{bd}\, K = L \cap S$, that is, $L \cap \mathrm{bd}\, K$ is a convex quadric curve itself. Being bounded, $L \cap \mathrm{bd}\, K$ should be an ellipse, again implying that $L \cap \mathrm{bd}\, K$ is centrally symmetric (not necessarily about p).

1) $\Rightarrow$ *2*) Without loss of generality, we put $p = 0$. The statement *1*) $\Rightarrow$ *2*) is established in [1, 12] for the case of convex bodies, when *2*) is equivalent to the condition "K is symmetric about p or $\mathrm{bd}\, K$ is an ellipsoid." Hence we may suppose that K is unbounded. Then $\mathrm{rec}\, K \neq \{0\}$ and there is a halfline h with apex 0 that lies in $\mathrm{rec}\, K$.

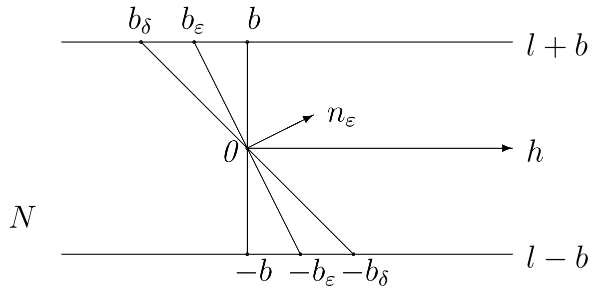
As in the proof of Theorem 1.6, we first exclude the trivial case when $K \subset p + \mathrm{rec}\, K$. In this case,

$$\mathrm{bd}\, K \setminus [(p + \mathrm{rec}\, K) \cup (p - \mathrm{rec}\, K)] = \varnothing.$$

On the other hand, the inclusion $K \subset p + \mathrm{rec}\, K$ obviously implies that all proper sections of K by 2-dimensional planes through p are unbounded, thus ensuring the trivial equivalence of conditions *1*) and *2*) of the theorem.

Our strategy is to show that all proper bounded sections of $\mathrm{bd}\, K$ by 2-dimensional subspaces are ellipses and then to apply Theorem 1.6. Assume, for contradiction, the existence of a 2-dimensional subspace $L \subset \mathbb{R}^n$ such that the section $L \cap \mathrm{bd}\, K$ is a bounded planar curve distinct from an ellipse. Then $L \cap \mathrm{rec}\, K = \{0\}$ and there is a hyperplane H containing L such that $H \cap \mathrm{rec}\, K = \{0\}$ (see (P4)), implying that $H \cap \mathrm{bd}\, K$ is bounded. Since $\dim (H \cap K) = n - 1 \geq 3$, and since every section of $H \cap K$ by a 2-dimensional subspace of H is centrally symmetric, $H \cap K$ is symmetric about 0 or $H \cap \mathrm{bd}\, K$ is an $(n-1)$-dimensional ellipsoid (see [1, 12]). Because $L \cap \mathrm{bd}\, K$ is not an ellipse, $H \cap \mathrm{bd}\, K$ cannot be an ellipsoid. Hence $H \cap K$ is symmetric about 0. Denote by l the line containing h.

1. First we claim that K lies in the both-way infinite cylinder $(H \cap K) + l$. Indeed, choose any 2-dimensional subspace N through l and consider the line segment $H \cap K \cap N$ (we observe that $N \not\subset H$ because of $l \not\subset H$). Since $H \cap K$ is symmetric about 0, we can write $H \cap K \cap N = [b, -b]$. We are going to show that $K \cap N$ is supported by the lines $l + b$ and $l - b$ (see the figure above). For any scalar $\varepsilon \in [0, \pi/2[$, denote by H_ε the hypersubspace of $\mathbb{R}^n$ whose unit normal n_ε lies in N and forms with h a positive angle of size ε according to the counterclockwise rotation about 0. Since $H \cap K$ is not an $(n-1)$-dimensional ellipsoid, the continuity argument implies the existence of a scalar



$\delta > 0$ such that the sections $H_\varepsilon \cap K$ are bounded and distinct from $(n-1)$-dimensional ellipsoids for all $\varepsilon \in]0,\delta[$. As above, the sections $H_\varepsilon \cap K$ are symmetric about 0. Hence the line segments $H_\varepsilon \cap K \cap N$ are centered at 0: $H_\varepsilon \cap K \cap N = [b_\varepsilon, -b_\varepsilon]$ for all $\varepsilon \in]0,\delta[$.

We state that for any $\varepsilon \in]0,\delta[$, the points b_ε and $-b_\varepsilon$ belong to the lines $l+b$ and $l-b$, respectively. Indeed, if b_ε were outside the closed slab of N between $l+b$ and $l-b$, then the inclusion $b_\varepsilon + h \subset K \cap N$ would imply that $b \in \operatorname{int} K$. Similarly, if b_ε were inside the open slab of N between $l+b$ and $l-b$, then, due to $h - b \subset K$, the point $-b_\varepsilon$ would be in $\operatorname{int} K$. Hence $b_\varepsilon \in l+b$, and, by symmetry, $-b_\varepsilon \in l-b$.

The argument above implies that both halflines $h+b$ and $h-b$ are in the relative boundary of $K \cap N$. Indeed, if for example, $h+b$ contained a point $x \in \operatorname{rint}(K \cap N)$, then $b \in]b_\varepsilon, x[\subset \operatorname{rint}(K \cap N)$, contradicting the choice of b. As a result, both lines $l+b$ and $l-b$ support $K \cap N$.

Since the subspace N through l was chosen arbitrarily, we conclude that $K \subset (H \cap K) + l$.

2. The inclusion $K \subset (H \cap K) + l$ implies that $\operatorname{rec} K = h$. Hence any hypersubspace transverse to h intersects K along a bounded set. Now, fixing a 2-dimensional subspace N through l, we continuously rotate the hypersubspace H_ε about 0 from the initial position $\varepsilon = 0$ until its unit normal vector $n_\varepsilon \in N$ reaches the limit position n_λ, $0 < \lambda < \pi/2$, where the section $H_\lambda \cap K$ is still symmetric about 0 but any further small rotation of H_ε results in a section $H_\varepsilon \cap K$, $\varepsilon > \lambda$, that is not symmetric about 0 (such a value λ exists because the line $l+b$ is not entirely in K). As above, all sections $H_\varepsilon \cap K$, $\varepsilon \in]\lambda, \pi/2[$, are $(n-1)$-dimensional ellipsoids. By the choice of λ,

$$H_\lambda \cap K = H_\lambda \cap \big((H \cap K) + l\big). \tag{9}$$

Since $H \cap \operatorname{bd} K$ is not an $(n-1)$-dimensional ellipsoid, the cylindric surface $(H \cap \operatorname{bd} K) + l$ is not ellipsoidal itself, and (9) implies that $H_\lambda \cap \operatorname{bd} K$ is also distinct from an $(n-1)$-dimensional ellipsoid. On the other hand, $H_\lambda \cap \operatorname{bd} K$ should be an $(n-1)$-dimensional ellipsoid as the limit position of $(n-1)$-dimensional ellipsoids $H_\varepsilon \cap \operatorname{bd} K$ when $\varepsilon \to \lambda^+$.

The obtained contradiction shows that all proper bounded sections of $\operatorname{bd} K$ by 2-dimensional planes through p are ellipses. By Theorem 1.6, $\operatorname{bd} K \setminus [(p + \operatorname{rec} K) \cup (p - \operatorname{rec} K)]$ lies in a convex quadric surface.

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