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A New Approach to Newton's Aerodynamic Problem*

Vladimir Janković
Mathematical Faculty, Studentski trg 16, 11000 Belgrade, Serbia
vjankovic@poincare.matf.bg.ac.yu

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We consider the following Generalized Aerodynamic Problem: *Find a convex body of the given length and of the given maximal section, which has the minimal resistance when moving in the rarefied medium.* We prove the existence of a solution for this problem.

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1. Introduction

In his book "Philosophiae Naturalis Principia Mathematica", in the part entitled "On moving of fluids and on resistance of thrown bodies", Newton studied the problem of resistance of various bodies moving in a "rarefied" medium. Among the other, he wrote the following:

Quod si figura $DNFG$ ejusmodi sit ut, si ab ejus puncto quovis N ad axem AB demittatur perpendiculum NM, et dicatur recta GP quae parallela sit rectae figuram tangenti in N, et axem productam sicet in P, fuerit MN ad GP ut GP^{cub} ad $4BP\times GB^q$, solidum quod figurae hujus revolutione circa axem AB describitur resistetur minime omnium ejusdem longitudinis & latitudinis.

The translation to English of this Latin text is the following one:

If the curve $DNFG$ satisfies the condition: if, from an arbitrary point N, we construct the orthogonal line to the axis AB and construct the line GP parallel to the tangent of the curve at the point N, which intersects the axis at the point P, then $MN : GP = GP^3 : (4BP \times GB^2)$; then the body obtained by the rotation of this curve around the axis AB will be submitted to the minimum resistance in the rarefied medium, compared to other bodies of the same length and width.

Suppose that the rotating body is obtained by the rotation around the axis x of the graph of the function $x(y)$, $y \in [0, a]$, and suppose that the body is moving in the sense opposite to the sense of axis x. If the space is filled with a rarefied fluid, then the

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resistance that the body encounters when moving is given by the formula

$$(R) \qquad F = 4\pi\rho v^2 \int_0^a \frac{dt}{1+\dot{x}(t)^2},$$

where ρ is the density of the fluid and v is the body's speed. The above formula can be derived by assuming that the fluid consists of identical balls, uniformly distributed in the space, and which are reflected from the body according to the law of reflection for the elastic bodies.

From the formula above one could get the impression that Newton's problem can be formalized as the problem of the calculus of variations in the following way:

$$(P) \qquad \int_0^a \frac{dt}{1+\dot{x}(t)^2} \to inf; \quad x(0)=0,\, x(a)=b.$$

It is not difficult to see that the value of this problem is equal to zero. If we put a saw-like function in the functional above, then its value will be positive and small, and it becomes smaller as the saw's teeth become sharper. This fact made some mathematicians to believe that the great scientist has made a mistake. For example, L. C. Young [4] thought that Newton's resistance law is physically absurd. V. M. Tihomirov [2] noted that the formalization (P) of the Aerodynamic Problem was not given by Newton, but by his successors. In the derivation of the law (R) it is assumed that every molecule of the fluid hits the body only once; therefore it is necessary to introduce the constraint that the function x, whose graph generates the rotating body, is monotone increasing. In this way Tihomirov obtained a new formalization of the Aerodynamic Problem (see [1], 1.2.3):

$$(T) \qquad \int_0^a \frac{tdt}{1+\dot{x}(t)^2} \to inf; \quad x(0)=0,\, x(a)=b,\, \dot{x}(t)\geq 0.$$

This is an optimal control problem. The solution of this problem is the function given by the following parametric equations (see [1], 1.6.2):

$$t(u) = \frac{\tau}{4}(u^3+2u+1/u),$$
$$x(u) = \frac{\tau}{16}(3u^4+4u^2-4\ln u-7),$$
$$u \geq 1.$$

It is interesting, but not surprising, that this curve coincides with the one described by Newton.

2. Generalized Aerodynamic Problem

By careful analyzing, one can see that it is not sufficient to assume that the function x, whose graph generates the rotating body, is monotone increasing, but it is necessary to impose the convexity condition as well. Only the convexity condition guarantees that every molecule of the fluid will hit the body at most once. The solution of the problem (T) is a convex function, and the same function will be the solution of the problem obtained from the problem (T) by adding the condition of convexity of the function x. This is why it is not easy to notice that it is necessary to impose the convexity condition too.

Newton's problem can be generalized to the bodies which are not rotating. It makes sense to study the problem to determine a convex body of the given length and of the given maximal cross section, which is submitted to the minimal resistance when moving in the rarefied medium. Mathematical formalization of this problem is the following one: Given a bounded open convex set $D \subseteq R^2$, find a convex function $z(x,y) : D \to R$, $0 \le z(x,y) \le b$, for which the body given by

$$\{(x,y,z) \,|\, (x,y) \in D, z(x,y) \le z \le b\},$$

is submitted to the minimal resistance when moving in the rarefied fluid in the sense opposite to the sense of the axis z. One can prove by a standard procedure that the resistance force is given by the formula

$$\vec{F} = 2\rho v^2 \iint_D \frac{z_x(x,y)\vec{i} + z_y(x,y)\vec{j} - \vec{k}}{1 + z_x(x,y)^2 + z_y(x,y)^2} dxdy.$$

Partial derivatives $z_x(x,y)$ and $z_y(x,y)$ exist almost everywhere on D and are measurable (see [2], 44, Theorems D and E). The integrant is measurable and bounded. Therefore this integral exists in the Lebesgue sense.

Several interesting questions can be made in connection with the Generalized Aerodynamic Problem. The first question is whether there exists a solution of this problem. The existence problem is solved by the following theorem.

Theorem. *The Generalized Aerodynamic Problem has a solution.*

The proof of this theorem is based on the following two lemmas. Lemma 2.1 is a variant of Theorem 10.9 of [3]. Lemma 2.2 can be obtained from Theorem 24.5 of [3]. Both lemmas are interesting in themselves and they both have short proofs, which we present here for completeness sake.

Lemma 2.1. *Let D be an open convex set in R^n, and let (f_k) be an uniformly bounded sequence of convex functions defined on D. There exists a subsequence of (f_k) which converges at each point of D.*

Lemma 2.2. *Let the sequence of convex functions (f_k), defined on the open interval I of the real line, converges to the function f. If each of the functions f_k, $k \in N$, and the function f are differentiable at the point $x \in I$, then $\lim_{k\to\infty} f_k'(x) = f'(x)$.*

Proof of Lemma 2.1. Let X be a countable everywhere dense set in D. Using the diagonal procedure we can prove that the sequence of functions (f_k) contains a subsequence which converges in every point from X. For the sake of simplicity we can suppose that actually the sequence (f_k) converges in every point of the set X. Let $a \in D$. The point a has a compact neighborhood U which is contained in D. All functions from the sequence (f_k) satisfy Lipschitz condition on U with the same Lipschitz constant L (see [2], Section 41, Theorem B). Let $\epsilon > 0$. There exists $x \in X \cap U$, such that $d(x,a) < \epsilon/3L$. Since $(f_k(x))$ is a convergent sequence, there exists $m \in N$ such that $d(f_k(x), f_l(x)) < \epsilon/3$, for $k, l \ge m$. If $k, l \ge m$, we have that

$$\begin{aligned} d(f_k(a), f_l(a)) &\le d(f_k(a), f_k(x)) + d(f_k(x), f_l(x)) + d(f_l(x), f_l(a)) \\ &\le Ld(a,x) + d(f_k(x), f_l(x)) + Ld(x,a) \le L\epsilon/3L + \epsilon/3 + L\epsilon/3L = \epsilon. \end{aligned}$$

Therefore $(f_k(a))$ is a Cauchy sequence and hence it is convergent. □

Proof of Lemma 2.2. Let all functions f_k, $k \in N$, and the function f be differentiable at the point $x \in I$. Further, let $y, z \in I$, $y < x < z$. The following inequalities hold

$$\frac{f_k(y) - f_k(x)}{y - x} \leq f_k'(x) \leq \frac{f_k(z) - f_k(x)}{z - x}.$$

If $k \to \infty$, then we have

$$\frac{f(y) - f(x)}{y - x} \leq \liminf_{k\to\infty} f_k'(x) \leq \limsup_{k\to\infty} f_k'(x) \leq \frac{f(z) - f(x)}{z - x}.$$

If $y \to x-$ and $z \to x+$, we obtain

$$f'(x) \leq \liminf_{k\to\infty} f_k'(x) \leq \limsup_{k\to\infty} f_k'(x) \leq f'(x).$$

It follows that

$$\lim_{k\to\infty} f_k'(x) = f'(x).$$

□

Proof of the Theorem. Let (z_k) be a minimizing sequence of functions. According to Lemma 2.1, this sequence has a convergent subsequence. We can suppose that the sequence (z_k) is convergent on the set D. Denote by z the function which is the limit of the sequence of functions (z_k) on the set D. The function z is also convex on the set D. Since every convex function on the open convex set is differentiable at almost every point of this set (see [2], Section 44, Theorem D), then at almost every point of the set D all functions z_k, $k \in N$, and the function z are differentiable. Let (x, y) be such a point. According to Lemma 2.2 we have that

$$\lim_{k\to\infty} \frac{\partial z_k(x,y)}{\partial x} = \frac{\partial z(x,y)}{\partial x}, \qquad \lim_{k\to\infty} \frac{\partial z_k(x,y)}{\partial y} = \frac{\partial z(x,y)}{\partial y}.$$

According to Dominated Convergence Theorem we obtain that

$$\lim_{k\to\infty} |\vec{F}(z_k)| = |\vec{F}(z)|.$$

Therefore the function z is the solution of the Generalized Aerodynamic Problem. □

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