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Locally Convex Lattice Cones

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We investigate lattice structures on locally convex cones, that is ordered cones that carry a locally convex topology. Examples include the extended reals $\overline{\mathbb{R}}$, cones of $\overline{\mathbb{R}}$-valued functions and cones of convex subsets of a locally convex vector space. The case of order completeness, where bounded below sets have suprema and infima, is of particular interest. It leads to the notion of order convergence and the introduction of the order topology and its comparison to the given topology of a completely ordered locally convex cone. The use of zero components of a given element allows a more subtle conceptualization of the cancellation law.

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1. Introduction

Some important mathematical settings, while close to the structure of vector spaces do not allow subtraction of their elements or multiplication by negative scalars. Examples are certain classes of functions that may take infinite values or are characterized through inequalities rather than equalities. They arise naturally in integration and in potential theory. Likewise, families of convex subsets of vector spaces which are of interest in various contexts, do not form vector spaces. If the cancellation law fails, domains of this type can not be embedded into larger vector spaces in order to apply results and techniques from classical functional analysis. The theory of locally convex cones, as developed in [3], uses order theoretical concepts to introduce a topological structure on ordered cones. In Sections 2, 3 and 4 of this paper we shall review some of the main concepts of this approach. We refer to [3] for details and proofs. In the subsequent sections we introduce lattice cones, complete lattice cones, zero components and the cancelation law, order convergence and the order topology. We demonstrate how every locally convex cone can be canonically embedded into a locally convex complete lattice cone.

2. Locally Convex Cones

A *cone* is a set $\mathcal{P}$ endowed with an addition $(a, b) \mapsto a + b$ and a scalar multiplication $(\alpha, a) \mapsto \alpha a$ for real numbers $\alpha \geq 0$. The addition is supposed to be associative and commutative, and there is a neutral element $0 \in \mathcal{P}$. For the scalar multiplication the usual associative and distributive properties hold, that is $\alpha(\beta a) = (\alpha\beta)a$, $(\alpha + \beta)a = \alpha a + \beta a$,

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$\alpha(a+b) = \alpha a + \alpha b$, $1a = a$ and $0a = 0$ for all $a, b \in \mathcal{P}$ and $\alpha, \beta \geq 0$. The *cancellation law*, stating that $a+c = b+c$ implies $a = b$, however, is not required in general. It holds if and only if the cone $\mathcal{P}$ can be embedded into a real vector space.

An *ordered cone* $\mathcal{P}$ carries a reflexive transitive relation $\leq$ such that $a \leq b$ implies $a+c \leq b+c$ and $\alpha a \leq \alpha b$ for all $a, b, c \in \mathcal{P}$ and $\alpha \geq 0$. Equality on $\mathcal{P}$ is obviously such an order. Note that anti-symmetry is not required for the relation $\leq$.

The theory of locally convex cones as developed in [3] uses order theoretical concepts to introduce a quasiuniform topological structure on an ordered cone. In a first approach, the resulting topological neighborhoods themselves will be considered to be elements of the cone. In this vein, a *full locally convex cone* $(\mathcal{P}, \mathcal{V})$ is an ordered cone $\mathcal{P}$ that contains an *abstract neighborhood system* $\mathcal{V}$, that is a subset of positive elements which is directed downward, closed for addition and multiplication by strictly positive scalars. The elements v of $\mathcal{V}$ define *upper* resp. *lower neighborhoods* for the elements of $\mathcal{P}$ by

$$v(a) = \{\, b \in \mathcal{P} \mid b \leq a + v \,\} \quad \text{resp.} \quad (a)v = \{\, b \in \mathcal{P} \mid a \leq b + v \,\},$$

creating the *upper* resp. *lower topologies* on $\mathcal{P}$. Their common refinement is called the *symmetric topology.* All elements of $\mathcal{P}$ are supposed to be *bounded below*, that is for every $a \in \mathcal{P}$ and $v \in \mathcal{V}$ we have $0 \leq a + \lambda v$ for some $\lambda \geq 0$. They need however not be bounded above (see Section 3 below). The presence of unbounded elements represents the main difference between locally convex cones and locally convex vector spaces and accounts for much of the richness and subtlety of this setting.

Finally, a *locally convex cone*$(\mathcal{P}, \mathcal{V})$ is a subcone of a full locally convex cone not necessarily containing the abstract neighborhood system $\mathcal{V}$. Every locally convex ordered topological vector space is a locally convex cone in this sense, as it may be canonically embedded into a full locally convex cone (see Examples 2.1(c) below and I.2.7 in [3]). It is shown in Chapter I.5.2 of [3] how a convex quasiuniform structure on a cone can be used to construct a full locally convex cone which contains the given one as a subcone and induces the given uniform structure. This yields a second, equivalent approach to locally convex cones.

Examples 2.1. (a) In the extended real number system $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ we consider the usual order and algebraic operations, in particular $a + \infty = +\infty$ for all $a \in \overline{\mathbb{R}}$, $\alpha \cdot (+\infty) = +\infty$ for all $\alpha > 0$ and $0 \cdot (+\infty) = 0$. Endowed with the neighborhood system $\mathcal{V} = \{\varepsilon \in \mathbb{R} \mid \varepsilon > 0\}$, $\overline{\mathbb{R}}$ is a full locally convex cone. For $a \in \mathbb{R}$ the intervals $(-\infty, a+\varepsilon]$ are the upper and the intervals $[a-\varepsilon, +\infty]$ are the lower neighborhoods, while for $a = +\infty$ the entire cone $\overline{\mathbb{R}}$ is the only upper neighborhood, and $\{+\infty\}$ is open in the lower topology. The symmetric topology is the usual topology on $\mathbb{R}$ with $+\infty$ as an isolated point. It is finer than the usual topology of $\overline{\mathbb{R}}$ where the intervals $[a, +\infty]$ for $a \in \mathbb{R}$ are the neighborhoods of $+\infty$.

(b) For the subcone $\overline{\mathbb{R}}_+ = \{a \in \overline{\mathbb{R}} \mid a \geq 0\}$ of $\overline{\mathbb{R}}$ we may also consider the singleton neighborhood system $\mathcal{V} = \{0\}$. The elements of $\overline{\mathbb{R}}_+$ are obviously bounded below even with respect to the neighborhood $v = 0$, hence $\overline{\mathbb{R}}_+$ is a full locally convex cone. For $a \in \overline{\mathbb{R}}$ the intervals $(-\infty, a]$ and $[a, +\infty]$ are the only upper and lower neighborhoods, respectively. The symmetric topology is the discrete topology on $\overline{\mathbb{R}}_+$.

(c) Let $(E, \leq)$ be a locally convex ordered topological vector space. Recall that equality is an order relation, hence this example will cover locally convex spaces in general. In order to interpret E as a locally convex cone we shall embed it into a larger full cone. This is done in a canonical way: Let $\mathcal{P}$ be the cone of all non-empty convex subsets of E, endowed with the usual addition and multiplication of sets by non-negative scalars, that is $\alpha A = \{\alpha a \mid a \in A\}$ and $A + B = \{a + b \mid a \in A \text{ and } b \in B\}$ for $A, B \in \mathcal{P}$ and $\alpha \geq 0$. We define the order on $\mathcal{P}$ by

$$A \leq B \quad \text{if } A \subset B + E_-,$$

where $E_- = \{x \in E \mid x \leq 0\}$ is the negative cone in E. The requirements for an ordered cone are easily checked. The neighborhood system in $\mathcal{P}$ is given by a basis $\mathcal{V} \subset \mathcal{P}$ of closed, convex, balanced and order convex neighborhoods of the origin in E. We observe that for every $A \in \mathcal{P}$ and $V \in \mathcal{V}$ there is $\rho > 0$ such that $\rho V \cap A \neq \emptyset$. This yields $0 \in A + \rho V$. Therefore $\{0\} \leq A + \rho V$, and every element $A \in \mathcal{P}$ is indeed bounded below. Thus $(\mathcal{P}, \mathcal{V})$ is a full locally convex cone. Via the embedding $x \mapsto \{x\} : E \to \mathcal{P}$ the space E itself is a subcone of $\mathcal{P}$. This embedding preserves the order structure of E, and on its image the symmetric topology of $\mathcal{P}$ coincides with the given vector space topology of E. Thus E is indeed a locally convex cone, but not a full cone. Other subcones of $\mathcal{P}$ that merit further investigation are those of all closed, closed and bounded, or compact convex sets in $\mathcal{P}$, respectively. Details on these and further related examples may be found in [3], I.1.7, I.2.7 and I.2.8.

(d) Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone, X a set and let $\mathcal{F}(X, \mathcal{P})$ be the cone of all $\mathcal{P}$-valued functions on X, endowed with the pointwise operations and order. If $\widetilde{\mathcal{P}}$ is a full cone containing both $\mathcal{P}$ and $\mathcal{V}$, then we may identify the elements $v \in \mathcal{V}$ with the constant functions $x \mapsto v$ for all $x \in X$, hence $\mathcal{V}$ is a subset and a neighborhood system for $\mathcal{F}(X, \widetilde{\mathcal{P}})$. A function $f \in \mathcal{F}(X, \widetilde{\mathcal{P}})$ is uniformly bounded below, if for every $v \in \mathcal{V}$ there is $\rho \geq 0$ such that $0 \leq f + \rho v$. These functions form a full locally convex cone $\left(\mathcal{F}_b(X, \widetilde{\mathcal{P}}), \mathcal{V}\right)$, carrying the topology of uniform convergence. As a subcone, $(\mathcal{F}_b(X, \mathcal{P}), \mathcal{V})$ is a locally convex cone. Alternatively, a more general neighborhood system $\mathcal{V}_{\mathcal{Y}}$ for $\mathcal{F}(X, \mathcal{P})$ may be created using a suitable family $\mathcal{Y}$ of subsets Y of X and the neighborhoods v_Y for $v \in \mathcal{V}$ and $Y \in \mathcal{Y}$, defined for functions $f, g \in \mathcal{F}(X, \mathcal{P})$ as $f \leq g + v_Y$ if $f(x) \leq g(x) + v$ for all $x \in Y$. In this case we consider the subcone $\mathcal{F}_{b_{\mathcal{Y}}}(X, \mathcal{P})$ of all functions in $\mathcal{F}(X, \mathcal{P})$ that are uniformly bounded below on the sets in $\mathcal{Y}$. Together with the neighborhood system $\mathcal{V}_{\mathcal{Y}}$, it forms a locally convex cone. $\left(\mathcal{F}_{b_{\mathcal{Y}}}(X, \mathcal{P}), \mathcal{V}_{\mathcal{Y}}\right)$ carries the topology of uniform convergence on the sets in $\mathcal{Y}$.

(e) For $x \in \overline{\mathbb{R}}$ denote $x^+ = \max\{x, 0\}$ and $x^- = -\min\{x, 0\}$. For $1 \leq p \leq +\infty$ and a sequence $(x_i)_{i \in \mathbb{N}}$ in $\overline{\mathbb{R}}$ let $\|(x_i)\|_p$ denote the usual l^p norm, that is $\|(x_i)\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{(1/p)} \in \overline{\mathbb{R}}$ for $p < +\infty$ and $\|(x_i)\|_\infty = \sup\{|x_i| \mid i \in \mathbb{N}\} \in \overline{\mathbb{R}}$. Now let $\mathcal{C}^p$ be the cone of all sequences $(x_i)_{i \in \mathbb{N}}$ in $\overline{\mathbb{R}}$ such that $\|(x_i^-)\|_p \leq +\infty$. We use the pointwise order in $\mathcal{C}^p$ and the neighborhood system $\mathcal{V}_p = \{\rho v_p \mid \rho > 0\}$, where

$$(x_i)_{i \in \mathbb{N}} \leq (y_i)_{i \in \mathbb{N}} + \rho v_p$$

means that $\|(x_i - y_i)^+\|_p \leq \rho$. (In this expression the l^p norm is evaluated only over the indexes $i \in \mathbb{N}$ for which $y_i < +\infty$.) It can be easily verified that $(\mathcal{C}^p, \mathcal{V}_p)$ is a locally convex cone. In fact $(\mathcal{C}^p, \mathcal{V}_p)$ can be embedded into a full cone following a procedure analogous to that in 2.1(c). The case for $p = +\infty$ is of course already covered by Example 2.1(d).

Linear operators 2.2. For cones $\mathcal{P}$ and $\mathcal{Q}$ a mapping $T : \mathcal{P} \to \mathcal{Q}$ is called a *linear operator* if $T(a+b) = T(a) + T(b)$ and $T(\alpha a) = \alpha T(a)$ holds for all $a, b \in \mathcal{P}$ and $\alpha \geq 0$. If both $\mathcal{P}$ and $\mathcal{Q}$ are ordered, then T is called *monotone,* if $a \leq b$ implies $T(a) \leq T(b)$. If both $(\mathcal{P}, \mathcal{V})$ and $(\mathcal{Q}, \mathcal{W})$ are locally convex cones, the operator T is called *(uniformly) continuous* if for every $w \in \mathcal{W}$ one can find $v \in \mathcal{V}$ such that $T(a) \leq T(b) + w$ whenever $a \leq b + v$ for $a, b \in \mathcal{P}$. It is immediate from the definition that uniform continuity implies and combines continuity for the operator $T : \mathcal{P} \to \mathcal{Q}$ with respect to the upper, lower and symmetric topologies on $\mathcal{P}$ and $\mathcal{Q}$, respectively.

A *linear functional* on $\mathcal{P}$ is a linear operator $\mu : \mathcal{P} \to \overline{\mathbb{R}}$. The *dual cone* $\mathcal{P}^*$ of a locally convex cone $(\mathcal{P}, \mathcal{V})$ consists of all continuous linear functionals on $\mathcal{P}$ and is the union of all *polars* v° of neighborhoods $v \in \mathcal{V}$, where $\mu \in v^\circ$ means that $\mu(a) \leq \mu(b) + 1$, whenever $a \leq b + v$ for $a, b \in \mathcal{P}$. Continuity implies that a linear functional μ is monotone, and for a full cone $\mathcal{P}$ it requires just that $\mu(v) \leq 1$ holds for some $v \in \mathcal{V}$ in addition. We endow $\mathcal{P}^*$ with the topology $w(\mathcal{P}^*, \mathcal{P})$ of pointwise convergence on the elements of $\mathcal{P}$, considered as functions on $\mathcal{P}^*$ with values in $\overline{\mathbb{R}}$ with its usual topology. As in locally convex topological vector spaces, the polar v° of a neighborhood $v \in \mathcal{V}$ is seen to be $w(\mathcal{P}^*, \mathcal{P})$-compact and convex ([3], Theorem II.2.4).

Examples 2.3. Revisiting the preceding Examples 2.1, we observe that the dual cone $\overline{\mathbb{R}}^*$ of $\overline{\mathbb{R}}$ (see 2.1(a)) consists of all positive reals (via the usual multiplication), and the singular functional $\bar{0}$ such that $\bar{0}(a) = 0$ for all $a \in \mathbb{R}$ and $\bar{0}(+\infty) = +\infty$. Likewise, in 2.1(b), the continuous linear functionals on $\overline{\mathbb{R}}_+$, endowed with the neighborhood system $\mathcal{V} = \{0\}$, are the positive reals together with $\bar{0}$, but further include the element $+\infty$, acting as $+\infty(0) = 0$ and $+\infty(a) = +\infty$ for all $0 \neq a \in \overline{\mathbb{R}}_+$. This functional is obviously contained in the polar of the neighborhood $0 \in \mathcal{V}$. In 2.1(c) and (d) on the other hand, due to the generality of the settings, a complete description for the respective dual cones is not immediately available. We may, however, identify some of their elements: In 2.1(c), let μ be a continuous monotone linear function on the locally convex ordered topological vector space $(E, \leq)$. Then the mapping

$$A \mapsto \sup\{\mu(a) \mid a \in A\} : Conv(E) \to \overline{\mathbb{R}}$$

is seen to be an element of $Conv(E)^*$. In 2.1(d), if $\mu \in \mathcal{P}^*$ and if $x \in Y$ for some $Y \in \mathcal{Y}$, then the mapping $\mu_x : \mathcal{F}_{b_\mathcal{Y}}(X, \mathcal{P}) \to \overline{\mathbb{R}}$ such that

$$\mu_x(f) = \mu\left(f(x)\right) \quad \text{for all } f \in \mathcal{F}_{b_\mathcal{Y}}(X, \mathcal{P})$$

is a continuous linear functional on $\mathcal{F}_{b_\mathcal{Y}}(X, \mathcal{P})$; more precisely: If $\mu \in v^\circ$ for $v \in \mathcal{V}$ and $x \in Y$ for $Y \in \mathcal{Y}$, then $\mu_x \in v_Y^\circ$. In 2.1(e) for $p < +\infty$ the dual cone of $\mathcal{C}^p$ consists of all sequences $(y_i)_{i \in \mathbb{N}}$ such that $y_i \geq 0$ for all $i \in \mathbb{N}$ and $\|(y_i)\|_q < +\infty$, where q is the conjugate index of p.

Hahn-Banach type extension and separation theorems for linear functionals are most important for the development of a powerful duality theory for locally convex cones. We shall mention a few results from [3] and [6]. A *sublinear functional* on a cone $\mathcal{P}$ is a mapping $p : \mathcal{P} \to \overline{\mathbb{R}}$ such that $p(\alpha a) = \alpha p(a)$ and $p(a + b) \leq p(a) + p(b)$ hold for all $a, b \in \mathcal{P}$ and $\alpha \geq 0$. Likewise, an *extended superlinear functional* on $\mathcal{P}$ is a mapping $q : \mathcal{P} \to \overline{\overline{\mathbb{R}}} = \mathbb{R} \cup \{+\infty, -\infty\}$ such that $q(\alpha a) = \alpha q(a)$ and $q(a + b) \geq q(a) + q(b)$ hold for all $a, b \in \mathcal{P}$ and $\alpha \geq 0$. (We set $\alpha + (-\infty) = -\infty$ for all $\alpha \in \overline{\mathbb{R}}$, $\alpha \cdot (-\infty) = -\infty$ for all $\alpha > 0$ and $0 \cdot (-\infty) = 0$ in this context.) We cite Theorem 3.1 from [6]:

Theorem 2.4 (Sandwich Theorem). *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone, and let $v \in \mathcal{V}$. For a sublinear functional $p : \mathcal{P} \to \overline{\mathbb{R}}$ and an extended superlinear functional $q : \mathcal{P} \to \overline{\mathbb{R}}$ there exists a linear functional $\mu \in v^\circ$ such that $q \leq \mu \leq p$ if and only if $q(a) \leq p(b) + 1$ holds whenever $a \leq b + v$ for $a, b \in \mathcal{P}$.*

This leads to a variety of extension and separation results, the most general ones being Theorems 4.1 and 4.4 in [6]. We shall only mention the following simplified version of 4.1 in [6] (Theorem II.2.9 in [3]).

Corollary 2.5. *Let $(\mathcal{N}, \mathcal{V})$ be a subcone of the locally convex cone $(\mathcal{P}, \mathcal{V})$. Every continuous linear functional on $\mathcal{N}$ can be extended to a continuous linear functional on $\mathcal{P}$; more precisely: For every $\mu \in v^\circ_{\mathcal{N}}$ there is $\tilde{\mu} \in v^\circ_{\mathcal{P}}$ such that $\tilde{\mu}$ coincides with μ on $\mathcal{N}$.*

3. Weak Preorder and the Relative Topology

In addition to the given order $\leq$ on a locally convex cone, one also considers the *weak preorder* $\preccurlyeq$ (see [7]) which is slightly weaker then the given order and defined for $a, b \in \mathcal{P}$ by

$$a \preccurlyeq b \quad \text{if } a \leq \gamma b + \varepsilon v$$

for all $v \in \mathcal{V}$ and $\varepsilon > 0$ with some $1 \leq \gamma \leq 1 + \varepsilon$. This order represents a closure of the given order with respect to the linear and topological structures of $\mathcal{P}$. It is obviously coarser than the given order, that is $a \leq b$ implies $a \preccurlyeq b$ for $a, b \in \mathcal{P}$. In the preceding Examples 2.1(a) and (b), however, both orders coincide. In 2.1(c), on the other hand, we have $A \preccurlyeq B$ if $A \subset \overline{B + E_-}$, the topological closure in E of the set $B + E_-$. In Example 2.1(d), since its order was defined pointwise, the locally convex cone $\big(\mathcal{F}_{b_\mathcal{Y}}(X, \mathcal{P}), \mathcal{V}_\mathcal{Y}\big)$ carries the weak preorder whenever $\mathcal{P}$ does.

The weak preorder on $\mathcal{P}$ is again compatible with the algebraic operations, as Lemma 3.3 below will imply. It may also be used in a full cone containing $\mathcal{P}$ and $\mathcal{V}$. Consequently, the respective relation involving the neighborhoods in $\mathcal{V}$ is defined for elements $a, b \in \mathcal{P}$ and $v \in \mathcal{V}$ as

$$a \preccurlyeq b + v \quad \text{if } a \leq \gamma(b + v) + \varepsilon u$$

for all $u \in \mathcal{V}$ and $\varepsilon > 0$ with some $1 \leq \gamma \leq 1 + \varepsilon$. Endowed with the weak preorder $(\mathcal{P}, \mathcal{V})$ forms again a locally convex cone. For details we refer to [7]. We omit the easy proof of the following:

Lemma 3.1. *Let $a, b \in \mathcal{P}$. Then $a \preccurlyeq b$ if and only if $a \preccurlyeq \gamma b + \varepsilon v$ for all $v \in \mathcal{V}$ and $\varepsilon > 0$ with some $1 \leq \gamma \leq 1 + \varepsilon$.*

Thus, if the locally convex cone $(\mathcal{P}, \mathcal{V})$ is endowed with the weak preorder, the subsequently derived (second) weak preorder coincides with the given (first) one.

Theorems 3.1 and 3.2 in [7] state that the weak preorder and neighborhoods in a locally convex cone $\mathcal{P}$ are entirely determined by its dual cone $\mathcal{P}^*$, that is $a \preccurlyeq b$ holds for $a, b \in \mathcal{P}$ if and only if $\mu(a) \leq \mu(b)$ for all $\mu \in \mathcal{P}^*$, and $a \preccurlyeq b + v$ holds for $a, b \in \mathcal{P}$ and a neighborhood $v \in \mathcal{V}$, if and only if $\mu(a) \leq \mu(b) + 1$ for all $\mu \in v^\circ$.

It is evident that for a linear operator T between locally convex cones $(\mathcal{P},\mathcal{V})$ and $(\mathcal{Q},\mathcal{W})$, continuity with respect to the given orders implies continuity and monotonicity with respect to the respective weak preorders on $\mathcal{P}$ and $\mathcal{Q}$, that is $a \preccurlyeq b+v$ implies $T(a) \preccurlyeq T(b)+w$ and $a \preccurlyeq b$ implies $T(a) \preccurlyeq T(b)$.

The weak preorder may be used to establish a representation for a locally convex cone $(\mathcal{P},\mathcal{V})$ as a cone of continuous $\overline{\mathbb{R}}$-valued functions on some topological space and as a cone of convex subsets of some locally convex topological vector space, respectively. We cite Theorem 4.1 from [7]:

Theorem 3.2. *Every locally convex cone $(\mathcal{P},\mathcal{V})$ can be embedded into*

(i) *a locally convex cone of continuous $\overline{\mathbb{R}}$-valued functions on some topological space X, endowed with the pointwise order and operations and the topology of uniform convergence on a family of compact subsets of X.*

(ii) *a locally convex cone of convex subsets of a locally convex topological vector space, endowed with the usual addition and multiplication by scalars, the set inclusion as order and the neighborhoods inherited from the vector space.*

These embeddings are linear and preserve the weak preorder and the neighborhoods of $(\mathcal{P},\mathcal{V})$.

While all elements of a locally convex cone are bounded below by definition, they need not to be bounded above. An element a of a locally convex cone $(\mathcal{P},\mathcal{V})$ is called *bounded (above)* if for every neighborhood $v \in \mathcal{V}$, there is $\lambda \geq 0$ such that $a \leq \lambda v$. The subset $\mathcal{B} \subset \mathcal{P}$ of all bounded elements is a subcone and even a face in $\mathcal{P}$ (see Proposition 4.1 below). All invertible elements of $\mathcal{P}$ are obviously bounded above, since their negatives are bounded below. Continuous linear functionals take only finite values on bounded elements. Similarly, a subset A of $\mathcal{P}$ is said to be *bounded above* (or *bounded below*) if for every $v \in \mathcal{V}$ there is $\lambda \geq 0$ such that $a \leq \lambda v$ (or $0 \leq a+\lambda v$) holds for all $a \in A$.

The presence of unbounded elements constitutes a significant difference between locally convex cones and locally convex topological vector spaces. It tends to make matters more interesting, but also considerably more complicated. If, for example, the element $a \in \mathcal{P}$ is not bounded, then the mapping $\alpha \mapsto \alpha a : [0,+\infty) \to \mathcal{P}$, is discontinuous if we consider the usual topology of $\mathbb{R}_+$ and any of the given (upper, lower or symmetric) topologies on $\mathcal{P}$, which therefore appear to be rather restrictive. We shall therefore introduce slightly coarser neighborhoods on $\mathcal{P}$ which take unbounded elements suitably into account. Given a neighborhood $v \in \mathcal{V}$ and $\varepsilon > 0$, we define the corresponding *upper* and *lower relative neighborhoods* $v_\varepsilon(a)$ and $(a)v_\varepsilon$ for an element $a \in \mathcal{P}$ by

$$\begin{aligned} v_\varepsilon(a) &= \{\, b \in \mathcal{P} \mid b \leq \gamma a + \varepsilon v \text{ for some } 1 \leq \gamma \leq 1+\varepsilon \,\} \\ (a)v_\varepsilon &= \{\, b \in \mathcal{P} \mid a \leq \gamma b + \varepsilon v \text{ for some } 1 \leq \gamma \leq 1+\varepsilon \,\}. \end{aligned}$$

Their intersection $v_\varepsilon^s(a) = v_\varepsilon(a) \cap (a)v_\varepsilon$ is the corresponding *symmetric relative neighborhood.* These are of course convex subsets of $\mathcal{P}$. Without proof, we cite Lemma 4.1 from [8]:

Lemma 3.3. *Let $a,b,c,a_i,b_i \in \mathcal{P}$, $v \in \mathcal{V}$, $\lambda \geq 0$ and $\varepsilon,\delta > 0$.*

(a) *If $a \in v_\varepsilon(b)$ and $b \in v_\delta(c)$, then $a \in v_{(\varepsilon+\delta+\varepsilon\delta)}(c)$.*

(b) *If $a \in v_\varepsilon(b)$ and $0 \leq b+\lambda v$, then $a \leq (1+\varepsilon)b + \varepsilon(1+\lambda)v$.*

(*c*) *If* $a \in v_\varepsilon(b)$ *and* $0 \le a + \lambda v$, *then* $a \le (1+\varepsilon)b + \varepsilon(1+\lambda+\varepsilon)v$ *and* $0 \le b + (\lambda+\varepsilon)v$.

(*d*) *If* $a_i \in v_\varepsilon(b_i)$ *and if* $0 \le b_i + \lambda v$ *for* $i = 1, \ldots, n$, *then* $(a_1 + \ldots + a_n) \in v_{\varepsilon n(1+\lambda)}(b_1 + \ldots + b_n)$.

Similar statements hold for the lower and for the symmetric relative neighborhoods. Part (*d*) shows compatibility of these neighborhoods with the addition. Compatibility with the multiplication by positive scalars is obvious. For elements $a, b \in \mathcal{P}$ the weak preorder on $\mathcal{P}$ as defined earlier in this section may be recovered as

$$a \preccurlyeq b \quad \text{if } a \in v_\varepsilon(b)$$

for all $v \in \mathcal{V}$ and $\varepsilon > 0$. Lemma 3.3(*d*) implies that this order is compatible with the algebraic operations in $\mathcal{P}$.

For varying $v \in \mathcal{V}$ and $\varepsilon > 0$ the neighborhoods $v_\varepsilon(\cdot)$, $(\cdot)v_\varepsilon$ and $v^s_\varepsilon(\cdot)$ create the *upper, lower* and *symmetric relative topologies* on $\mathcal{P}$, respectively. These topologies are obviously coarser than the corresponding given topologies, but coincide locally with the latter on bounded elements of $\mathcal{P}$, as for $a \le \lambda v$ we have

$$(\varepsilon v)(a) \subset v_\varepsilon(a) \subset (\rho v)(a) \qquad \text{and} \qquad (a)(\varepsilon v) \subset (a)v_\varepsilon(a) \subset (a)(\rho v).$$

with $\rho = (1+\lambda)\varepsilon$. However, while the relative neighborhoods form convex subsets of $\mathcal{P}$, they do not create a locally convex cone topology. Indeed, the sets $\big\{(a,b) \mid a \in v_\varepsilon(b)\big\}$ are not necessarily convex in $\mathcal{P}^2$, hence do not establish a convex semiuniform structure on $\mathcal{P}$. The symmetric relative topology is induced by the family of pseudometrics $\{d_v \mid v \in \mathcal{V}\}$, defined by

$$d_v(a,b) = \inf\big\{1, \sqrt{\varepsilon} \mid a \in v^s_\varepsilon(b)\big\}.$$

The required properties of pseudometrics (see Section 2.1 in [11]) are readily checked.

Remarks and Examples 3.4. (a) Let $\mathcal{P} = \overline{\mathbb{R}}$, endowed with the neighborhood system $\mathcal{V} = \{\varepsilon \in \mathbb{R} \mid \varepsilon > 0\}$ (see Example 2.1(a)). For the neighborhood $v = 1$ and $\varepsilon > 0$ the relative neighborhoods of an element $a \in \overline{\mathbb{R}}$ are

$$v_\varepsilon(a) = (-\infty,\ (1+\varepsilon)a + \varepsilon] \qquad \text{or} \qquad v_\varepsilon(a) = (-\infty,\ a+\varepsilon]$$

if $a \ge 0$ or if $a < 0$, respectively. The weak preorder therefore coincides with the given order of $\overline{\mathbb{R}}$. The lower relative neighborhoods are given by

$$(a)v_\varepsilon = \left[\tfrac{a-\varepsilon}{1+\varepsilon},\ +\infty\right] \qquad \text{or} \qquad (a)v_\varepsilon = [a-\varepsilon,\ +\infty]$$

if $a \ge \varepsilon$ or if $a < \varepsilon$, respectively. This yields

$$v^s_\varepsilon(a) = \left[\tfrac{a-\varepsilon}{1+\varepsilon},\ (1+\varepsilon)a+\varepsilon\right], \qquad v^s_\varepsilon(a) = [a-\varepsilon,\ (1+\varepsilon)a+\varepsilon],$$

or

$$v^s_\varepsilon(a) = [a-\varepsilon, a+\varepsilon]$$

if $a \ge \varepsilon$, if $0 \le a < \varepsilon$ or if $a < 0$, respectively. The upper, lower and symmetric relative topologies of $\overline{\mathbb{R}}$ therefore coincide with the corresponding given topologies. (see 1.1(a)). The symmetric relative topology, in particular, is the usual topology on $\mathbb{R}$ with $+\infty$ as an isolated point.

(b) Let $\mathcal{P} = \overline{\mathbb{R}}_+ = \{a \in \overline{\mathbb{R}} \mid a \geq 0\}$, endowed with the neighborhood system $\mathcal{V} = \{0\}$ (see Example 1.1(b)). For the only neighborhood $v = 0 \in \mathcal{V}$ and $\varepsilon > 0$ the relative neighborhoods of an element $a \in \overline{\mathbb{R}}_+$ are

$$v_\varepsilon(a) = [0,\ (1+\varepsilon)a]\,, \quad (a)v_\varepsilon = \left[\tfrac{a}{1+\varepsilon},\ +\infty\right] \quad \text{and} \quad v^s_\varepsilon(a) = \left[\tfrac{a}{1+\varepsilon},\ (1+\varepsilon)a\right].$$

The weak preorder therefore coincides with the given order of $\overline{\mathbb{R}}_+$, and the symmetric relative topology is the Euclidean topology on $(0,+\infty)$, but renders $0 \in \mathcal{P}$ and $+\infty \in \mathcal{P}$ as isolated points. Recall from Example 2.1(b) that the symmetric given topology on $\overline{\mathbb{R}}_+$, in contrast, is the discrete topology.

(c) In Example 2.1(d), that is the locally convex cone $(\mathcal{F}_b(X,\mathcal{P}),\mathcal{V})$ of $\mathcal{P}$-valued functions on a set X, the (upper, lower and symmetric) relative topologies are generally coarser than the given (upper, lower and symmetric) topologies of $\mathcal{F}_b(X,\mathcal{P})$. Indeed, let $X = \mathcal{P} = \mathbb{R}$. For the function $f(x) = x^2$ in $\mathcal{F}_b(X,\mathbb{R})$ and the neighborhood $v = 1 \in \mathbb{R}$ the upper neighborhood $v(f)$ consists of all functions $g \in \mathcal{F}_b(X,\mathbb{R})$ such that $g \leq f + 1$. Every relative upper neighborhood $v_\varepsilon(f)$ of f, on the other hand, contains the function $g(x) = (1+\varepsilon)x^2$ which is not contained in $v(f)$. Thus $v_\varepsilon(f) \not\subset v(f)$. A similar observation can be made for Examples 2.1(c).

(d) It is worthwhile to notice that a continuous linear operator T between two locally convex cones $(\mathcal{P},\mathcal{V})$ and $(\mathcal{Q},\mathcal{W})$ is also continuous if we endow both $\mathcal{P}$ and $\mathcal{Q}$ with either their respective upper, lower or symmetric relative topologies. This can be easily verified. A continuous linear functional μ on $\mathcal{P}$, in particular, is therefore also continuous mapping between $\mathcal{P}$ and $\overline{\mathbb{R}}$, if we endow $\mathcal{P}$ with either the upper, lower or symmetric relative v-topology and, correspondingly, $\overline{\mathbb{R}}$ with its given upper, lower or symmetric topology (see Part (a)).

(e) For a locally convex cone $(\mathcal{P},\mathcal{V})$ the mapping

$$(\alpha, a) \mapsto \alpha a \ :\ [0,+\infty) \times \mathcal{P} \to \mathcal{P},$$

is generally not continuous with respect to the given topologies of $\mathbb{R}$ and $\mathcal{P}$. However, it is easily verified that this mapping is continuous if we consider the respective symmetric relative topologies of $\overline{\mathbb{R}}_+$ (see 3.4(b)) and $\mathcal{P}$ instead.

(f) Let $(\mathcal{P},\mathcal{V})$ be a locally convex cone. For a subset A of $\mathcal{P}$ we denote by $\overline{A}^{(l)}$ its closure with respect to the lower relative topology of $\mathcal{P}$. We list the following observations:

(i) The set $\overline{A}^{(l)}$ consists of all elements $b \in \mathcal{P}$ such that for every $v \in \mathcal{V}$ and $\varepsilon > 0$ there is some $a \in A$ such that $b \in v_\varepsilon(a)$. Indeed, we have $b \in \overline{A}^{(l)}$ if and only if $(b)v_\varepsilon \cap A \neq \emptyset$ for all $v \in \mathcal{V}$ and $\varepsilon > 0$, that is if there is $a \in A$ such that $b \in v_\varepsilon(a)$. For a singleton set $\{a\}$ in particular we have $b \in \overline{\{a\}}^{(l)}$ if and only if $b \in v_\varepsilon(a)$ for all$v \in \mathcal{V}$ and $\varepsilon > 0$, that is $b \preccurlyeq a$. Thus $\overline{\{a\}}^{(l)} = \{b \in \mathcal{P} \mid b \preccurlyeq a\}$.

(ii) The set $\overline{A}^{(l)}$ is convex whenever A is convex. Indeed let $a, b \in \overline{A}^{(l)}$ and $c = \alpha a + (1-\alpha)b$ for some $0 \leq \alpha \leq 1$. Given $v \in \mathcal{V}$ and $\varepsilon > 0$ there is $\lambda \geq 0$ such that both $0 \leq a + \lambda v$ and $0 \leq b + \lambda v$. Choose $\delta > 0$ such that $\delta(1+\lambda+\delta) \leq \varepsilon$. Then $a \leq (1+\delta)a' + \varepsilon v$ and $b \leq (1+\delta)b' + \varepsilon v$ for some $a', b' \in A$ by (i) together with 4.1(c). This shows

$c \leq (1+\delta)(\alpha a' + (1-\alpha)b') + \varepsilon v$, hence $c \in v_\varepsilon(c')$, where $c' = \alpha a' + (1-\alpha)b' \in A$. Using (i) we infer that $c \in \overline{A}^{(l)}$.

(iii) Every subset A of $\mathcal{P}$ which is closed with respect to the lower topology is decreasing with respect to the weak preorder. Indeed, let $b \preccurlyeq a$ for some $b \in \mathcal{P}$ and $a \in A$. Then $b \in v_\varepsilon(a)$ for all $v \in \mathcal{V}$ and $\varepsilon > 0$. Thus $b \in \overline{A}^{(l)} = A$ by (i) as claimed.

Let $\mathcal{Q}$ be the family of all non-empty convex subsets of $\mathcal{P}$ which are closed with respect to the lower topology. If we use the standard multiplication for sets by non-negative scalars and a slightly modified addition, that is

$$A \oplus B = \overline{(A+B)}^{(l)} \quad \text{for } A, B \in \mathcal{Q},$$

then $\mathcal{Q}$ becomes a cone. Its neutral element is $\overline{\{0\}}^{(l)}$. We use the set inclusion as the order on $\mathcal{Q}$ and define neighborhoods corresponding to those in $\mathcal{P}$: We set

$$A \leq B \oplus v \quad \text{for } A, B \in \mathcal{Q} \text{ and } v \in \mathcal{V},$$

if for every $a \in A$ and $\varepsilon > 0$ there is $b \in B$ such that $a \leq \gamma b + (1+\varepsilon)v$ for some $1 \leq \gamma \leq 1+\varepsilon$. First we observe that for every $A \in \mathcal{Q}$ and a fixed element $a \in A$ $v \in \mathcal{V}$ there is $\lambda \geq 0$ such that $0 \leq a + \lambda v$. Since $\overline{\{0\}}^{(l)} = \{b \in \mathcal{P} \mid b \preccurlyeq 0\}$, this yields $\overline{\{0\}}^{(l)} \leq A \oplus (\lambda+1)v$. Indeed, for every $b \preccurlyeq 0$, we have $b \leq v$, hence $b \leq a + (\lambda+1)v$. Thus every element $A \in \mathcal{Q}$ is seen to be bounded below and $(\mathcal{Q}, \mathcal{V})$ satisfies the requirements for a locally convex cone. Next we observe that the weak preorder on $(\mathcal{Q}, \mathcal{V})$ coincides with the given order. Indeed, suppose that $A \preccurlyeq B$, and let $a \in A$. Given $v \in \mathcal{V}$ and $\varepsilon > 0$ we set $\delta = \min\{\varepsilon/3, 1\}$ and have $A \leq \gamma B \oplus \delta v$ for some $1 \leq \gamma \leq 1+\delta$. According to Lemma 3.1 there is $1 \leq \gamma' \leq 1+\delta$ such that $a \leq (\gamma'\gamma)b + (1+\delta)\delta v$ for some $b \in B$. Since $(1+\delta)\delta \leq \varepsilon$, this yields $a \leq (\gamma'\gamma)b + \varepsilon v$, and since $1 \leq \gamma\gamma' \leq (1+\delta)^2 \leq 1+\varepsilon$ we have $a \in v_\varepsilon(b)$ and infer from (i) that $a \in \overline{B}^{(l)} = B$, hence $A \leq B$. Therefore $A \preccurlyeq B$ holds if and only if $A \leq B$. A similar argument shows that $A \preccurlyeq B \oplus v$ holds for $A, B \in \mathcal{Q}$ and $v \in \mathcal{V}$ if and only if $A \leq B \oplus v$. An element $A \in \mathcal{Q}$ is bounded above in $\mathcal{Q}$ if for every $v \in \mathcal{V}$ there is $\lambda \geq 0$ such that $A \leq \lambda v$, that is $a \leq (\lambda+1)v$ holds for all $a \in A$.

(g) Similarly, for a subset A of a locally convex cone $(\mathcal{P}, \mathcal{V})$ let us denote by $\overline{A}^{(u)}$ its closure with respect to the upper relative topology of $\mathcal{P}$. In an analogous way as in (d) on can verify: (i) The set $\overline{A}^{(u)}$ consists of all elements $b \in \mathcal{P}$ such that for every $v \in \mathcal{V}$ and $\varepsilon > 0$ there is some $a \in A$ such that $b \in (a)v_\varepsilon$. For a singleton set $\{a\}$ in particular we have $\overline{\{a\}}^{(u)} = \{b \in \mathcal{P} \mid a \preccurlyeq b\}$. (ii) The set $\overline{A}^{(u)}$ is convex whenever A is convex. (iii) A set $A \subset \mathcal{P}$ is said to be *bounded below* if for every $v \in \mathcal{V}$ there is $\lambda \geq 0$ such that $0 \leq a + \lambda v$ for all $a \in A$. It is straightforward to verify that $\overline{A}^{(u)}$ is bounded below whenever A is bounded below. (iv) Every subset of $\mathcal{P}$ which is closed with respect to the upper relative topology is increasing with respect to the weak preorder.

Let $\mathcal{Q}$ be the family of all convex subsets of $\mathcal{P}$ which are closed with respect to the upper topology and bounded below in the sense of (iii). If we use the standard multiplication for sets by non-negative scalars and the addition

$$A \oplus B = \overline{(A+B)}^{(u)} \quad \text{for } A, B \in \mathcal{Q},$$

then $\mathcal{Q}$ becomes a cone with the neutral element $\overline{\{0\}}^{(u)} = \{b \in \mathcal{P} \mid 0 \preccurlyeq b\}$. We use the inverse set inclusion as the order on $\mathcal{Q}$, that is

$$A \leq B \quad \text{if } B \subset A$$

and define neighborhoods corresponding to those in $\mathcal{P}$ by

$$A \leq B \oplus v \quad \text{for } A, B \in \mathcal{Q} \text{ and } v \in \mathcal{V},$$

if for every $b \in B$, and $\varepsilon > 0$ there is $a \in A$ such that $a \leq \gamma b + (1+\varepsilon)v$ for some $1 \leq \gamma \leq 1+\varepsilon$. Because for every $A \in \mathcal{Q}$ and $v \in \mathcal{V}$ there is $\lambda > 0$ such that $0 \leq a + \lambda v$ for all $a \in A$, we have $\overline{\{0\}}^{(u)} \leq A \oplus \lambda v$, and every element $A \in \mathcal{Q}$ is bounded below. Hence $(\mathcal{Q}, \mathcal{V})$ is a locally convex cone. A similar argument than in (d) yields that $(\mathcal{Q}, \mathcal{V})$ carries its weak preorder. Note that other than in (d) the empty set is a member of $\mathcal{Q}$, indeed its maximal element. We set $A \oplus \emptyset = \emptyset$, $\alpha \cdot \emptyset = \emptyset$ and $0 \cdot \emptyset = \overline{\{0\}}^{(u)}$ for all $A \in \mathcal{Q}$ and $\alpha > 0$. An element $A \in \mathcal{Q}$ is bounded above in $\mathcal{Q}$ if for every $v \in \mathcal{V}$ there is $\lambda \geq 0$ such that $A \leq \lambda v$, that is there is $a \in A$ such that $a \leq \lambda v$.

In both Examples (f) and (g) the given locally convex cone $\mathcal{P}$ may be considered as a subcone of $\mathcal{Q}$ via the embedding $a \mapsto \overline{\{a\}}$. This embedding preserves the order structure and the neighborhoods of $\mathcal{P}$, provided that $\mathcal{P}$ is endowed with the weak preorder, that is $\overline{\{a\}} \leq \overline{\{b\}} + v$ holds if and only if $a \preccurlyeq b + v$ for $a, b \in \mathcal{P}$ and $v \in \mathcal{V}$.

It is worthwhile to notice that a continuous linear operator T between two locally convex cones $(\mathcal{P}, \mathcal{V})$ and $(\mathcal{Q}, \mathcal{W})$ remains continuous if we endow both $\mathcal{P}$ and $\mathcal{Q}$ with either their respective upper, lower or symmetric relative topologies. A continuous linear functional μ on $\mathcal{P}$, in particular, is therefore also continuous mapping between $\mathcal{P}$ and $\overline{\mathbb{R}}$, if we endow $\mathcal{P}$ with either the upper, lower or symmetric relative v-topology. The corresponding relative topologies of $\overline{\mathbb{R}}$ coincide with the given ones ((see 1.1(a) and 3.4(a)).

Proposition 3.5. *The symmetric relative topology on a locally convex cone $(\mathcal{P}, \mathcal{V})$ is Hausdorff if and only if the weak preorder is antisymmetric.*

Proof. If $a \preccurlyeq b$ and $b \preccurlyeq a$ for $a, b \in \mathcal{P}$, then $a \in v^s_\varepsilon(b)$ for all $v \in \mathcal{V}$ and $\varepsilon > 0$. If the symmetric relative topology on $\mathcal{P}$ is Hausdorff, then this implies $a = b$. If on the other hand the weak preorder is antisymmetric, then for distinct elements $a, b \in \mathcal{P}$ we have either $a \not\preccurlyeq b$ or $b \not\preccurlyeq a$. Thus there are $v \in \mathcal{V}$ and $\varepsilon > 0$ such that either $a \notin v_\varepsilon(b)$ or $b \notin v_\varepsilon(a)$. Set $\delta = \min\{1, \varepsilon/3\}$ and assume that there is some $c \in v^s_\delta(a) \cap v^s_\delta(b)$. This means $a \leq \gamma c + \delta v$ and $c \leq \rho b + \delta v$ for some $1 \leq \gamma, \rho \leq 1 + \delta$. Thus $a \leq \gamma\rho b + \delta(1+\gamma)v$. Since $1 \leq \gamma\rho \leq 1 + \varepsilon$ and $\delta(1+\gamma) \leq \varepsilon$, this shows $a \in v_\varepsilon(b)$. Similarly, one verifies $b \in v_\varepsilon(a)$, contradicting the above. □

4. Boundedness Components

For an element $a \in \mathcal{P}$ we define the *upper* and *lower boundedness components* of a as

$$\mathcal{B}(a) = \bigcap_{v \in \mathcal{V}} \bigcup_{\varepsilon > 0} v_\varepsilon(a) \qquad \text{and} \qquad (a)\mathcal{B} = \bigcap_{v \in \mathcal{V}} \bigcup_{\varepsilon > 0} (a)v_\varepsilon,$$

respectively. We shall list a few of their basic properties. The detailed proofs for the following statements can be found in [8, Propositions 5.1 and 5.2].

Proposition 4.1. *Let $a, b, c \in \mathcal{P}$. Then*

(*a*) $b \in \mathcal{B}(a)$ *if and only if for every* $v \in \mathcal{V}$ *there are* $\alpha, \beta \geq 0$ *such that* $b \leq \alpha a + \beta v$.
(*b*) $\mathcal{B}(a)$ *is a subcone of* $\mathcal{P}$, *and* $\mathcal{B} \subset \mathcal{B}(a)$.
(*c*) $\mathcal{B}(a)$ *is a face in* $\mathcal{P}$, *that is* $b + c \in \mathcal{B}(a)$ *implies both* $b, c \in \mathcal{B}(a)$.
(*d*) $\mathcal{B}(\alpha a) = \mathcal{B}(a)$ *for* $\alpha > 0$, *and* $\mathcal{B}(a) + \mathcal{B}(b) \subset \mathcal{B}(a + b)$.
(*e*) $b \in \mathcal{B}(a)$ *if and only if* $\mathcal{B}(b) \subset \mathcal{B}(a)$.
(*f*) $b \in \mathcal{B}(a)$ *if and only if for all* $\mu \in \mathcal{P}^*$, $\mu(a) < +\infty$ *implies* $\mu(b) < +\infty$.
(*g*) $\mathcal{B}(a)$ *is closed in* $\mathcal{P}$ *with respect to the lower relative topology of* $\mathcal{P}$.

The elements of $\mathcal{B}(a)$ are called *bounded (above) relative to a*. By the definition of a locally convex cone we have $0 \in \mathcal{B}(a)$ for all $a \in \mathcal{P}$, and $\mathcal{B}(0) = \mathcal{B}$ consists of all bounded elements of $\mathcal{P}$. There are corresponding properties of the lower boundedness components. The *symmetric boundedness components* of $\mathcal{P}$ are the sets

$$\mathcal{B}^s(a) = \mathcal{B}(a) \cap (a)\mathcal{B} \quad \text{for } a \in \mathcal{P}.$$

According to Propositions 5.3 and 5.4 in [8], the symmetric boundedness components are closed for addition and multiplication by strictly positive scalars. They satisfy a version of the cancellation law, that is $a + c \preccurlyeq b + c$ for elements a, b and c of the same boundedness component implies that $a \preccurlyeq b$. Furthermore, the symmetric boundedness components furnish a partition of $\mathcal{P}$ into disjoint convex subsets that are both closed and connected in the symmetric relative topology.

5. Locally Convex Lattice Cones

A *topological vector lattice* is a vector lattice and a locally convex topological vector space over $\mathbb{R}$ that possesses a neighborhood base of solid sets. (See for example Chapter V.7 in [10], also [4] or [9].) Some of the following definitions and results are adaptations of classical concepts. The presence of unbounded elements and the general unavailability of negatives in locally convex cones, however, requires a more delicate approach. We shall say that $(\mathcal{P}, \mathcal{V})$ is a *locally convex upward lattice cone* if its order is antisymmetric and if for any two elements $a, b \in \mathcal{P}$ their supremum $a \vee b$ exists in $\mathcal{P}$ and if

(UL1) $(a + c) \vee (b + c) = a \vee b + c$ *holds for all* $a, b, c \in \mathcal{P}$.
(UL2) $a \leq c+v$ *and* $b \leq c+w$ *for* $a, b, c \in \mathcal{P}$ *and* $v, w \in \mathcal{V}$ *implies that* $a\vee b \leq c+(v+w)$.

Likewise, $(\mathcal{P}, \mathcal{V})$ is a *locally convex downward lattice cone* if its order is antisymmetric and if for any two elements $a, b \in \mathcal{P}$ their infimum $a \wedge b$ exists in $\mathcal{P}$ and if

(DL1) $(a + c) \wedge (b + c) = a \wedge b + c$ *holds for all* $a, b, c \in \mathcal{P}$.
(DL2) $c \leq a+v$ *and* $c \leq b+w$ *for* $a, b, c \in \mathcal{P}$ *and* $v, w \in \mathcal{V}$ *implies that* $c \leq a\wedge b+(v+w)$.

If both sets of the above conditions hold, then $(\mathcal{P}, \mathcal{V})$ is called a *locally convex lattice cone.* In case that $(\mathcal{P}, \mathcal{V})$ is indeed a locally convex topological vector space, the existence of suprema implies the existence of infima and vice versa, as $a \wedge b = -\left((-a) \vee (-b)\right)$. Conditions (UL1) and (UL2) then are equivalent to (DL1) and (DL2) and consistent with the above mentioned definition of a topological vector lattice. Indeed, $a \leq c + v$ and $b \leq c + w$ means that $a \leq c + s$ $b \leq c + t$ in this case, for some elements s and t of the neighborhoods v and w, respectively. Because these neighborhoods are supposed to be solid, we have $s \vee 0 \leq v$ and $t \vee 0 \leq w$ as well. Now $a \leq c + s \vee 0 + t \vee 0$ and $b \leq c + s \vee 0 + t \vee 0$ implies

$$a \vee b \leq c + s \vee 0 + t \vee 0 \leq c + (v + w)$$

as required in (UL2).

Proposition 5.1. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex upward (or downward) lattice cone. The lattice operation $(a, b) \mapsto a \vee b$ (or $(a, b) \mapsto a \wedge b$) is a continuous mapping from $\mathcal{P} \times \mathcal{P}$ to $\mathcal{P}$ if $\mathcal{P}$ is endowed with the symmetric relative topology.*

Proof. Suppose that $(\mathcal{P}, \mathcal{V})$ is a locally convex upward lattice cone, and let $a \in v_\varepsilon(b)$ and $c \in v_\varepsilon(d)$ for $a, b, c, d \in \mathcal{P}$, $v \in \mathcal{V}$ and $\varepsilon > 0$. There is $\lambda \geq 0$ such that both $0 \leq b + \lambda v$ and $0 \leq d + \lambda v$. Then $a \leq (1+\varepsilon)b + \varepsilon(1+\lambda)v$ and $c \leq (1+\varepsilon)d + \varepsilon(1+\lambda)v$ by Lemma 3.3(b). Thus

$$a \leq (1+\varepsilon)(b \vee d) + \varepsilon(1+\lambda)v \qquad \text{and} \qquad c \leq (1+\varepsilon)(b \vee d) + \varepsilon(1+\lambda)v.$$

hence

$$a \vee c \leq (1+\varepsilon)(b \vee d) + 2\varepsilon(1+\lambda)v$$

by (UL2). This shows $a \vee c \in v_{(2\varepsilon(1+\lambda))}(b \vee d)$. Similarly, using 3.3($c$) one verifies that $a \in (b)v_\varepsilon$ and $c \in (d)v_\varepsilon$ implies $a \vee c \in (b \vee d)v_{(2\varepsilon(1+\lambda+\varepsilon))}(b \vee d)$. Combining these observations for both the upper and lower relative neighborhoods then demonstrates that $a \in v^s_\varepsilon(b)$ and $c \in v^s_\varepsilon(d)$ implies $a \vee c \in v^s_{(2\varepsilon(1+\lambda+\varepsilon))}(b \vee d)$, hence our claim. A similar argument yields our claim for locally convex downward lattice cones. □

Proposition 5.2. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex lattice cone. Then $a + b = a \vee b + a \wedge b$ for all $a, b \in \mathcal{P}$.*

Proof. We observe that

$$a + b \leq \inf\big\{a + a \vee b\,,\ b + a \vee b\big\} = a \wedge b + a \vee b$$

by (DL1), and by (UL1)

$$a \vee b + a \wedge b = \sup\big\{a + a \wedge b\,,\ b + a \wedge b\big\} \leq a + b.$$

As the order of $\mathcal{P}$ is supposed to by antisymmetric, this yields our claim. □

Proposition 5.2 implies in particular that $a = a \vee 0 + a \wedge 0$ for all elements a of a locally convex lattice cone.

Examples of locally convex lattice cones include topological vector lattices and the cones $\overline{\mathbb{R}}$ and $\overline{\mathbb{R}}_+$ from Examples 2.1(a) and (b). If $(\mathcal{P}, \mathcal{V})$ is a locally convex (upward, downward) lattice cone, and if $\mathcal{Y}$ is a family of subsets of a set X, then the locally convex cone $\big(\mathcal{F}_{b_\mathcal{Y}}(X, \mathcal{P}), \mathcal{V}_\mathcal{Y}\big)$ of $\mathcal{P}$-valued functions from Example 2.1(d) is also a lattice cone of the same type. Suprema and infima are formed pointwise in this case. The cones $(\mathcal{C}^p, \mathcal{V}_p)$ from 2.1(e) are locally convex lattice cones.

6. Locally Convex Complete Lattice Cones

At instances, for example in integration theory, we require considerably stronger properties concerning the existence of suprema and infima in a locally convex cone. A locally convex cone $(\mathcal{P}, \mathcal{V})$ is said to be a *locally convex upward complete lattice cone* if $\mathcal{P}$ carries the weak preorder, this order is antisymmetric and if

(UC1) *Every non-empty subset* $A \subset \mathcal{P}$ *has a supremum* $\sup A \in \mathcal{P}$ *and* $\sup(A+b) = \sup A + b$ *holds for all* $b \in \mathcal{P}$.
(UC2) *Let* $\emptyset \neq A \subset \mathcal{P}$, $b \in \mathcal{P}$ *and* $v \in \mathcal{V}$. *If* $a \leq b + v$ *for all* $a \in A$, *then* $\sup A \leq b + v$.

In particular, every upward complete lattice cone $\mathcal{P}$ contains a largest element, that is $+\infty = \sup \mathcal{P}$, which may be adjoined as a maximal element to any locally convex cone with the convention that $a + \infty = +\infty$, $\alpha \cdot (+\infty) = +\infty$, $0 \cdot (+\infty) = 0$ and that $a \leq +\infty$ for all $a \in \mathcal{P}$ and $\alpha > 0$. Likewise, $(\mathcal{P}, \mathcal{V})$ is said to be a *locally convex downward complete lattice cone* if $\mathcal{P}$ carries the weak preorder, this order is antisymmetric and if

(DC1) *Every subset* $A \subset \mathcal{P}$ *that is bounded below has an infimum* $\inf A \in \mathcal{P}$ *and* $\inf(A+b) = \inf A + b$ *holds for all* $b \in \mathcal{P}$.
(DC2) *Let* $A \subset \mathcal{P}$ *be bounded below,* $b \in \mathcal{P}$ *and* $v \in \mathcal{V}$. *If* $b \leq a + v$ *for all* $a \in A$, *then* $b \leq \inf A + v$.

These requirements are obviously stronger then the corresponding ones in Section 5, so every locally convex upward (or downward) complete lattice cone is an upward (or downward) lattice. Requirements (UC2) and (DC2) mean that the upper or lower neighborhoods in $\mathcal{P}$ are closed for suprema or infima of their subsets, respectively. This corresponds to the properties of M-topologies in locally convex vector lattices. If $(\mathcal{P}, \mathcal{V})$ is a full cone, then (UC2) is evident, and (DC2) follows from (DC1). Recall from 3.4(g) that a subset A of $\mathcal{P}$ is *bounded below* if for every $v \in \mathcal{V}$ there is $\lambda \geq 0$ such that $0 \leq a + \lambda v$ for all $v \in \mathcal{V}$. This condition does in general not imply the existence of a lower bound in $\mathcal{P}$. However, if A has a lower bound $b \in \mathcal{P}$, that is $b \leq a$ for all $a \in A$, then A is bounded below in the above sense. Indeed, for every $v \in \mathcal{V}$ there is $\lambda \geq 0$ such that $0 \leq b + \lambda v$, hence $0 \leq a + \lambda v$ holds for all $a \in A$. Note that the empty set $\emptyset \subset \mathcal{P}$ is bounded below, and we have $\inf \emptyset = +\infty$ (see the remark above).

A locally convex cone $(\mathcal{P}, \mathcal{V})$ is called a *locally convex complete lattice cone* if it is both upward and downward complete.

For subsets A, B of a locally convex vector upward (or downward) complete lattice $\mathcal{P}$ let us denote

$$A \vee B = \{a \vee b \mid a \in A,\ b \in B\} \qquad (\text{or } A \wedge B = \{a \wedge b \mid a \in A,\ b \in B\}.)$$

Lemma 6.1. *Let* $(\mathcal{P}, \mathcal{V})$ *be a locally convex upward complete lattice cone and let* $A, B \subset \mathcal{P}$ *be non-empty subsets of* $\mathcal{P}$. *Then*

(*a*) $\sup(A+B) = \sup A + \sup B$.
(*b*) $\sup(A \cup B) = \sup(A \vee B) = \sup A \vee \sup B$.

Proof. For part (*a*), we observe that $a + b \leq \sup A + \sup B$ holds for all $a \in A$ and $b \in B$, hence $\sup(A+B) \leq \sup A + \sup B$. On the other hand, for any fixed $b \in B$ we have $\sup(A+B) \geq \sup(A+b) = \sup A + b$ by (UC1). This shows $\sup(A+B) \geq \sup(\sup A + B) = \sup A + \sup B$, again by (UC1).

For part (b) we observe that $a \leq a \vee b \leq \sup(A \vee B)$ and $b \leq a \vee b \leq \sup(A \vee B)$ holds for all $a \in A$ and $b \in B$. Thus $\sup(A \cup B) \leq \sup(A \vee B)$. Moreover, $a \vee b \leq \sup A \vee \sup B$ for all $a \in A$ and $b \in B$ demonstrates that $\sup(A \vee B) \leq \sup A \vee \sup B$. Because both $\sup A \leq \sup(A \cup B)$ and $\sup B \leq \sup(A \cup B)$, we have $\sup A \vee \sup B \leq \sup(A \cup B)$ as well. □

A similar argument in the downward case yields:

Lemma 6.2. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex downward complete lattice cone and let $A, B \subset \mathcal{P}$ be non-empty bounded below subsets of $\mathcal{P}$. Then*

(a) $\inf(A + B) = \inf A + \inf B$.
(b) $\inf(A \cup B) = \inf(A \wedge B) = \inf A \wedge \inf B$.

Remarks and Examples 6.3. (a) Every locally convex downward complete lattice cone has also suprema for all of its subsets, as $\sup A$ is the infimum of the set of all upper bounds for A. However, requirement (UC1) does not necessarily follow (see (e) below). Likewise, every locally convex upward complete lattice cone has infima for subsets with lower bounds in $\mathcal{P}$, but again, requirement (DC1) does not follow (see (d) below).

(b) The locally convex cones $\overline{\mathbb{R}}$ and $\overline{\mathbb{R}}_+$ (Examples 2.1(a) and (b)) are of course complete lattices.

(c) If $(\mathcal{P}, \mathcal{V})$ is a locally convex upward (or downward) complete lattice cone, and if $\mathcal{Y}$ is a family of subsets of a set X, then the locally convex cone $\big(\mathcal{F}_{b_{\mathcal{Y}}}(X, \mathcal{P}), \mathcal{V}_{\mathcal{Y}}\big)$ of $\mathcal{P}$-valued functions from Example 2.1(d) is also upward (or downward) complete. Suprema and infima are formed pointwise.

(d) Example 3.4(f) yields an locally convex upward complete lattice cone. The cone $(Q, \mathcal{V})$ of all non-empty closed (with respect to the lower relative topology) convex subsets of a locally convex cone $(\mathcal{P}, \mathcal{V})$ is ordered by set inclusion and carries the weak preorder which is antisymmetric (see 3.4(f)). For a family $\mathcal{A} \subset \mathcal{Q}$ its supremum is given by $\sup \mathcal{A} = \overline{conv\big(\bigcup_{A \in \mathcal{A}} A\big)}^{(u)}$, where $conv(C)$ denotes the convex hull of a set $C \subset \mathcal{P}$. Condition (DC1) may be checked easily: Let $B \in \mathcal{P}$. Clearly $A \oplus B \subset \sup \mathcal{A} \oplus B$ for all $A \in \mathcal{A}$, hence $\sup\{A \oplus B \mid A \in \mathcal{A}\} \leq \sup \mathcal{A} \oplus B$. For the converse inequality let $c \in \sup \mathcal{A} \oplus B = \overline{\big(conv\big(\bigcup_{A \in \mathcal{A}} A\big) + B\big)}^{(l)}$. Then for every lower relative neighborhood $(c)v_\varepsilon$ there is $d \in (c)v_\varepsilon \cap \big(conv\big(\bigcup_{A \in \mathcal{A}} A\big) + B\big)$. This means $d = \sum_{i=1}^n \alpha_i a_i + b$ for some $a_i \in A_i \in \mathcal{A}$, $b \in B$ and $0 \leq \alpha_i$ such that $\sum_{i=1}^n \alpha_i = 1$. Thus $d = \sum_{i=1}^n \alpha_i(a_i + b) \in \sup\{A \oplus B \mid A \in \mathcal{A}\}$. This implies $c \in \sup\{A \oplus B \mid A \in \mathcal{A}\}$ as well, since this set is closed in the lower topology. Our claim follows.

(e) A similar argument shows that Example 3.4(g) yields a locally convex downward complete lattice cone.

(f) Let X be a topological space, and let $\mathcal{P}$ be the cone of all $\overline{\mathbb{R}}$-valued lower semicontinuous functions on X, where $\overline{\mathbb{R}}$ is endowed with the usual, that is the one-point compactification topology. $\mathcal{P}$ is endowed with the pointwise operations and order and neighborhoods $v \in \mathcal{V}$ for $\mathcal{P}$ are given by the strictly positive constant functions. Then $(\mathcal{P}, \mathcal{V})$ forms a locally convex upward complete lattice cone. Similarly, the cone of all $\overline{\mathbb{R}}$ -valued upper semicontinuous functions on X forms a locally convex downward complete lattice cone.

7. Zero Components

Throughout this section we shall assume that $(\mathcal{P}, \mathcal{V})$ is a locally convex downward complete lattice cone. We define the *zero component* of an element a of a locally convex downward complete lattice cone $\mathcal{P}$ by

$$\mathfrak{O}(a) = \inf\big\{b \geq 0 \mid a \in \mathcal{B}(b)\big\}.$$

This expression is well defined, and $\mathfrak{O}(a) \geq 0$ for all $a \in \mathcal{P}$. Recall from Proposition 4.1(a) that $a \in \mathcal{B}(b)$ if and only if for every $v \in \mathcal{V}$ there are $\alpha, \beta \geq 0$ such that $a \leq \alpha b + \beta v$.

The introduction of zero components is especially useful for the investigation of variations of the cancellation law in downward complete lattice cones.

Proposition 7.1. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex downward complete lattice cone, and let $a, b, c \in \mathcal{P}$.*

(a) *If $a + c \leq b + c$, then $a \leq b + \mathfrak{O}(c)$.*
(b) *If $a \in \mathcal{B}(b)$, then $\mathfrak{O}(a) \leq \mathfrak{O}(b)$*
(c) *If a is bounded, then $\mathfrak{O}(a) = 0$.*

Proof. Let $a, b, c \in \mathcal{P}$. For part (a), suppose that $a + c \leq b + c$. Following Lemma I.4.1 in [3], the above implies $a + \rho c \leq b + \rho c$ for all $\rho > 0$. Given $v \in \mathcal{V}$ and $\varepsilon > 0$ there is $\lambda \geq 0$ such that both $0 \leq b + \lambda v$ and $0 \leq c + \lambda v$. Thus $0 \leq (b + \rho c) + 2\lambda v$ for all $0 < \rho \leq 1$. This yields

$$a \leq a + \rho(c + \lambda v) \leq (b + \rho c) + \rho\lambda v.$$

Let $d \geq 0$ such that $c \in \mathcal{B}(d)$. Then $c \leq \alpha d + \beta v$ holds for some $\alpha, \beta \geq 0$. Consequently, for all $\rho > 0$ such that $\rho \leq \max\{\frac{\varepsilon}{2\lambda+1}, \frac{1}{\alpha+1}, \frac{\varepsilon}{2\beta+1}\}$ we have $\rho c \leq (\rho\alpha)d + (\rho\beta)v \leq d + (\varepsilon/2)v$, hence

$$a \leq (b + \rho c) + \frac{\varepsilon}{2} v \leq (b + d) + \varepsilon v.$$

Now we may use rules (DC1) and (DC2) and take the infimum over the right-hand side of this inequality with respect to all $d \geq 0$ such that $c \in \mathcal{B}(d)$. This yields

$$a \leq b + \mathfrak{O}(c) + \varepsilon v.$$

This last inequality holds true for all $v \in \mathcal{V}$ and $\varepsilon > 0$. Since $\mathcal{P}$ carries the weak preorder, this yields $a \leq b + \mathfrak{O}(c)$, as claimed. For part (b) suppose that $a \in \mathcal{B}(b)$. Then for every $c \geq 0$ such that $b \in \mathcal{B}(c)$ we have $\mathcal{B}(b) \subset \mathcal{B}(b)$ by 4.1(e), hence $a \in \mathcal{B}(c)$ as well. This yields $\mathfrak{O}(a) \leq \mathfrak{O}(b)$. Part (c) follows from part (b) with $b = 0$. □

Proposition 7.2. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex downward complete lattice cone, and let $a, b, \in \mathcal{P}$. Then*

(a) $\mathfrak{O}(a + b) = \mathfrak{O}(a) + \mathfrak{O}(b)$.
(b) $\mathfrak{O}(\alpha a) = \alpha\mathfrak{O}(a) = \mathfrak{O}(a)$ *for all $\alpha > 0$.*
(c) *If $\alpha a = a$ for all $\alpha > 0$, then $\mathfrak{O}(a) = a$.*

Proof. Let $a, b, \in \mathcal{P}$. Part (b) is obvious since for every $\alpha > 0$ and every $c \in \mathcal{P}$ we have $\alpha a \in \mathcal{B}(c)$ if and only if $a \in \mathcal{B}(c)$ by 4.1(b). For part (a) let $a \in \mathcal{B}(c)$ and $b \in \mathcal{B}(d)$ for $c, d \geq 0$. Then $a + b \in \mathcal{B}(c + d)$ by 4.1(d). This shows $\mathfrak{O}(a + b) \leq \mathfrak{O}(a) + \mathfrak{O}(b)$. For the converse, given $v \in \mathcal{V}$ there is $\lambda \geq 0$ such that $0 \leq b + \lambda v$. Hence $a \leq (a + b) + \lambda v$, and we infer that $a \in \mathcal{B}(a + b)$. Thus $\mathfrak{O}(a) \leq \mathfrak{O}(a + b)$ by 7.1(b), and likewise $\mathfrak{O}(b) \leq \mathfrak{O}(a + b)$. This yields $\mathfrak{O}(a) + \mathfrak{O}(b) \leq 2\mathfrak{O}(a + b) = \mathfrak{O}(a + b)$. For part (c) let $a \in \mathcal{P}$ such that $\alpha a = a$ for all $\alpha \geq 0$. For every $v \in \mathcal{V}$ there is $\lambda > 0$ such that $0 \leq a + \lambda v$, hence $0 \leq (1/\lambda)a + v = a + v$. This shows $0 \leq a$, since $\mathcal{P}$ carries the weak preorder. Thus $\mathfrak{O}(a) \leq a$. If on the other hand $a \in \mathcal{B}(c)$ for some $c \geq 0$, then there are $\alpha, \beta \geq 0$ such that $a \leq \alpha c + \beta v$. Since $\varepsilon \alpha c \leq c$ for all $0 < \varepsilon \leq 1/(\alpha + 1)$, this implies

$$a = \varepsilon a \leq \varepsilon \alpha c + \varepsilon \beta v \leq c + \varepsilon \beta v$$

for all such ε. This yields $a \leq c$ since $\mathcal{P}$ carries the weak preorder, and we also have $a \leq \mathfrak{O}(a)$. □

Proposition 7.2(b) implies in particular that a linear functional $\mu \in \mathcal{P}^*$ can attain only the values 0 or $+\infty$ at a zero component.

Some additional properties can be derived if $(\mathcal{P}, \mathcal{V})$ is also an upward or indeed an upward complete lattice.

Lemma 7.3. *Suppose $(\mathcal{P}, \mathcal{V})$ is a locally convex lattice cone and downward complete. Then the zero component of an element $a \in \mathcal{P}$ may be alternatively expressed as*

$$\mathfrak{O}(a) = \inf_{\varepsilon > 0} \big\{ \varepsilon \, (a \vee 0) \big\}.$$

Proof. Let $a \in \mathcal{P}$. Then $0 \leq a \vee 0$ and $a \leq a \vee b$. Thus $a \in \mathcal{B}(a \vee b)$. This implies $a \in \mathcal{B}\left(\varepsilon \, (a \vee 0)\right)$ for all $\varepsilon > 0$ by 4.1(d). Hence $\inf \big\{ b \geq 0 \mid a \in \mathcal{B}(b) \big\} \leq \inf_{\varepsilon > 0} \big\{ \varepsilon \, (a \vee 0) \big\}$. For the converse inequality let $b \geq 0$ such that $a \in \mathcal{B}(b)$. Given $v \in \mathcal{V}$ and $\varepsilon > 0$ there are $\alpha, \beta \geq 0$ such that $a \leq \alpha b + \beta v$ (see 4.1(a)). (UL2) then yields $a \vee 0 \leq \alpha b + 2\beta v$. Thus for $0 < \delta \leq \min\{\frac{\varepsilon}{2\beta + 1}, \frac{1}{\alpha + 1}\}$ we have

$$\delta (a \vee 0) \leq \delta \alpha b + 2\delta \beta v \leq b + \varepsilon v,$$

since $b \geq 0$ and $\delta \alpha \leq 1$ implies $(\delta \alpha) b \leq b$. This shows $\inf_{\varepsilon > 0} \big\{ \varepsilon \, (a \vee 0) \big\} \leq b + \varepsilon v$, hence

$$\inf_{\varepsilon > 0} \big\{ \varepsilon \, (a \vee 0) \big\} \leq \inf \big\{ b \geq 0 \mid a \in \mathcal{B}(b) \big\} + \varepsilon v$$

by (DC2). Because this holds for all $v \in \mathcal{V}$ and for all $\varepsilon > 0$, and because $\mathcal{P}$ carries the weak preorder, we conclude that $\inf_{\varepsilon > 0} \big\{ \varepsilon \, (a \vee 0) \big\} \leq \inf \big\{ b \geq 0 \mid a \in \mathcal{B}(b) \big\}$. □

Proposition 7.4. *Suppose $(\mathcal{P}, \mathcal{V})$ is a locally convex lattice cone and downward complete. Then $b + \mathfrak{O}(a) = b$ holds for all $a, b \in \mathcal{P}$ whenever $a \in \mathcal{B}(b)$.*

Proof. Let $a, b \in \mathcal{P}$ such that $a \in \mathcal{B}(b)$. Then $\mathfrak{O}(a) \leq \mathfrak{O}(b)$ by Proposition 7.1(b). Thus we only have to verify that $b + \mathfrak{O}(b) = b$. Clearly $b \leq b + \mathfrak{O}(b)$. For the reverse inequality let $v \in \mathcal{V}$. There is $\lambda \geq 0$ such that $0 \leq b + \lambda v$, hence also $0 \leq b \wedge 0 + \lambda v$ by (DC2). Then using Proposition 5.2 we observe that

$$b + \mathfrak{O}(b) \leq b + \varepsilon \, (b \vee 0) \leq b + \varepsilon \, (b \vee 0) + \varepsilon \, ((b \wedge 0) + \lambda v) = (1 + \varepsilon) b + \varepsilon \lambda v$$

holds for all $\varepsilon > 0$. This shows $b + \mathfrak{O}(b) \leq b$. □

Proposition 7.5. *Let $(\mathcal{P},\mathcal{V})$ be a locally convex complete lattice cone, and let A,B be non-empty subsets of $\mathcal{P}$. Then*

(a) $\inf(A\vee B)=\inf A\vee\inf B$ *if both A and B are bounded below.*
(b) $\sup(A\wedge B)\le\sup A\wedge\sup B\le\sup(A\wedge B)+\mathfrak{O}\left(\sup(A\vee B)\right).$

Proof. We first observe that

$$\inf A\vee\inf B\le a\vee b\qquad\text{and}\qquad a\wedge b\le\sup A\wedge\sup B$$

holds for all $a\in A$ and $b\in B$. Thus

$$\inf A\vee\inf B\le\inf(A\vee B)\qquad\text{and}\qquad\sup(A\wedge B)\le\sup B\wedge\sup A.$$

For part (a) we assume that both sets A and B are bounded below and use Proposition 5.2 for

$$\begin{aligned}\inf(A\vee B)+\inf(A\wedge B)\le\ &\inf\{a\vee b+a\wedge b\mid a\in a,\ b\in B\}\\ =\ &\inf(A+B)\\ =\ &\inf A+\inf B\\ =\ &\inf A\vee\inf B+\inf A\wedge\inf B.\end{aligned}$$

As $\inf(A\wedge B)=\inf A\wedge\inf B$, the cancellation law in Proposition 7.1(a) yields

$$\inf(A\vee B)\le\inf A\ \vee\ \inf B+\mathfrak{O}\left(\inf(A\wedge B)\right).$$

Similarly, one obtains

$$\sup A\wedge\sup B\le\sup(A\wedge B)+\mathfrak{O}\left(\sup(A\vee B)\right),$$

that is part (b). Finally, as $\inf(A\wedge B)=\inf A\wedge\inf B\le\inf A\vee\inf B$, Proposition 7.5 shows

$$\inf A\ \vee\ \inf B+\mathfrak{O}\left(\inf A\wedge\inf B\right)=\inf A\ \vee\ \inf B.$$

This completes our proof of part (a).

Examples 7.6. (a) If $\mathcal{P}=\overline{\mathbb{R}}$ or $\mathcal{P}=\overline{\mathbb{R}}_+$ (see 2.1(a) and 2.1(b)), then $\mathfrak{O}(a)=0$ for all $a<+\infty$, and $\mathfrak{O}(+\infty)=+\infty$.

(b) If $(\mathcal{P},\mathcal{V})$ is downward complete and $f\in\mathcal{F}_{b_{\mathcal{Y}}}(X,\mathcal{P})$ (see Example 2.1(d)), then $\mathfrak{O}(f)$ is the mapping $x\mapsto\mathfrak{O}\left(f(x)\right)$. For $\mathcal{P}=\overline{\mathbb{R}}$, in particular, the zero component of an $\overline{\mathbb{R}}$-valued function $f\in\mathcal{F}_{b_{\mathcal{Y}}}(X,\overline{\mathbb{R}})$ is the mapping $\mathfrak{O}\left(f\right)(x)=0$ if $f(x)<+\infty$, and $\mathfrak{O}\left(f\right)(x)=+\infty$ else. The same observation applies to Example 6.3(f), that is the cone of $\overline{\mathbb{R}}$-valued upper semicontinuous functions on a topological space with the positive constants as neighborhoods.

(c) Let us consider Example 3.4(g)(e) (see also 6.3(e)), that is the downward complete lattice cone $(\mathcal{Q},\mathcal{V})$ of all convex subsets locally convex cone $(\mathcal{P},\mathcal{V})$, which are bounded below and closed with respect to the upper relative topology. The order in $\mathcal{Q}$ is the inverse set inclusion and the neighborhoods are given by $A\le B\oplus v$ for $A,B\in\mathcal{Q}$ and $v\in\mathcal{V}$, if for every $b\in B$, and $\varepsilon>0$ there is $a\in A$ such that $a\le\gamma b+(1+\varepsilon)v$ for some $1\le\gamma\le1+\varepsilon$.

The closed convex subsets (including the empty set) of $\overline{\{0\}}^{(u)} = \{b \in \mathcal{P} \mid 0 \preccurlyeq b\}$ are the positive elements in $\mathcal{Q}$. We claim that for an element $A \in \mathcal{Q}$ we have

$$\mathfrak{O}(A) = \{b \succcurlyeq 0 \mid \mathcal{B}_v(b) \cap A \neq \emptyset \text{ for all } v \in \mathcal{V}\}.$$

We shall argue for this using the following steps: Let B denote the set on the right-hand side of the above equation.

(i) The set $B \subset \mathcal{P}$ is convex. Indeed, let $b_1, b_2 \in B$, $0 \leq \lambda_1, \lambda_2 \leq 1$ such that $\lambda_1 + \lambda_2 = 1$ and $b = \lambda_1 b_1 + \lambda_2 b_2$. Given $v \in \mathcal{V}$ there are $a_1 \in \mathcal{B}_v(b_1) \cap A$ and $a_2 \in \mathcal{B}_v(b_2) \cap A$. Set $a = \lambda_1 a_1 + \lambda_2 a_2 \in A$ and choose $\alpha_1, \alpha_2, \beta, \rho \geq 0$ such that

$$a_1 \leq \alpha_1 b_1 + \beta v, \qquad a_2 \leq \alpha_2 b_2 + \beta v, \qquad 0 \leq b_1 + \rho v \quad \text{and} \quad 0 \leq b_2 + \rho v.$$

Setting $\alpha = \max\{\alpha_1, \alpha_2\}$ we have

$$a_1 \leq (\alpha_1 b_1 + \beta v) + (\alpha - \alpha_1)(b_1 + \rho v) + \alpha_1 \rho v = \alpha b_1 + (\beta + \alpha\rho) v$$

and, likewise

$$a_2 \leq (\alpha_2 b_1 + \beta v) + (\alpha - \alpha_2)(b_2 + \rho v) + \alpha_2 \rho v = \alpha b_2 + (\beta + \alpha\rho) v.$$

Thus

$$a \leq \lambda_1 (\alpha b_1 + (\beta + \alpha\rho) v) + \lambda_2 (\alpha b_1 + (\beta + \alpha\rho) v) = \alpha b + (\beta + \alpha\rho) v.$$

We infer that $a \in \mathcal{B}_v(b) \cap A$, hence $\mathcal{B}_v(b) \cap A \neq \emptyset$. Since this holds for all $v \in \mathcal{V}$ and since $b \succcurlyeq 0$ is evident from $b_1, b_2 \succcurlyeq 0$, we conclude that $b \in B$.

(ii) The set $B \subset \mathcal{P}$ is closed with respect to the upper topology. Indeed, let $c \in \overline{B}^{(u)}$ and let $v \in \mathcal{V}$. There is $b \in v_1(c) \cap B$, that is $b \leq \gamma c + v$ for some $1 \leq \gamma \leq 2$. There is $a \in \mathcal{B}_v(b) \cap A$, that is $a \leq \alpha b + \beta v$ for some $\alpha, \beta \geq 0$. Combining these yields $a \leq \alpha\gamma c + (\alpha + \beta) v$. This shows $\mathcal{B}_v(c) \cap A \neq \emptyset$ for all $v \in \mathcal{V}$. Furthermore, since $B \subset \overline{\{0\}}^{(u)} = \{b \in \mathcal{P} \mid 0 \preccurlyeq b\}$ which is closed with respect to the upper relative topology, we have $c \in \overline{\{0\}}^{(u)}$ as well, hence $c \succcurlyeq 0$. Together with the above this yields $c \in B$. Since $B \subset \mathcal{P}$ is obviously bounded below (we have $0 \leq b + v$ for all $b \in \mathcal{P}$), we conclude from (i) and (ii) that $B \in \mathcal{Q}$.

(iii) We have $A \in \mathcal{B}\left(\overline{\{b\}}^{(u)}\right)$ for all $b \in B$. Indeed, let $v \in \mathcal{V}$. Given $b \in B$ there is some $a \in \mathcal{B}_v(b) \cap A$, that is there are $\alpha, \beta, \lambda \geq 0$ such that $a \leq \alpha b + \beta v$ and $0 \leq b + \lambda v$. Then for every $c \in \overline{\{b\}}^{(u)}$, that is $b \preccurlyeq c$, we have $b \in v_1(c)$, hence $b \leq 2c + (2 + \lambda) v$ (see Proposition 4.1(c) with $\varepsilon = 1$). This yields $a \leq 2\alpha c + (2\alpha + \lambda\alpha + \beta) v$ and $A \leq 2\alpha\, \overline{\{b\}}^{(u)} \oplus (2\alpha + \lambda\alpha + \beta) v$, hence $A \in \mathcal{B}\left(\overline{\{b\}}^{(u)}\right)$. Consequently,

$$\mathfrak{O}(A) \leq \inf\left\{\overline{\{b\}}^{(u)} \mid b \in B\right\} = \overline{conv\left(\bigcup_{b \in B} \overline{\{b\}}^{(u)}\right)}^{(u)} = B.$$

(iv) On the other hand, let $C \in \mathcal{Q}$ such that $C \geq 0$, that is $C \subset \overline{\{0\}}^{(u)}$, and $A \in \mathcal{B}(C)$. Let $c \in C$. Given $v \in \mathcal{V}$ there are $\alpha, \beta \geq 0$ such that $A \leq \alpha C \oplus \beta v$. According to ro our definitions of the neighborhoods in $\mathcal{Q}$ (see 3.4(g)), for $\varepsilon = 1$ we find $a \in A$ such that $a \leq \gamma(\alpha c) + 2(\beta v)$ with some $1 \leq \gamma \leq 2$. This yields $\mathcal{B}_v(b) \cap A \neq \emptyset$ for all $v \in \mathcal{V}$, hence $c \in B$ since $c \succcurlyeq 0$. Thus

$$C = \overline{conv\left(\bigcup_{c\in C} \overline{\{c\}}^{(u)}\right)}^{(u)} \subset B.$$

This shows $\mathfrak{O}(A) \subset B$, that is $\mathfrak{O}(A) \geq B$, and our claim follows. In particular, we have $\mathfrak{O}(A) = \overline{\{0\}}^{(u)}$ if and only if $\mathcal{B}_v(0) \cap A \neq \emptyset$ for all $v \in \mathcal{V}$, that is if and only if for every $v \in \mathcal{V}$ there are $a \in A$ and $\lambda \geq 0$ such that $a \leq \lambda v$, that is if and only if the element $A \in \mathcal{Q}$ is bounded above (see 3.4(g)).

For a concrete example let $\mathcal{P}$ be the cone of all real-valued bounded below continuous functions on the open interval $(0,1)$, endowed with the positive constants as neighborhoods (see 2.1(d)) and let $\mathcal{Q}$ be as before. Consider the subset

$$C = \left\{ f \in \mathcal{P} \;\middle|\; f(x) \geq \frac{1}{x} - 2 \text{ for all } x \in (0,1) \right\}.$$

This set is convex, bounded below and closed with respect to the upper relative topology, hence $C \in \mathcal{Q}$. For a function $g \geq 0$ in $\mathcal{P}$, we have $\mathcal{B}(g) \cap C \neq \emptyset$ if and only if there are $\alpha, \beta \geq 0$ such that $1/x \leq \alpha g(x) + \beta$ for all $x \in (0,1)$, that is if and only if the inferior limit of $xg(x)$ at 0 is greater than 0. Thus

$$\mathfrak{O}(C) = \left\{ g \in \mathcal{P} \;\middle|\; g \geq 0 \text{ and } \varliminf_{x\to 0} xg(x) > 0 \right\}.$$

Now according to the cancellation rule in Proposition 7.1(a), if $A, B \in \mathcal{Q}$ such that $A + C \leq B + C$, that is $B + C \subset A + C$, then $A \leq B + \mathfrak{O}(c)$, that is $B + \mathfrak{O}(C) \subset A$.

8. Order Convergence in Locally Convex Complete Lattice Cones

We proceed to define order convergence for nets in a locally convex complete lattice cone $(\mathcal{P}, \mathcal{V})$. A net $(a_i)_{i\in\mathcal{I}}$ in $\mathcal{P}$ is called *bounded below* if there is $i_0 \in \mathcal{I}$ such that the set $\{a_i \mid i \geq i_0\}$ is bounded below in the sense of 3.4(g). We define the superior and inferior limits of a bounded below net $(a_i)_{i\in\mathcal{I}}$ in $\mathcal{P}$ by

$$\varliminf_{i\in\mathcal{I}} a_i = \sup_{i\in\mathcal{I}} \left(\inf_{k\geq i} a_k \right) \qquad \text{and} \qquad \varlimsup_{i\in\mathcal{I}} a_k = \inf_{i\in\mathcal{I}} \left(\sup_{k\geq i} a_k \right).$$

Because the order of $\mathcal{P}$ is supposed to be antisymmetric, both limits are uniquely defined. Obviously, $\varliminf_{i\in\mathcal{I}} a_i \leq \varlimsup_{i\in\mathcal{I}} a_i$. If $\varliminf_{i\in\mathcal{I}} a_i$ and $\varlimsup_{i\in\mathcal{I}} a_i$ coincide, we shall denote their common value by $\lim_{i\in\mathcal{I}} a_i$ and say that the net $(a_i)_{i\in\mathcal{I}}$ is *order convergent.* Obviously, every increasing or decreasing bounded below net is order convergent in this sense, converging towards the supremum or the infimum of the set of its elements, respectively.

Lemma 8.1. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex complete lattice cone, and let $(a_i)_{i\in\mathcal{I}}$ and $(b_i)_{i\in\mathcal{I}}$ be bounded below nets in $\mathcal{P}$. Then*

$$\varliminf_{i\in\mathcal{I}} a_i + \varliminf_{i\in\mathcal{I}} b_i \leq \varliminf_{i\in\mathcal{I}} (a_i + b_i) \leq \varlimsup_{i\in\mathcal{I}} a_i + \varliminf_{i\in\mathcal{I}} b_i \leq \varlimsup_{i\in\mathcal{I}} (a_i + b_i) \leq \varlimsup_{i\in\mathcal{I}} a_i + \varlimsup_{i\in\mathcal{I}} b_i.$$

Proof. For any bounded below net $(c_i)_{i\in\mathcal{I}}$ in $\mathcal{P}$, for $i \in \mathcal{I}$, let

$$s_i^{(c)} = \inf_{k\geq i} c_k \qquad \text{and} \qquad S_i^{(c)} = \sup_{k\geq i} c_k.$$

The nets $(s_i^{(c)})_{i\in\mathcal{I}}$ and $(S_i^{(c)})_{i\in\mathcal{I}}$ are increasing and decreasing, respectively, and

$$\underline{\lim}_{i\in\mathcal{I}} c_i = \sup_{i\in\mathcal{I}} s_i^{(c)} \qquad \text{and} \qquad \overline{\lim}_{i\in\mathcal{I}} c_i = \inf_{i\in\mathcal{I}} S_i^{(c)}.$$

Now, using the nets $(a_i)_{i\in\mathcal{I}}$, $(b_i)_{i\in\mathcal{I}}$ and $(a_i + b_i)_{i\in\mathcal{I}}$ in place of $(c_i)_{i\in\mathcal{I}}$ we observe that

$$s_i^{(a+b)} \geq s_i^{(a)} + s_i^{(b)} \qquad \text{and} \qquad S_i^{(a+b)} \leq S_i^{(a)} + S_i^{(b)}$$

for all $i \in \mathcal{I}$. For every $k \in \mathcal{I}$ we have by (UC1)

$$s_k^{(a)} + \sup_{i\in\mathcal{I}} s_i^{(b)} = \sup_{i\in\mathcal{I}}(s_k^{(a)} + s_i^{(b)}) \leq \sup_{l\in\mathcal{I}}(s_l^{(a)} + s_l^{(b)}),$$

as $s_k^{(a)} + s_i^{(b)} \leq s_l^{(a)} + s_l^{(b)}$ whenever $i, k \leq l$. This shows

$$\underline{\lim}_{i\in\mathcal{I}} a_i + \underline{\lim}_{i\in\mathcal{I}} b_i = \sup_{k\in\mathcal{I}} s_k^{(a)} + \sup_{i\in\mathcal{I}} s_k^{(b)} \leq \sup_{l\in\mathcal{I}}(s_l^{(a)} + s_l^{(b)}) = \underline{\lim}_{i\in\mathcal{I}}(a_i + b_i),$$

the first part of our claim. A similar argument using the decreasing nets $(S_i^{(c)})_{i\in\mathcal{I}}$ yields

$$\overline{\lim}_{i\in\mathcal{I}} a_i + \overline{\lim}_{i\in\mathcal{I}} b_i = \inf_{k\in\mathcal{I}} S_k^{(a)} + \inf_{i\in\mathcal{I}} S_k^{(b)} \geq \inf_{l\in\mathcal{I}}(S_l^{(a)} + S_l^{(b)}) = \overline{\lim}_{i\in\mathcal{I}}(a_i + b_i).$$

Finally, for all $i, l \in \mathcal{I}$ and $j \geq i, l$ we have

$$s_i^{(a+b)} = \inf_{k\geq i}(a_k + b_k) \leq \inf_{k\geq j}(S_l^{(a)} + b_k) = S_l^{(a)} + \inf_{k\geq j} b_k \leq S_l^{(a)} + \underline{\lim}_{i\in\mathcal{I}} b_i,$$

hence

$$\underline{\lim}_{i\in\mathcal{I}}(a_i + b_i) = \sup_{i\in\mathcal{I}} s_i^{(a+b)} \leq \inf_{l\in\mathcal{I}} S_l^{(a)} + \underline{\lim}_{i\in\mathcal{I}} b_i = \overline{\lim}_{i\in\mathcal{I}} a_i + \underline{\lim}_{i\in\mathcal{I}} b_i.$$

A similar argument shows that

$$\overline{\lim}_{i\in\mathcal{I}} a_i + \underline{\lim}_{i\in\mathcal{I}} b_i \leq \overline{\lim}_{i\in\mathcal{I}}(a_i + b_i).$$

□

Note that Lemma 8.1 implies in particular that

$$\underline{\lim}_{i\in\mathcal{I}}(a + b_i) = a + \underline{\lim}_{i\in\mathcal{I}} b_i \qquad \text{and} \qquad \overline{\lim}_{i\in\mathcal{I}}(a + b_i) = a + \overline{\lim}_{i\in\mathcal{I}} b_i$$

holds for $a \in \mathcal{P}$ and a bounded below net $(b_i)_{i\in\mathcal{I}}$.

Lemma 8.2. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex complete lattice cone, let $(a_i)_{i\in\mathcal{I}}$ and $(b_i)_{i\in\mathcal{I}}$ be bounded below nets in $\mathcal{P}$, and let $v \in \mathcal{V}$. If $a_i \leq b_i + v$ for all $i \in \mathcal{I}$, then $\underline{\lim}_{i\in\mathcal{I}} a_i \leq \underline{\lim}_{i\in\mathcal{I}} b_i + v$ and $\overline{\lim}_{i\in\mathcal{I}} a_i \leq \overline{\lim}_{i\in\mathcal{I}} b_i + v$.*

Proof. For every $i \in \mathcal{I}$ we have $\inf_{k \geq i} a_k \leq b_j + v$ for all $j \geq i$, hence

$$\inf_{k \geq i} a_k \leq \inf_{j \geq i} b_j + v \leq \underline{\lim}_{j \in \mathcal{J}} b_j + v$$

by (DC2). Thus $\underline{\lim}_{i \in \mathcal{I}} a_i \leq \underline{\lim}_{j \in \mathcal{J}} b_j + v$ by (UC2). A similar argument yields the second statement. $\square$

Lemma 8.2 implies in particular that $\lim_{i \in \mathcal{I}} a_i \leq \lim_{i \in \mathcal{I}} b_i$ holds for order convergent nets $(a_i)_{i \in \mathcal{I}}$ and $(b_i)_{i \in \mathcal{I}}$ whenever $a_i \leq b_i$ for all $i \in \mathcal{I}$. We omit the proof for the following:

Lemma 8.3. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex complete lattice cone. Let $(a_i)_{i \in \mathcal{I}}$ be a bounded below net in $\mathcal{P}$, and let $(\alpha_i)_{i \in \mathcal{I}}$ be a bounded net of non-negative reals such that $\lim_{i \in \mathcal{I}} \alpha_i > 0$. Then*

$$\left(\lim_{i \in \mathcal{I}} \alpha_i\right) \left(\underline{\lim}_{i \in \mathcal{I}} a_i\right) \leq \underline{\lim}_{i \in \mathcal{I}} (\alpha_i a_i) \leq \overline{\lim}_{i \in \mathcal{I}} (\alpha_i a_i) \leq \left(\lim_{i \in \mathcal{I}} \alpha_i\right) \left(\overline{\lim}_{i \in \mathcal{I}} a_i\right).$$

Corollary 8.4. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex complete lattice cone. Let $(a_i)_{i \in \mathcal{I}}$ and $(b_i)_{i \in \mathcal{I}}$ be order convergent nets in $\mathcal{P}$, and let $(\alpha_i)_{i \in \mathcal{I}}$ be a bounded net of non-negative reals such that $\lim_{i \in \mathcal{I}} \alpha_i > 0$. Then*

$$\lim_{i \in \mathcal{I}} (a_i + b_i) = \lim_{i \in \mathcal{I}} a_i + \lim_{i \in \mathcal{I}} b_i \qquad \textit{and} \qquad \lim_{i \in \mathcal{I}} (\alpha_i a_i) = \left(\lim_{i \in \mathcal{I}} \alpha_i\right) \left(\lim_{i \in \mathcal{I}} a_i\right).$$

This is an obvious consequence of our previous results 8.1 and 8.3. Note that the requirement that $\lim_{i \in \mathcal{I}} \alpha_i > 0$ may not be omitted if the elements of the net $(a_i)_{i \in \mathcal{I}}$ are not bounded in $\mathcal{P}$: In the locally convex complete lattice cone $\overline{\mathbb{R}}$ choose $a_n = n$ and $\alpha_n = (1/n)$. Then $\lim_{n \to \infty} (\alpha_n a_n) = 1$, but $\left(\lim_{n \to \infty} \alpha_n\right) \left(\lim_{n \to \infty} a_n\right) = 0 \cdot (+\infty) = 0$.

We proceed to investigate continuity of the lattice operations with respect to order convergence (c.f. Proposition 5.1).

Proposition 8.5. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex complete lattice cone and let $(a_i)_{i \in \mathcal{I}}$ and $(b_i)_{i \in \mathcal{I}}$ be convergent nets in $\mathcal{P}$. Then*

(*a*) $\lim_{i \in \mathcal{I}} (a_i \vee b_i) = \left(\lim_{i \in \mathcal{I}} a_i\right) \vee \left(\lim_{i \in \mathcal{I}} b_i\right).$

(*b*) $\overline{\lim}_{i \in \mathcal{I}} (a_i \wedge b_i) \leq \left(\lim_{i \in \mathcal{I}} a_i\right) \wedge \left(\lim_{i \in \mathcal{I}} b_i\right) \leq \underline{\lim}_{i \in \mathcal{I}} (a_i \wedge b_i) + \mathfrak{O}\left(\lim_{i \in \mathcal{I}} (a_i \vee b_i)\right).$

Proof. (*a*) Let $(a_i)_{i \in \mathcal{I}}$ and $(b_i)_{i \in \mathcal{I}}$ be convergent nets. Then

$$\overline{\lim}_{i \in \mathcal{I}} (a_i \vee b_i) = \inf_{i \in \mathcal{I}} \left(\sup_{l \geq i} (a_l \vee b_l)\right) \leq \inf_{i \in \mathcal{I}} \left(\left(\sup_{l \geq i} a_l\right) \vee \left(\sup_{j \geq i} b_j\right)\right).$$

Because for any choice of $i, k \in \mathcal{I}$ and any $p \in \mathcal{I}$ such that both $i \leq p$ and $k \leq p$ we have

$$\left(\sup_{l \geq p} a_l\right) \vee \left(\sup_{j \geq p} b_j\right) \leq \left(\sup_{l \geq i} a_l\right) \vee \left(\sup_{j \geq k} b_j\right),$$

we realize that

$$\inf_{i\in\mathcal{I}}\left(\left(\sup_{l\geq i} a_l\right)\vee\left(\sup_{j\geq i} b_j\right)\right)\leq \inf_{i,k\in\mathcal{I}}\left(\left(\sup_{l\geq i} a_l\right)\vee\left(\sup_{j\geq k} b_j\right)\right).$$

Now we use Proposition 7.5(a) for

$$\inf_{i,k\in\mathcal{I}}\left(\left(\sup_{l\geq i} a_l\right)\vee\left(\sup_{j\geq k} b_j\right)\right)=\inf_{i\in\mathcal{I}}\left(\sup_{l\geq i} a_l\right)\vee\inf_{k\in\mathcal{I}}\left(\sup_{j\geq k} b_j\right)=\left(\overline{\lim_{i\in\mathcal{I}}}\, a_i\right)\vee\left(\overline{\lim_{i\in\mathcal{I}}}\, b_i\right).$$

Both nets $(a_i)_{i\in\mathcal{I}}$ and $(b_i)_{i\in\mathcal{I}}$ are supposed to be convergent. So we have

$$\left(\overline{\lim_{i\in\mathcal{I}}}\, a_i\right)\vee\left(\overline{\lim_{i\in\mathcal{I}}}\, b_i\right)=\left(\underline{\lim_{i\in\mathcal{I}}}\, a_i\right)\vee\left(\underline{\lim_{i\in\mathcal{I}}}\, b_i\right)\leq\underline{\lim_{i\in\mathcal{I}}}(a_i\vee b_i).$$

Summarizing, the above yields

$$\overline{\lim_{i\in\mathcal{I}}}(a_i\vee b_i)\leq\left(\lim_{i\in\mathcal{I}} a_i\right)\vee\left(\lim_{i\in\mathcal{I}} b_i\right)\leq\underline{\lim_{i\in\mathcal{I}}}(a_i\vee b_i)$$

as claimed in part (a). Similarly, one verifies part (b): The inequality

$$\overline{\lim_{i\in\mathcal{I}}}(a_i\wedge b_i)\leq\left(\overline{\lim_{i\in\mathcal{I}}}\, a_i\right)\wedge\left(\overline{\lim_{i\in\mathcal{I}}}\, b_i\right)=\left(\lim_{i\in\mathcal{I}} a_i\right)\wedge\left(\lim_{i\in\mathcal{I}} b_i\right)$$

is obvious. Next we use part (a), Proposition 5.2 and the limit rules from Lemma 8.1 for

$$\begin{aligned}\left(\lim_{i\in\mathcal{I}} a_i\right)\wedge\left(\lim_{i\in\mathcal{I}} b_i\right)+\lim_{i\in\mathcal{I}}(a_i\vee b_i) &= \left(\lim_{i\in\mathcal{I}} a_i\right)\wedge\left(\lim_{i\in\mathcal{I}} b_i\right)+\left(\lim_{i\in\mathcal{I}} a_i\right)\vee\left(\lim_{i\in\mathcal{I}} b_i\right)\\ &= \lim_{i\in\mathcal{I}} a_i+\lim_{i\in\mathcal{I}} b_i=\lim_{i\in\mathcal{I}}(a_i+b_i)\\ &= \lim_{i\in\mathcal{I}}(a_i\wedge b_i+a_i\vee b_i)\\ &\leq \underline{\lim_{i\in\mathcal{I}}}(a_i\wedge b_i)+\overline{\lim_{i\in\mathcal{I}}}(a_i\vee b_i)\\ &= \underline{\lim_{i\in\mathcal{I}}}(a_i\wedge b_i)+\lim_{i\in\mathcal{I}}(a_i\vee b_i).\end{aligned}$$

Now the cancellation rule from Proposition 7.1(a) yields the remaining part of (b). □

Series 8.6. A series $\sum_{i=1}^{\infty} a_i$ with terms a_i in a locally convex complete lattice cone $(\mathcal{P},\mathcal{V})$ is said to be order convergent with limit $s\in\mathcal{P}$ if the sequence $s_n=\sum_{i=1}^{n} a_i$ of its partial sums is order convergent to s. We write $\sum_{i=1}^{\infty} a_i=s$ in this case.

Proposition 8.7. *Let $(\mathcal{P},\mathcal{V})$ be a locally convex complete lattice cone and let $a_i,b_i\in\mathcal{P}$ for $i\in\mathbb{N}$. If the series $\sum_{i=1}^{\infty} a_i$ is convergent and if $a_i\leq b_i$ for all $i\in\mathbb{N}$, then the series $\sum_{i=1}^{\infty} b_i$ is also convergent.*

Proof. Let $a_i, b_i \in \mathcal{P}$ such that $a_i \leq b_i$ for all $i \in \mathbb{N}$. Let $s_n = \sum_{i=1}^n a_i$ and $r_n = \sum_{i=1}^n b_i$ be the partial sums of the series $\sum_{i=1}^\infty a_i$ and $\sum_{i=1}^\infty b_i$, and let $s = \sum_{i=1}^\infty a_i$. Then $s_n \leq r_n$ for all $n \in \mathbb{N}$, hence $s \leq \underline{\lim}_{n\to\infty} r_n$. For $m \geq n$ we have

$$r_n + s_m = r_n + s_n + \sum_{i=n+1}^{m} a_i \leq r_n + s_n + \sum_{i=n+1}^{m} b_i = r_m + s_n.$$

For a fixed $n \in \mathbb{N}$ and $m \to \infty$ this leads to

$$r_n + s = r_n + \underline{\lim}_{m\to\infty} s_m = \underline{\lim}_{m\to\infty} (r_n + s_m) \leq \underline{\lim}_{m\to\infty} (s_n + r_m) = \underline{\lim}_{m\to\infty} r_m + s_n.$$

Now we let $n \to \infty$ and obtain

$$\begin{aligned} \overline{\lim}_{n\to\infty} r_n + s = \ \overline{\lim}_{n\to\infty} (r_n + s) &\leq \overline{\lim}_{n\to\infty} \left(\underline{\lim}_{m\to\infty} r_m + s_n \right) \\ &= \underline{\lim}_{m\to\infty} r_m + \overline{\lim}_{n\to\infty} s_n = \underline{\lim}_{m\to\infty} r_m + s. \end{aligned}$$

The cancellation law from Proposition 7.1(a) now yields

$$\overline{\lim}_{n\to\infty} r_n \leq \underline{\lim}_{n\to\infty} r_n + \mathfrak{O}(s).$$

But $s \leq \underline{\lim}_{n\to\infty} r_n$, as we observed before, and therefore $\underline{\lim}_{n\to\infty} r_n + \mathfrak{O}(s) = \underline{\lim}_{n\to\infty} r_n$ by Proposition 7.4. This yields

$$\overline{\lim}_{n\to\infty} r_n \leq \underline{\lim}_{n\to\infty} r_n,$$

hence convergence of the sequence $(r_n)_{n\in\mathbb{N}}$, that is the partial sums of the series $\sum_{i=1}^\infty b_i$. □

9. Order Continuous Linear Operators

Let $(\mathcal{P}, \mathcal{V})$ and $(\mathcal{Q}, \mathcal{W})$ be locally convex complete lattice cones. We shall say that a continuous linear operator $T : \mathcal{P} \to \mathcal{Q}$ is *order continuous* if it is continuous with respect to order convergence, that is if

$$T\left(\lim_{i\in\mathcal{I}} a_i\right) = \lim_{i\in\mathcal{I}} T(a_i)$$

holds for every order convergent net $(a_i)_{i\in\mathcal{I}}$ in $\mathcal{P}$. The limits refer to order convergence in $\mathcal{P}$ and $\mathcal{Q}$, respectively. For every bounded below net $(a_i)_{i\in\mathcal{I}}$ in $\mathcal{P}$ and every order continuous linear operator $T : \mathcal{P} \to \mathcal{Q}$ we have

$$T\left(\underline{\lim}_{i\in\mathcal{I}} a_i\right) = T\left(\lim_{i\in\mathcal{I}} \inf_{k\geq i} a_k\right) = \lim_{i\in\mathcal{I}} T\left(\inf_{k\geq i} a_k\right) \leq \lim_{i\in\mathcal{I}} \inf_{k\geq i} T(a_k) = \underline{\lim}_{i\in\mathcal{I}} T(a_i)$$

and, likewise

$$T\left(\overline{\lim}_{i\in I} a_i\right) = T\left(\lim_{i\in\mathcal{I}} \sup_{k\geq i} a_k\right) = \lim_{i\in\mathcal{I}} T\left(\sup_{k\geq i} a_k\right) \geq \lim_{i\in\mathcal{I}} \sup_{k\geq i} T(a_k) = \overline{\lim}_{i\in\mathcal{I}} T(a_i),$$

that is

$$T\left(\underline{\lim}_{i\in\mathcal{I}} a_i\right) \le \underline{\lim}_{i\in\mathcal{I}} T(a_i) \le \overline{\lim}_{i\in\mathcal{I}} T(a_i) \le T\left(\overline{\lim}_{i\in\mathcal{I}} a_i\right).$$

The order continuous linear functionals in $\mathcal{P}^*$, that is the order continuous linear operators from $\mathcal{P}$ into the completely ordered locally convex cone $\overline{\mathbb{R}}$form a subcone of $\mathcal{P}^*$.

10. Lattice Homomorphisms

Let both $(\mathcal{P},\mathcal{V})$ and $(\mathcal{Q},\mathcal{W})$ be locally convex upward (or downward) lattice cones. A continuous linear operator $T:\mathcal{P}\to\mathcal{Q}$ is a called *upward (or downward) lattice homomorphism* if it is compatible with the lattice operations in $\mathcal{P}$ and $\mathcal{Q}$, that is if

$$T(a\vee b) = T(a)\vee T(b) \qquad (\text{or} \quad T(a\wedge b) = T(a)\wedge T(b))$$

holds for all $a,b\in\mathcal{P}$. If $(\mathcal{P},\mathcal{V})$ and $(\mathcal{Q},\mathcal{W})$ are locally convex lattice cones and $T:\mathcal{P}\to\mathcal{Q}$ is both an upward and downward lattice homomorphism then T is called a *lattice homomorphism.* Linear operators that are both order continuous and lattice homomorphisms are of particular interest. Suppose that both $(\mathcal{P},\mathcal{V})$ and $(\mathcal{Q},\mathcal{W})$ are locally convex complete lattice cones. Because the supremum or infimum of any subset A of $\mathcal{P}$ is the limit with respect to order convergence of the net of suprema or infima of finite subsets of A, we have

$$T(\sup A) = \sup\big\{T(a)\mid a\in A\big\} \qquad \text{and} \qquad T(\inf B) = \inf\big\{T(b)\mid b\in B\big\}$$

for an order continuous lattice homomorphism $T:\mathcal{P}\to\mathcal{Q}$ and for all subsets A and bounded below subsets B of $\mathcal{P}$. This implies in particular that

$$T\left(\underline{\lim}_{i\in\mathcal{I}} a_i\right) = \underline{\lim}_{i\in\mathcal{I}} T(a_i) \qquad \text{and} \qquad T\left(\overline{\lim}_{i\in\mathcal{I}} a_i\right) = \overline{\lim}_{i\in\mathcal{I}} T(a_i)$$

holds for every bounded below net $(a_i)_{i\in\mathcal{I}}$ in $\mathcal{P}$.

Examples 10.1. (a) Theorem II.6.7 in [3] states that for every neighborhood $v\in\mathcal{V}$ in a locally convex upward (or downward) complete lattice cone $(\mathcal{P},\mathcal{V})$ all the extreme points of its polar $v^\circ\subset\mathcal{P}^*$ are upward (or downward) lattice homomorphisms from $\mathcal{P}$ into $\overline{\mathbb{R}}$.

(b) Let $(\mathcal{P},\mathcal{V})$ be a locally convex cone with dual $\mathcal{P}^*$ and let $(\mathcal{Q},\mathcal{V})$ be the cone of all non-empty convex subsets of $\mathcal{P}$ which are closed with respect to the lower topology (see Example 3.4(f)). In 6.3(d) we showed that $(\mathcal{Q},\mathcal{V})$ is a locally convex upward complete lattice cone ordered by the set inclusion. There is a natural embedding $\mu\mapsto\tilde{\mu}:\mathcal{P}^*\to\mathcal{Q}^*$, where

$$\tilde{\mu}(A) = \sup\{\mu(a)\mid a\in A\}$$

for $\mu\in\mathcal{P}^*$ and $A\in\mathcal{Q}$. Indeed, if $\mu\in v^\circ$ for some $v\in\mathcal{V}$, then $A\le B\oplus v$ for $A,B\in\mathcal{Q}$ means that for every $a\in A$ and $\varepsilon\ge 0$ there is $b\in B$ such that $a\le\gamma b+(1+\varepsilon)v$ (see 3.4(g)) with some $1\le\gamma\le 1+\varepsilon$. This yields $\mu(a)\le\gamma\mu(b)+(1+\varepsilon)\le\gamma\tilde{\mu}(b)+(1+\varepsilon)$ for all $\varepsilon>0$, hence $\mu(a)\le\tilde{\mu}(B)+1$. This yields $\tilde{\mu}(A)\le\tilde{\mu}(B)+1$, and therefore $\tilde{\mu}\in v^\circ\subset\mathcal{Q}^*$. Moreover, $\tilde{\mu}$ is an upward lattice homomorphism even with respect to arbitrary suprema in $\mathcal{Q}$: Let $\mathcal{A}$ be a subset of $\mathcal{Q}$ and let c be an element of $conv\left(\bigcup_{A\in\mathcal{A}} A\right)$, the convex hull of the union of all elements of $\mathcal{A}$. Then $c = \sum_{i=1}^{n} \alpha_i a_i$ for some $a_i \in A_i \in \mathcal{A}$ and $\alpha_i \geq 0$ such that $\sum_{i=1}^{n} \alpha_i = 1$. Thus

$$\mu(c) = \sum_{i=1}^{n} \alpha_i \mu(a_i) \leq \sum_{i=1}^{n} \alpha_i \tilde{\mu}(A_i) \leq \sup_{A\in\mathcal{A}} \tilde{\mu}(A).$$

Since the functional $\mu : \mathcal{P} \to \overline{\mathbb{R}}$ is also continuous with respect to the lower relative topology on $\mathcal{P}$, we can conclude that

$$\begin{aligned}\tilde{\mu}(\sup \mathcal{A}) &= \sup\left\{ \mu(a) \;\middle|\; a \in \overline{conv\left(\bigcup_{A\in\mathcal{A}} A\right)}^{(l)} \right\} \\ &= \sup\left\{ \mu(a) \;\middle|\; a \in conv\left(\bigcup_{A\in\mathcal{A}} A\right) \right\} \leq \sup_{A\in\mathcal{A}} \tilde{\mu}(A).\end{aligned}$$

The converse inequality is obvious.

(c) Similarly one argues for the locally convex cone $(\mathcal{Q}, \mathcal{V})$ of all bounded below convex subsets of $\mathcal{P}$ which are closed with respect to the upper topology (see Examples 3.4(f) and 6.3(e)). $(\mathcal{Q}, \mathcal{V})$ is a locally convex downward complete lattice cone, ordered by the inverse set inclusion. There is a natural embedding $\mu \mapsto \tilde{\mu} : \mathcal{P}^* \to \mathcal{Q}^*$, where

$$\tilde{\mu}(A) = \inf\{\mu(a) \mid a \in A\}$$

for $\mu \in \mathcal{P}^*$ and $A \in \mathcal{Q}$. As similar argument as in (b) shows that $\tilde{\mu}(\inf \mathcal{A}) = \inf_{A\in\mathcal{A}} \tilde{\mu}(A)$ holds for every bounded below family of sets $\mathcal{A} \subset \mathcal{Q}$.

(d) Let $(\mathcal{P}, \mathcal{V})$ be a locally convex upward (or downward) lattice cone, X a set, and consider the locally convex cone $\left(\mathcal{F}_{b_\mathcal{Y}}(X, \mathcal{P}), \mathcal{V}_\mathcal{Y}\right)$ of $\mathcal{P}$-valued functions on X endowed with the topology of uniform convergence on the sets in a family $\mathcal{Y}$ of subsets of X (Example 2.1(d)). This was seen to be again an upward (or downward) lattice cone. For $\mu \in \mathcal{P}^*$ and $x \in Y$ for some $Y \in \mathcal{Y}$ the mapping $\mu_x : \mathcal{F}_{b_\mathcal{Y}}(X, \mathcal{P}) \to \overline{\mathbb{R}}$ such that $\mu_x(f) = \mu(f(x))$ for all $\in \mathcal{F}_{b_\mathcal{Y}}(X, \mathcal{P})$ is a continuous linear functional on $\mathcal{F}_{b_\mathcal{Y}}(X, \mathcal{P})$. Moreover, if μ is an upward (or downward) lattice homomorphism for $\mathcal{P}$, then μ_x is a lattice homomorphism of the same type for $\mathcal{F}_{b_\mathcal{Y}}(X, \mathcal{P})$.

11. Comparison of Topologies

We shall proceed probing different patterns of convergence in a locally convex complete lattice cone $(\mathcal{P}, \mathcal{V})$. For a net $(a_i)_{i\in\mathcal{I}}$ in $\mathcal{P}$, *convergence with respect to the symmetric relative topology* of $\mathcal{P}$ towards $a \in \mathcal{P}$ means that for every $v \in \mathcal{V}$ and $\varepsilon > 0$ there is $i_0 \in I$ such that $a_i \in v_\varepsilon^s(a)$ for all $i \geq i_0$. $(a_i)_{i\in\mathcal{I}}$ is a *Cauchy net* if for every $v \in \mathcal{V}$ and $\varepsilon > 0$ there is $i_0 \in I$ such that $a_i \in v_\varepsilon(a_k)$ for all $i, k \geq i_0$. Obviously, convergence implies that $(a_i)_{i\in\mathcal{I}}$ is a Cauchy net. The converse, that is topological completeness holds also true:

Proposition 11.1. *Every locally convex complete lattice cone is complete with respect to the symmetric relative topology.*

Proof. Suppose that $(a_i)_{i\in\mathcal{I}}$ is a Cauchy net in $\mathcal{P}$. We shall first demonstrate that $(a_i)_{i\in\mathcal{I}}$ is order convergent. Let $v \in \mathcal{V}$ and $0 < \varepsilon \le 1$. There is $i_0 \in \mathcal{I}$ such that $a_i \in v_\varepsilon(a_k)$ for all $i, k \ge i_0$. Choose $\lambda \ge 0$ such that $0 \le a_{i_0} + \lambda v$. Following Lemma 3.3(*b*) and (*c*) this implies

$$a_i \le (1+\varepsilon)a_{i_0} + \varepsilon(1+\lambda)v \qquad \text{and} \qquad a_{i_0} \le (1+\varepsilon)a_i + \varepsilon(2+\lambda)v$$

for all $i \ge i_0$. This shows in particular that $(a_i)_{i\in\mathcal{I}}$ is bounded below and also that

$$a_i \le (1+\varepsilon)^2 a_k + 3\varepsilon(2+\lambda)v$$

for all $i, k \ge i_0$. We infer that

$$\overline{\lim_{i\in\mathcal{I}}}\, a_i \le (1+\varepsilon)^2 \underline{\lim_{k\in I}}\, a_k + 3\varepsilon(2+\lambda)v.$$

As this holds for all $v \in \mathcal{V}$ and $0 < \varepsilon \le 1$, and as $\mathcal{Q}$ carries the weak preorder which is supposed to be antisymmetric, we infer that $\overline{\lim}_{i\in\mathcal{I}}\, a_i = \underline{\lim}_{k\in I}\, a_k$, hence order convergence towards an element $a \in \mathcal{P}$. Moreover, the above shows that

$$a_i \le (1+\varepsilon^2)a + 3\varepsilon(2+\lambda)v \qquad \text{and} \qquad a \le (1+\varepsilon^2)a_i + 3\varepsilon(2+\lambda)v$$

holds for all $i \ge i_0$. Thus the net $(a_i)_{i\in\mathcal{I}}$ converges to a in the symmetric relative topology as well. □

In fact, we just verified that every Cauchy net, hence every convergent net in the symmetric relative topology of $(\mathcal{P}, \mathcal{V})$ is indeed order convergent with the same limit. We shall formulae this as a separate proposition:

Proposition 11.2. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex complete lattice cone. Convergence of a net $(a_i)_{i\in\mathcal{I}}$ in $\mathcal{P}$ towards $a \in \mathcal{P}$ in the symmetric relative topology implies order convergence towards a.*

While convergence in the symmetric relative topology implies order convergence, the converse is not necessarily true, as a simple example may show: In the completely ordered locally convex cone $\overline{\mathbb{R}}$ order convergence means convergence in the usual (one-point compactification) topology of $\overline{\mathbb{R}}$ which for the element $+\infty$ does not coincide with the symmetric relative topology of $\overline{\mathbb{R}}$. The sequence $(n)_{n\in\mathbb{N}}$, for example, is order convergent towards $+\infty \in \overline{\mathbb{R}}$, but does not converge in the symmetric relative topology, as $+\infty$ is an isolated point in this topology.

11.1. Order topology

While order convergence in a locally convex complete lattice cone $(\mathcal{P}, \mathcal{V})$ does not necessarily correspond to a topology on $\mathcal{P}$ in the sense that order and topological convergence for nets coincide (see 1.1.9 in [4]), there is a finest topology $\mathcal{O}(\mathcal{P})$ on $\mathcal{P}$ with the following properties (see also V.6 in [10]):

(OT1) *Every very element in $\mathcal{P}$ admits a basis of both convex and order convex neighborhoods. For every neighborhood U of $0 \in \mathcal{P}$ and every bounded element $a \in \mathcal{P}$ there is $\varepsilon > 0$ such that $\varepsilon a \in U$. If $a \in U$ is invertible, then $-a \in U$ as well.*

(OT2) *The mappings* $(a,b) \mapsto a+b : \mathcal{P}^2 \to \mathcal{P}$ *and* $(\alpha,a) \mapsto \alpha a : (0,+\infty) \times \mathcal{P} \to \mathcal{P}$ *are continuous.*

(OT3) *All order convergent nets in* $\mathcal{P}$ *are topologically convergent with the same limit.*

Indeed, the family $\mathfrak{T}$ of all topologies on $\mathcal{P}$ with these properties is not empty as it contains the discrete topology. Let $\mathcal{O}(\mathcal{P})$ be the supremum of this family in the lattice of topologies on $\mathcal{P}$. A neighborhood basis in $\mathcal{O}(\mathcal{P})$ for a point $a \in \mathcal{P}$ is generated by the intersections of finitely many neighborhoods for a taken from topologies in $\mathfrak{T}$. This shows that $\mathcal{O}(\mathcal{P})$ again satisfies (OT1) to (OT3), hence is the finest topology with these properties. We shall call $\mathcal{O}(\mathcal{P})$ the *(strong) order topology* on $\mathcal{P}$. It is easy to verify that the symmetric relative topology of $\mathcal{P}$ satisfies (OT1) and (OT2), however not (OT3) in general. To the contrary, we observed earlier in 11.2 that convergence for a net in the symmetric relative topology implies order convergence, hence convergence in $\mathcal{O}(\mathcal{P})$. Since the closure in any topology of a given subset A of $\mathcal{P}$ can be described as the set of all limit points of convergent nets in this subset, Proposition 11.2 implies that the closure of A with respect to the symmetric relative topology is contained in the closure of A with respect to $\mathcal{O}(\mathcal{P})$. We infer that $\mathcal{O}(\mathcal{P})$ is generally coarser than the symmetric relative topology. Note that$\mathcal{O}(\mathcal{P})$ is however not necessarily a locally convex cone topology. For $\mathcal{P} = \overline{\mathbb{R}}$, for example, the order topology is the usual topology of $\overline{\mathbb{R}}$, where $+\infty$ is not an isolated point.

We observe that Conditions (OT1) to (OT3) imply the following for $\mathcal{O}(\mathcal{P})$:

(OT4) *Let* $\mathcal{P}_0$ *be the subcone of all invertible elements of* $\mathcal{P}$. *The mapping* $(\alpha,a) \mapsto \alpha a : \mathbb{R} \times \mathcal{P}_0 \to \mathcal{P}_0$ *is continuous.*

We shall make this argument in several short steps: First suppose that $a_i \to 0$ for $a_i \in \mathcal{P}_0$ in any topology satisfying (OT1) to (OT3). Given a neighborhood U of 0 there is i_0 such that $a_i \in U$ for all $i \geq i_0$. This implies $-a_i \in U$ as well by the last part of (OT1). Thus $(-a_i) \to 0$. Next suppose that $a_i \to a$ for $a_i, a \in \mathcal{P}_0$. Then $(a_i + (-a)) \to 0$ by (OT2), hence $((-a_i) + a) \to 0$ by the preceding step, and $(-a_i) \to (-a)$ by (OP2). In a third step, suppose that $\alpha_i \to 0$ for $0 \leq a_i \in \mathbb{R}$ and $a_i \to a$ for $a_i, a \in \mathcal{P}_0$. Given a neighborhood U of $0 \in \mathcal{P}$ there is a second neighborhood V of $0 \in \mathcal{P}$ such that U is a neighborhood for all elements of V. Following (OT1) there is $\varepsilon > 0$ such that $\varepsilon a \in V$, and according to (OT2) we have $(\alpha_i + \varepsilon)a_i \to \varepsilon a$. As U is a neighborhood of εa, there is an index i_0 such that $(\alpha_i+\varepsilon)a_i \in U$ for all $i \geq i_0$. Now the convexity of the neighborhood U guarantees that $\alpha_i a_i \in U$ holds as well. This demonstrates that $\alpha_i a_i \to 0$. Summarizing, in combination with (OT2) we have $\alpha_i a_i \to \alpha a$ whenever $\alpha_i \to \alpha$ for $0 \leq \alpha_i \in \mathbb{R}$ and $a_i \to a$ for $a_i, a \in \mathcal{P}_0$. Now in the fourth and final step of our argument, let $\alpha_i \to \alpha$ in $\mathbb{R}$ and $a_i \to a$ for $a_i, a \in \mathcal{P}_0$. Let $\beta_i = \alpha_i \vee 0$ and $\gamma_i = -(\alpha_i \wedge 0)$. Then $\beta_i, \gamma_i \geq 0$ and $\alpha_i = \beta_i - \gamma_i$. We have $\beta_i a_i \to \beta a$ and $\gamma_i(-a_i) \to \gamma(-a)$, where $\beta = \alpha \vee 0$ and $\gamma = -(\alpha \wedge 0)$, by the second and third steps of our argument. Thus

$$\alpha_i a_i = \beta_i a_i + \gamma_i(-a_i) \;\to\; \beta a + \gamma(-a) = \alpha a,$$

again by (OT2), as claimed.

12. Extensions of Linear Operators

A short inspection of the Hahn-Banach type extension results for linear functionals in [6] (see also Section 2 before) shows that they are still valid if the range $\overline{\mathbb{R}}$ for the functionals is replaced by some locally convex cone $(\mathcal{Q},\mathcal{W})$, provided that

(i) $(\mathcal{Q},\mathcal{W})$ is a full locally convex complete lattice cone,
(ii) all elements of $\mathcal{Q}$, with the exception of the element $+\infty = \sup\mathcal{Q}$, are invertible,
(iii) the neighborhood system $\mathcal{W}$ consist of all (strictly) positive multiples of a single neighborhood $w \in \mathcal{W}$.

Requirement (ii) means of course that $\mathcal{Q}$ is a Dedekind complete Riesz space with an adjoint maximal element $+\infty$. Results about the extension of monotone linear operators between vector spaces and Dedekind complete Riesz spaces are due to Kantorovič ([1] and [2]) and may for example be found in Section 1.5 of [4]. Without furnishing the details of this, we reformulate Corollary 4.1 in [6] (see also Corollary 2.5).

Theorem 12.1. *Let $(\mathcal{N},\mathcal{V})$ be a subcone of the locally convex cone $(\mathcal{P},\mathcal{V})$. Suppose that $(\mathcal{Q},\mathcal{W})$ is a full locally convex complete lattice cone, that all elements of $\mathcal{Q}$ other than $+\infty$ are invertible, and that $\mathcal{W} = \{\alpha w \mid \alpha > 0\}$ for some $w \in \mathcal{W}$. Then every continuous linear operator $T : \mathcal{N} \to \mathcal{Q}$ can be extended to a continuous linear operator $\overline{T} : \mathcal{P} \to \mathcal{Q}$.*

Unfortunately, a similar result is not generally available if the completely ordered locally convex cone $(\mathcal{Q},\mathcal{W})$ does not meet the stringent additional requirements of Theorem 12.1. However, we have the following:

Theorem 12.2. *Let $\mathcal{N}$ be a subcone of the locally convex cone $(\mathcal{P},\mathcal{V})$ and let $(\mathcal{Q},\mathcal{W})$ be a locally convex complete lattice cone. Every continuous linear operator $T : \mathcal{N} \to \mathcal{Q}$ can be uniquely extended to $\overline{\mathcal{N}}$, the closure of $\mathcal{N}$ in $\mathcal{P}$ with respect to the symmetric relative topology.*

Proof. Let $T : \mathcal{N} \to \mathcal{Q}$ be a continuous linear operator and let $a \in \overline{\mathcal{N}}$. There is a net $(a_i)_{i\in\mathcal{I}}$ in $\mathcal{N}$ converging to a in the symmetric relative topology. Given $w \in \mathcal{W}$ and $\varepsilon > 0$ there is $v \in \mathcal{V}$ such that $T(b) \le T(c) + w$ whenever $b \le c + v$ for $b, c \in \mathcal{N}$. Because $(a_i)_{i\in\mathcal{I}}$ is a Cauchy net in $\mathcal{N}$, there is $i_0 \in \mathcal{I}$ such that $a_i \in v_\varepsilon(a_k)$ for all $i, k \ge i_0$. This implies $T(a_i) \in w_\varepsilon(T(a_k))$ for all $i, k \ge i_0$, hence $(T(a_i))_{i\in\mathcal{I}}$ is a Cauchy net in $\mathcal{Q}$ as well. Proposition 11.1 shows that this net converges in $\mathcal{Q}$. Moreover, if $(b_j)_{j\in\mathcal{J}}$ is a second net in $\mathcal{N}$ converging toward the same element a, given $w \in \mathcal{W}$ and $\varepsilon > 0$ we choose $v \in \mathcal{V}$ as above and find $i_0 \in \mathcal{I}$ and $j_0 \in \mathcal{J}$ such that both $a_i \in v_\varepsilon(b_j)$ and $b_j \in v_\varepsilon(a_i)$, hence $T(a_i) \in w_\varepsilon(T(b_j))$ and $T(b_j) \in v_\varepsilon(T(a_i))$, for all $i \ge i_0$ and $j \ge j_0$. This shows that both nets $(T(a_i))_{i\in\mathcal{I}}$ and $(T(b_j))_{j\in\mathcal{J}}$ have the same limit in $\mathcal{Q}$ which we denote $T(a)$. It is now straightforward to verify that this procedure results in a bounded linear extension $\overline{T} : \overline{\mathcal{N}} \to \mathcal{Q}$ of the operator T. Uniqueness of this extension is obvious. □

13. The Lattice Completion of a Locally Convex Cone

Every locally convex cone $(\mathcal{P},\mathcal{V})$ can be canonically embedded into a full locally convex complete lattice cone. For this, we use the representation for $(\mathcal{P},\mathcal{V})$ as a cone of $\overline{\mathbb{R}}$-valued functions on its dual cone $\mathcal{P}^*$, that is, with the element $a \in \mathcal{P}$ we associate the function φ_a on $\mathcal{P}^*$ such that $\varphi_a(\mu) = \mu(a)$. We use the pointwise algebraic operations and order for functions on $\mathcal{P}^*$. Corresponding to the neighborhoods $v \in \mathcal{V}$ we consider the $\overline{\mathbb{R}}$-valued functions ψ_v on $\mathcal{P}^*$ such that

$$\psi_v(\mu) = \inf\{\alpha > 0 \mid \mu \in \alpha v^\circ\}$$

for all $\mu \in \mathcal{P}^*$ (as usual, we set $\inf \emptyset = +\infty$.) By $\widehat{V}$ we denote the family of all functions $\sum_{i=1}^{n} \psi_{v_i}$ for $v_i \in \mathcal{V}$. An $\overline{\mathbb{R}}$-valued function φ on $\mathcal{P}^*$ is bounded below relative to $\widehat{\mathcal{V}}$ if for every $\psi \in \widehat{\mathcal{V}}$ there is $\lambda \geq 0$ such that $0 \leq \varphi + \lambda\psi$. Let $\widehat{\mathcal{P}}$ denote the cone of all $\overline{\mathbb{R}}$-valued functions φ on $\mathcal{P}^*$ that are bounded below relative to $\widehat{\mathcal{V}}$ and positive homogeneous, that is $\varphi(\rho\mu) = \rho\varphi(\mu)$ holds for all $\mu \in \mathcal{P}^*$ and $\rho \geq 0$. Then $(\widehat{\mathcal{P}}, \widehat{\mathcal{V}})$ is a full locally convex cone. We claim that $a \preccurlyeq b + v$ holds for $a, b \in \mathcal{P}$ and $v \in \mathcal{V}$ if and only if $\varphi_a \leq \varphi_b + \psi_v$. Indeed, suppose that $a \preccurlyeq b + v$. Then for $\mu \in \mathcal{P}^*$ and $\alpha > 0$ such that $\mu \in \alpha v^\circ$ we have $\mu(a) \leq \mu(b) + \alpha$, hence $\varphi_a \leq \varphi_b + \psi_v$. Conversely, if $a \not\preccurlyeq b + v$, then following Theorem 3.2 in [7] there is $\mu \in v^\circ \subset \mathcal{P}^*$ such that $\mu(a) > \mu(b) + 1$. The former implies $\psi_v(\mu) \leq 1$, hence $\varphi_a(\mu) > \varphi_b(\mu) + \psi_v(\mu)$. We infer in particular that the functions φ_a are contained in $\widehat{\mathcal{P}}$ for all $a \in \mathcal{P}$. Indeed, given $v \in \mathcal{V}$ there is $\lambda \geq 0$ such that $0 \leq a + \lambda v$, hence $0 \leq \varphi_a + \lambda\psi_a$. Since every neighborhood in $\mathcal{V}$ is a sum of functions ψ_{v_i}, the function φ_a is bounded below relative to $\widehat{\mathcal{V}}$ as claimed. The canonical embedding

$$a \mapsto \varphi_a : \mathcal{P} \to \widehat{\mathcal{P}}.$$

is linear and by the preceding investigations preserves the neighborhoods with respect to the weak preorder of $\mathcal{P}$. The mapping $v \mapsto \psi_v : \mathcal{V} \to \widehat{\mathcal{P}}$ is however not linear. But we obviously have $\psi_v \leq \psi_u$ whenever $v \leq u$, $\max\{\psi_v, \psi_u\} \leq \psi_{(v+u)}$ and $\psi_{(\alpha v)} = \alpha\psi_v$ for all $u, v \in \mathcal{V}$ and $\alpha > 0$. Together with the above, this yields that the locally convex cone topology induced by the neighborhood system $\widehat{\mathcal{V}}$ on the embedding of $\mathcal{P}$ into $\widehat{\mathcal{P}}$ is equivalent to the given topology on $\mathcal{P}$ which is induced by $\mathcal{V}$, if considered with respect to the weak preorder.

It is straightforward to check that the full locally convex cone $(\widehat{\mathcal{P}}, \widehat{\mathcal{V}})$ is indeed a complete lattice cone. Because infima and suprema of subsets of $\widehat{\mathcal{P}}$, that is sets of $\overline{\mathbb{R}}$-valued functions on $\mathcal{P}^*$, are formed pointwise, it follows that order convergence for bounded below nets in $\widehat{\mathcal{P}}$ is indeed pointwise convergence (with respect to the usual, that is the order topology of $\overline{\mathbb{R}}$) of the involved functions. An unbounded below net in $\widehat{\mathcal{P}}$, however, may be pointwise convergent without being order convergent. This phenomenon is remedied by the use of the order topology $\mathcal{O}(\mathcal{P})$. Indeed, let $(\varphi_i)_{i \in \mathcal{I}}$ be a (not necessarily bounded below) net in $\widehat{\mathcal{P}}$ converging pointwise to $\varphi \in \widehat{\mathcal{P}}$. Let $\psi_i = \varphi_i \vee 0 \in \widehat{\mathcal{P}}$ and $\omega_i = -(\varphi_i \wedge 0) \in \widehat{\mathcal{P}}_{\Upsilon}$ Then $\psi_i, \omega_i \geq 0$, the elements ω_i are invertible in $\widehat{\mathcal{P}}$, and we have $\varphi_i = \psi_i - \omega_i$. Moreover, the nets $(\psi_i)_{i \in \mathcal{I}}$ and $(\omega_i)_{i \in \mathcal{I}}$ are bounded below and are pointwise, hence order convergent towards $\psi = \varphi \vee 0 \in \widehat{\mathcal{P}}$ and $\omega = -(\varphi \wedge 0) \in \widehat{\mathcal{P}}$. This implies convergence in the order topology $\mathcal{O}(\mathcal{P})$ by (OT3). (OT4) yields that the net $(-\omega_i)_{i \in \mathcal{I}}$ converges to $-\omega \in \widehat{\mathcal{P}}$ with respect to $\mathcal{O}(\mathcal{P})$. Thus $\varphi_i = \psi_i + (-\omega_i)$ converges to $\psi + (-\omega) = \varphi$ in $\mathcal{O}(\mathcal{P})$ by (OT2). Summarizing, pointwise convergence for any net in $\widehat{\mathcal{P}}$ implies convergence in the order topology. The topology of pointwise convergence is therefore finer than $\mathcal{O}(\mathcal{P})$. The topology of pointwise convergence, on the other hand, satisfies (OT1), (OT2) and (OT3), and is therefore also coarser than $\mathcal{O}(\mathcal{P})$. Thus both topologies coincide and are obviously Hausdorff.

We call $(\widehat{\mathcal{P}}, \widehat{\mathcal{V}})$ the *standard lattice completion* of the locally convex cone $(\mathcal{P}, \mathcal{V})$. It is distinguished in the following way: Let $\Phi : \mathcal{P} \to \widehat{\mathcal{P}}$, that is $a \mapsto \varphi_a$ be the embedding of $\mathcal{P}$ into $\widehat{\mathcal{P}}$ as established above, and suppose that $(\mathcal{Q}, \mathcal{W})$ is another locally convex cone with an antisymmetric order, and that there is a similar isomorphic embedding Ψ of $\mathcal{P}$ into $\mathcal{Q}$. Let A be a subset of $\mathcal{P}$ such that $\sup\Psi(A)$ exists in $\mathcal{Q}$, and let us compare this to $\sup\Phi(A)$ taken in $\widehat{\mathcal{P}}$. For this let $\mu\in\mathcal{P}^*$. Let $\varepsilon_\mu\in\widehat{\mathcal{P}}^*$ be the point evaluation at μ, that is $\varepsilon_\mu(\varphi)=\varphi(\mu)$ for all $\varphi\in\widehat{\mathcal{P}}$, and let $\tilde{\mu}\in\mathcal{Q}^*$ be any extension of μ to a functional on $\mathcal{Q}$, that is $\tilde{\mu}(\Psi(a))=\varepsilon_\mu(\Phi(a))=\mu(a)$ for all $a\in\mathcal{P}$. Because the lattice operations for the functions in $\widehat{\mathcal{P}}$ are performed pointwise on the elements of $\mathcal{P}^*$ we have

$$\varepsilon_\mu(\sup\Phi(A))=\sup\big\{\Phi(a)(\mu)\mid a\in A\big\}=\sup\big\{\Psi(a)(\mu)\mid a\in A\big\}\le\tilde{\mu}(\sup\Psi(A)).$$

If $\inf\Psi(A)$ exists in $\mathcal{Q}$ a similar argument yields

$$\varepsilon_\mu(\inf\Phi(A))=\inf\big\{\Phi(a)(\mu)\mid a\in A\big\}=\inf\big\{\Psi(a)(\mu)\mid a\in A\big\}\ge\tilde{\mu}(\inf\Psi(A)).$$

In this sense $(\widehat{\mathcal{P}},\widehat{\mathcal{V}})$ is a locally convex complete lattice cone extension which yields the smallest possible suprema and the largest possible infima of subsets of $\mathcal{P}$. In particular, if $\sup A$ (or $\inf A$) for a subset A of $\mathcal{P}$ is already contained as an element of $\mathcal{P}$, then $\sup\Phi(A)\le\Phi(\sup A)$ (or $\inf\Phi(A)\ge\Phi(\inf A)$) in $\widehat{\mathcal{P}}$.

It is often preferable to realize the lattice completion of a locally convex cone $(\mathcal{P},\mathcal{V})$ as a cone of $\overline{\mathbb{R}}$-valued functions on a suitable subset of $\mathcal{P}^*$ rather than on the whole of $\mathcal{P}^*$. A subset Υ of $\mathcal{P}^*$ *supports the separation property* if for $a,b\in\mathcal{P}$ and $v\in\mathcal{V}$ such that $a\not\preccurlyeq b+v$ there is $\alpha\ge 0$ and $\mu\in\Upsilon\cap\alpha v^\circ$ such that $\mu(a)>\mu(b)+\alpha$. Following Theorem 3.2 in [7] this holds of course true for $\Upsilon=\mathcal{P}^*$. Let us denote by $\widehat{\mathcal{P}}_\Upsilon$ and $\widehat{\mathcal{V}}_\Upsilon$ the restrictions to Υ of the functions in $\widehat{\mathcal{P}}$ and $\widehat{\mathcal{V}}$. Then $(\widehat{\mathcal{P}}_\Upsilon,\widehat{\mathcal{V}}_\Upsilon)$ is again a full locally convex complete lattice cone and the restriction map $\Lambda:\widehat{\mathcal{P}}\to\widehat{\mathcal{P}}_\Upsilon$ is an isomorphism on the embedding of $\mathcal{P}$. Indeed, if $\varphi_a\le\varphi_b+\psi_v$ for $a,b\in\mathcal{P}$ and $v\in\mathcal{V}$, then $\Lambda(\varphi_a)\le\Lambda(\varphi_b)+\Lambda(\psi_v)$ holds as well in $\widehat{\mathcal{P}}_\Upsilon$. Conversely, if $\varphi_a\not\le\varphi_b+\psi_v$, then $a\not\preccurlyeq b+v$ and by our assumption there is $\alpha\ge 0$ and $\mu\in\Upsilon\cap\alpha v^\circ$ such that $\mu(a)>\mu(b)+\alpha$. Then $\psi_v(\mu)\le\alpha$, hence $\mu(a)>\mu(b)+\psi_v(\mu)$ and $\Lambda(\varphi_a)\not\le\Lambda(\varphi_b)+\Lambda(\psi_v)$ in $\widehat{\mathcal{P}}_\Upsilon$. Since the lattice operations are performed pointwise, we have $\Lambda(\sup A)=\sup(\Lambda(A))$ for every non-empty subset A of $\widehat{\mathcal{P}}$ and $\Lambda(\inf A)=\inf(\Lambda(A))$ for every non-empty bounded below subset A of $\widehat{\mathcal{P}}$. Let us illustrate in two simple examples how a suitable subset set $\Upsilon\subset\mathcal{P}^*$ can be chosen:

(i) If $\mathcal{V}$ consists of the multiples of a single neighborhood v, that is for example, if $\mathcal{P}$ is normed vector space, then we may choose $\Upsilon=\{\mu\in\mathcal{P}^*\mid\psi_v(\mu)=1\}$, that is the dual unit sphere. The lattice completion $\widehat{\mathcal{P}}_\Upsilon$ of $\mathcal{P}$ then consists of all $\overline{\mathbb{R}}$-valued bounded below functions on Υ, endowed with the topology of uniform convergence.

(ii) For a second example let X be a compact set and let $\mathcal{P}=\mathcal{C}(X)$ be the space of all continuous real-valued functions on X, endowed with the pointwise operations and order. The neighborhood system $\mathcal{V}$ consisting of all positive constants generates the topology of uniform convergence. Then the set of all point evaluations ε_x for $x\in X$ is a suitable choice for Υ, rather than the whole dual $\mathcal{P}^*$ of $\mathcal{P}$ which consists of all positive regular Borel measures on X. Υ obviously supports the separation property, and the standard lattice completion of $(\mathcal{P},\mathcal{V})$ can be realized as a cone of $\overline{\mathbb{R}}$-valued functions on X.

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