Journal of Convex Analysis
Volume 16 (2009), No. 1, 239–260

Semiconvex Functions: Representations as Suprema of Smooth Functions and Extensions

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Received: October 25, 2007
Revised manuscript received: March 3, 2008

We prove results on representations of semiconvex functions with an arbitrary modulus (equivalently: strongly paraconvex functions) in superreflexive Banach spaces as suprema of families of differentiable functions. Also, results on extensions of semiconvex functions are proved. Further, characterizations of semiconvex functions by uniform Fréchet subdifferentiability and (global) $[\alpha]$-subdifferentiability are given. We also show that weakly convex functions in Nurminskii's sense coincide with locally semiconvex functions.

Keywords: Semiconvex function, strongly paraconvex function, generalized subdifferentials, suprema of smooth functions

2000 Mathematics Subject Classification: Primary 26B25; Secondary 46T99

1. Introduction

The notion of a convex function has been generalized in many different ways. One direction of generalizations yields a class of semiconvex functions with an arbitrary modulus ([2], [1]) and a wider class of approximately convex functions in the sense of [21] (different from the older class of ε-convex functions in the sense of Hyers and Ulam). In this article, we will consider mainly real continuous functions defined on an open convex subset of a Banach space; in particular, the following remarks concern such functions.

Note that, on a Banach space, strongly paraconvex functions [26] essentially coincide with semiconvex functions (see Remark 2.11 below), and uniformly Fréchet subdifferentiable functions ([33] and [18]) coincide with semiconvex functions (see Theorem 4.12 below). Lower C^2 functions [25] and weakly convex functions in Vial's sense [31] coincide in $\mathbb{R}^n$ with locally semiconvex functions with linear modulus. Further, lower C^1 functions in $\mathbb{R}^n$ [30] coincide with approximately convex functions [8] (i.e., with locally semiconvex functions, see [35, Remark 2.6]), and also with weakly convex functions in Nurminskii's

*The research of the second author was partially supported by the grant GAČR 201/06/0198 from the Grant Agency of Czech Republic and partially supported by the grant MSM 0021620839 from the Czech Ministry of Education.

ISSN 0944-6532 / $ 2.50

sense [23] (see Corollary 4.15 below).

The definitions of the above mentioned classes are of one of the following three types:

(a) *"Inequality definition"*, which weakens the classical inequality from the definition of convex functions (used for definitions of semiconvexity, strong paraconvexity, weak convexity in Vial's sense, and approximate convexity).
(b) *"Supremum definition"*, which weakens the alternative definition of a convex function as a supremum of a family of affine functions, using more general families of smooth functions (used for definitions of lower C^1 and lower C^2 functions).
(c) *"Subdifferentiability definition"*, which weakens the alternative definition of a convex function as a subdifferentiable function, using more general notions of subgradient (used for definitions of weakly convex functions in Nurminskii's sense, of uniformly Fréchet subdifferentiable functions, and of uniformly upper subdifferentiable functions [20], [19]).

Semiconvex functions (with a modulus φ) are defined (see Definition 2.9) using an "inequality definition". *The main aim of the present article is to study the possibility of characterizing these functions in Banach spaces using versions of the "supremum definition".*

Our results generalize those contained in [7] and [9]. A result from [9] says that locally semiconvex functions in $\mathbb{R}^n$ with modulus of the form $\varphi(t) = Ct^\alpha$ $(0 < \alpha \leq 1)$ can also be equivalently defined by the "supremum definition". (The case $\alpha = 1$ is implicitly contained in [25] for functions in $\mathbb{R}^n$, and in [24] and [5] for functions in a Hilbert space).

In Theorem 5.5, we give a "supremum characterization" of semiconvex functions (with an arbitrary modulus) in a superreflexive Banach space, which generalizes [7, Theorem 3.4.2] (working with functions in $\mathbb{R}^n$).

However, our main interest lies in characterizing semiconvex functions with a modulus of the form $\psi(t) = C\,\varphi(t)$, where φ is fixed and $C > 0$ is arbitrary. The subdifferentiability characterization of these functions is possible in all Banach spaces (see Theorem 4.16), and the proof is easy. But the "supremum characterization" of these functions in general Banach spaces is a much more difficult task. The results of [9] suggest the following question:

Question. Let X be a Banach space, $\Omega \subset X$ be a convex open set, f be a real function on Ω, and $\varphi \in \mathcal{M}$ (see Definition 2.1). Are the following statements equivalent?

(i) There is $C_1 > 0$ such that f is semiconvex on Ω with modulus $C_1\varphi$.
(ii) There exist $C_2 > 0$ and a family $\{g_\alpha : \alpha \in A\}$ of Fréchet differentiable functions on Ω such that $f(x) = \sup\{g_\alpha(x) : \alpha \in A\}$, $x \in \Omega$, and the derivatives g'_α are uniformly continuous on Ω with modulus $C_2\varphi$.

The affirmative answer to this question in the case $X = \mathbb{R}^n$ and $\varphi(t) = t^\alpha$ $(0 < \alpha \leq 1)$ can be proved by the method of [9] (where a corresponding local result is proved).

Our main result (see Theorem 5.8) gives an affirmative answer for each $\varphi \in \mathcal{M}$ if X admits an equivalent norm with modulus of smoothness of power type 2 (in particular, if X is Hilbert or $X = L_p(\mu)$, $2 \leq p < \infty$).

The proof is based on a careful computation (Lemma 5.3) and on a new Corollary 3.6

about moduli of semiconvex functions.

We do not believe that the answer is positive in general (even if Ω is bounded), but we do not have any counterexample.

Our method of proof gives immediately results on extensions of semiconvex functions from bounded convex open sets (Theorem 5.7 and Theorem 5.10), and also from bounded non-convex sets, on which the "subdifferentiability definition" of semiconvex functions must be used (Proposition 5.12). This proposition will be applied in a forthcoming article [11], where the strongest possible results on smallness of the set of points of Gâteaux non-differentiability of a semiconvex function (with modulus $C\varphi$) on a separable Hilbert space will be proved.

2. Preliminaries

2.1. Basic notation

In the following, X will always be a (real) Banach space and we set $S_X := \{x \in X : \|x\| = 1\}$. By $B(x, r)$ we denote the open ball with center x and radius r. For $t \in \mathbb{R}$, we set $t^+ := \max(t, 0)$. The symbol $\langle \cdot, \cdot \rangle$ is used for the usual duality on $X \times X^*$. We will use the following terminology concerning moduli of continuity.

Definition 2.1. We denote by $\mathcal{M}$ the set of all functions $\omega : [0, \infty) \to [0, \infty)$ with $\omega(0) = 0$ which are non-decreasing and right continuous at 0.

Definition 2.2. Let (A, ρ), (B, σ) be metric spaces and $f : A \to B$. Then:

(i) The *minimal modulus of continuity of* f is the function $\omega_f^c : [0, \infty) \to [0, \infty]$ defined in the usual way:

$$\omega_f^c(t) = \sup\{\sigma(f(x), f(y)) : x, y \in A,\ \rho(x, y) \leq t\}.$$

(ii) We say that f is *uniformly continuous with modulus* $\varphi \in \mathcal{M}$ if $\omega_f^c \leq \varphi$.

(iii) We say that f is α*-Hölder* ($\alpha > 0$) if f is uniformly continuous with modulus $\varphi(t) = Ct^\alpha$ for some $C > 0$.

Clearly, ω_f^c is nondecreasing, and the function f is uniformly continuous if and only if $\lim_{t \to 0+} \omega_f^c(t) = 0$.

We will need the following lemma which is due to Stechkin, but was not published by him. For a proof, see [12, p. 78] or [15, p. 670].

(Recall that $\omega : [0, \infty) \to [0, \infty]$ is *subadditive*, if $\omega(t_1 + t_2) \leq \omega(t_1) + \omega(t_2)$.)

Lemma 2.3. *Let $\varphi \in \mathcal{M}$ be subadditive. Then there exists a concave $\tilde{\varphi} \in \mathcal{M}$ such that $\varphi \leq \tilde{\varphi} \leq 2\varphi$.*

Remark 2.4. It is easy to prove that each concave $\varphi \in \mathcal{M}$ is subadditive, see [13, Theorem 7.2.5].

We now recall some facts we need about superreflexive Banach spaces. For the original definition of superreflexive spaces, and a number of their characterizations, see, e.g., [4] or [10]. For example, X is superreflexive if and only if it admits an equivalent norm, which is uniformly rotund (resp. uniformly smooth).

If X is a Banach space, then the *modulus of smoothness* of the norm of X (or of the space X) is defined as the function

$$\rho_X(\tau) = \rho(\tau) = \sup\left\{\frac{\|x+\tau y\| + \|x-\tau y\|}{2} - 1 : \|x\| = \|y\| = 1\right\}, \quad \tau > 0.$$

We say ([10, p. 157]) that X has the *modulus of smoothness of power type* p, if there exists $K > 0$ such that $\rho(\tau) \leq K\tau^p,\ t > 0$. For the following Pisier's result see, e.g., [4, p. 412] or [10].

Theorem P. *A Banach space X is superreflexive if and only if X admits an equivalent norm which has modulus of smoothness of power type p for some $1 < p \leq 2$.*

Remark 2.5. It is well-known (see [10, Corollary V.1.2]) that, if μ is an arbitrary measure, then $L_p(\mu)$ has a modulus of smoothness of power type p (resp. 2), if $1 < p \leq 2$ (resp. $p > 2$). In particular, each Hilbert space has a modulus of smoothness of power type 2.

Finally, we will need the following well-known fact.

Lemma 2.6. *If $0 < \beta \leq 1$ and a Banach space $(X, \|\cdot\|)$ has the modulus of smoothness of power type $1+\beta$, then the Fréchet derivative $\|\cdot\|'$ exists and is β-Hölder on S_X.*

Proof. Using [10, Lemma IV.5.1] with $\varepsilon = 1/2$, we obtain that the duality mapping J is β-Hölder on S_X. Since $J(x) = \|\cdot\|'(x)$ for $x \in S_X$ (cf. [10, p. 7]), we obtain that $\|\cdot\|'$ is β-Hölder on S_X. □

Definition 2.7. Let X be a Banach space, $x \in X$, $v \in X$, and f be a real function defined on a subset of X. Then we denote

$$d^+f(x;v) := \limsup_{h\to 0+}(f(x+hv) - f(x))h^{-1}.$$

We will need the following lemma which is an easy consequence of the classical Dini's theorem on Dini's derivatives. Because of the lack of a reference, we supply a proof.

Lemma 2.8. *Let X be a Banach space, $a \in X$, $v \in X$, $\|v\| = 1$, and $h > 0$. Let f be a real continuous function on the segment $\{a + tv : t \in [0,h]\}$. Then there exist $0 < t_1 < h$ and $0 < t_2 < h$ such that*

$$d^+f(a+t_1v;v) \geq \frac{f(a+h) - f(a)}{h} \quad \textit{and} \quad d^+f(a+t_2v;v) \leq \frac{f(a+h) - f(a)}{h}.$$

Proof. Let $g(t) := f(a+tv),\ 0 \leq t \leq h$. Then g is continuous on $[0,h]$, $Q := (f(a+h) - f(a))/h = (g(h) - g(0))/h$, and $d^+f(a+tv;v) = D^+g(t)$ for $t \in [0,h)$, where $D^+g(t)$ is the upper right Dini derivative of g at x (see [6, p. 39] for the definition). Suppose, to the contrary, that no t_1 with the desired property exists; then we have

$$D^+g(t) < Q \ \text{ for each } t \in (0,h). \tag{1}$$

Choose a point $z \in (0,h)$. By the definition of D^+g and (1), we can choose $z^* \in (z,h)$ such that $g(z^*) - g(z) < Q(z^* - z)$. Now, for each $\Delta \in (0, \min(z, h - z^*))$, the classical

Dini's theorem (see [6, Theorem 1.2., p. 39]) and (1) give that $g(z)-g(\Delta) \leq Q(z-\Delta)$ and $g(h-\Delta)-g(z^*) \leq Q(h-\Delta-z^*)$. Using the continuity of g, we obtain $g(z)-g(0) \leq Qz$ and $g(h)-g(z^*) \leq Q(h-z^*)$. So,

$$\begin{aligned} Qh &= g(h)-g(0) = (g(h)-g(z^*)) + (g(z^*)-g(z)) + (g(z)-g(0)) \\ &< Q(h-z^*) + Q(z^*-z) + Qz = Qh, \end{aligned}$$

which is a contradiction. The proof of the existence of t_2 is almost the same; it is sufficient to replace all "<" by ">" and all "≤" by "≥" in the argument above. □

The following definition is taken from [7]; the definitions in [2] and [1] are slightly different, but essentially equivalent.

Definition 2.9. A continuous real valued function f on an open convex set $\Omega \subset X$ is called *semiconvex with modulus* $\omega \in \mathcal{M}$ if

$$f(\lambda x+(1-\lambda)y) \leq \lambda f(x)+(1-\lambda)f(y)+\lambda(1-\lambda)\omega(\|x-y\|)\|x-y\|, \tag{2}$$

whenever $\lambda \in [0,1]$ and $x, y \in \Omega$.

A function is called *semiconvex on* Ω if it is semiconvex on Ω with some modulus $\omega \in \mathcal{M}$.

The following definition is taken from [29]; the definitions in [26] and [28] are slightly different, but essentially equivalent.

Definition 2.10. Let $\alpha \in \mathcal{M}$ be such that $\lim_{t\to 0+} \alpha(t)/t = 0$ and $\Omega \subset X$ be a convex open set. A continuous function $f : \Omega \to \mathbb{R}$ is called *strongly* $\alpha(\cdot)$*-paraconvex* if

$$f(\lambda x+(1-\lambda)y) \leq \lambda f(x)+(1-\lambda)f(y)+\min(\lambda, 1-\lambda)\alpha(\|x-y\|), \tag{3}$$

whenever $\lambda \in [0,1]$ and $x, y \in \Omega$. We will say that f is *strongly paraconvex* if it is strongly $\alpha(\cdot)$-paraconvex for some $\alpha \in \mathcal{M}$ with $\lim_{t\to 0+} \alpha(t)/t = 0$.

Remark 2.11. (a) Let $\omega \in \mathcal{M}$, let $\Omega \subset X$ be a convex open set, and f a continuous function on Ω. Then:

(i) If f is semiconvex with modulus ω, then f is strongly $\alpha(\cdot)$-paraconvex for $\alpha(t) := t\omega(t)$.

(ii) If f is strongly $\alpha(\cdot)$-paraconvex for $\alpha(t) := t\omega(t)$, then f is semiconvex with modulus 2ω.

(iii) f is semiconvex if and only if f is strongly paraconvex.

The statements (i) and (ii) are obvious, if we observe that $\lambda(1-\lambda) \leq \min(\lambda, 1-\lambda) \leq 2\lambda(1-\lambda)$ for $\lambda \in [0,1]$. To show (iii), suppose that f is strongly $\alpha(\cdot)$-paraconvex for some $\alpha \in \mathcal{M}$. Since $\lim_{t\to 0+} \alpha(t)/t = 0$, there exists $\omega \in \mathcal{M}$ such that $2(\alpha(t)/t) \leq \omega(t),\ t \geq 0$. Using (ii), we obtain that f is semiconvex with modulus ω.

(b) The definition of strong $\alpha(\cdot)$-paraconvexity in [26] and [28] is given for nondecreasing $\alpha\colon [0,\infty) \to [0,\infty]$ (which can be somewhere infinite). However (for continuous functions f), the definition with $\alpha \in \mathcal{M}$ gives the same notion of a strongly paraconvex function (see Remark 3.8).

Now we define approximately convex functions in the sense of Ngai, Luc and Théra [21]. (Note that the term "approximately convex functions" is used for a long time for another type of functions, namely for ε-convex functions in the sense of Hyers and Ulam.)

Definition 2.12 ([21]). A real valued function f on an open set $\Omega \subset X$ is called *approximately convex at* $x_0 \in \Omega$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y) + \varepsilon\lambda(1-\lambda)\|x-y\| \tag{4}$$

whenever $\lambda \in [0,1]$ and $x, y \in B(x_0, \delta)$. We say that f is *approximately convex* on Ω if it is approximately convex at each $x_0 \in \Omega$.

We say that f is *uniformly approximately convex* on Ω if for every $\varepsilon > 0$ there exists $\delta > 0$ such that (4) holds whenever $\lambda \in [0,1]$, $x, y \in \Omega$, and $\|x - y\| < \delta$.

Remark 2.13. (i) Obviously, each semiconvex (strongly paraconvex) function is approximately convex (even uniformly approximately convex).

(ii) Each approximately convex function on an open set $\Omega \subset X$ which is finite and lower semicontinuous is locally Lipschitz (see [21, Proposition 3.2 and Corollary 3.5]). So, each function, which is semiconvex (strongly paraconvex) by our definition is locally Lipschitz.

3. Properties of the minimal modulus of semiconvexity

In the proof of our main results, we will need the fact (see Corollary 3.6) that if f is continuous and semiconvex on an open convex set with a modulus $\varphi \in \mathcal{M}$, then it is semiconvex with a *concave* modulus $\omega \in \mathcal{M}$ with $\omega \le 4\varphi$. We have chosen a proof which requires almost no results about semiconvex functions (namely, only local Lipschitzness of semiconvex functions is used).

Definition 3.1. Let f be a real valued function on an open convex subset Ω of a Banach space X. Put $\omega_f(0) := 0$ and, for $t > 0$, set (recall that b^+ is the positive part of b)

$$\begin{aligned}\omega_f(t) := \sup\Big\{ \Big(\frac{f(\lambda x + (1-\lambda)y) - \lambda f(x) - (1-\lambda) f(y)}{\lambda(1-\lambda)\|x-y\|} \Big)^+ :\\ x, y \in \Omega, 0 < \|x-y\| \le t, \lambda \in (0,1) \Big\}.\end{aligned} \tag{5}$$

It is easy to see that $\omega_f : [0,\infty) \to [0,\infty]$ is nondecreasing and f is semiconvex if and only if f is continuous and $\omega_f \in \mathcal{M}$. Then clearly ω_f is the minimal modulus of semiconvexity of f.

It is well-known that ω_f can be expressed in a more geometric form: for $t > 0$,

$$\begin{aligned}\omega_f(t) = \sup\Big\{ \Big(\frac{f(z) - f(z - hv)}{h} - \frac{f(z + kv) - f(z)}{k} \Big)^+ :\\ \|v\| = 1, k, h > 0, z, z - hv, z + kv \in \Omega, h + k \le t \Big\}.\end{aligned} \tag{6}$$

Indeed, denoting $x := z + kv$, $y := z - hv$, $\lambda := h/(h+k)$, we have $1 - \lambda = k/(k+h)$, $z = \lambda x + (1-\lambda)y$, $\|x - y\| = h + k$, and an easy computation shows that

$$\frac{f(z) - f(z - hv)}{h} - \frac{f(z + kv) - f(z)}{k} = \frac{f(\lambda x + (1-\lambda)y) - \lambda f(x) - (1-\lambda)f(y)}{\lambda(1-\lambda)\|x - y\|}.$$

Definition 3.2. Let f be a locally Lipschitz real valued function on an open convex subset Ω of a Banach space X. For $t \geq 0$, set

$$\omega_f^*(t) := \sup\left\{ \left(d^+ f(x; v) - d^+ f(x + \tau v; v)\right)^+ : \|v\| = 1,\ x, x + \tau v \in \Omega,\ 0 \leq \tau \leq t \right\}. \tag{7}$$

Lemma 3.3. *Let f be a locally Lipschitz real valued function on an open convex subset Ω of a Banach space X. Then*

(*i*) ω_f^* *is nondecreasing and subadditive.*
(*ii*) *If* $\lim_{t\to 0+} \omega_f^*(t) = 0$, *then* $\omega_f^* \in \mathcal{M}$ *and there exists a concave* $\tilde{\omega} \in \mathcal{M}$ *such that* $\omega_f^* \leq \tilde{\omega} \leq 2\omega_f^*$.

Proof. Obviously, ω_f^* is nondecreasing. To prove the subadditivity, suppose that $t_1, t_2 > 0$ are given. Consider arbitrary $x \in \Omega$, unit vector $v \in X$, and $0 < \tau \leq t_1 + t_2$ such that $x + \tau v \in \Omega$. Write $\tau = \tau_1 + \tau_2$ so that $0 < \tau_1 \leq t_1$ and $0 < \tau_2 \leq t_2$. Since always $(a+b)^+ \leq a^+ + b^+$, we have

$$\begin{aligned} &\left(d^+ f(x; v) - d^+ f(x + \tau v; v)\right)^+ \\ \leq\ &\left(d^+ f(x; v) - d^+ f(x + \tau_1 v; v)\right)^+ + \left(d^+ f(x + \tau_1 v; v) - d^+ f((x + \tau_1 v) + \tau_2 v; v)\right)^+ \\ \leq\ &\omega_f^*(t_1) + \omega_f^*(t_2). \end{aligned}$$

Consequently, $\omega_f^*(t_1 + t_2) \leq \omega_f^*(t_1) + \omega_f^*(t_2)$.

If $\lim_{t\to 0+} \omega_f^*(t) = 0$, then ω_f^* is bounded on a right neigbourhood of 0. So, the subadditivity of ω_f^* cleary implies that ω_f^* is finite. The existence of $\tilde{\omega}$ now follows from Stechkin's Lemma 2.3. □

Lemma 3.4. *Let f be a locally Lipschitz real valued function on an open convex subset Ω of a Banach space X. Then $\omega_f \leq \omega_f^* \leq 2\,\omega_f$.*

Proof. Consider $z \in \Omega$, a unit vector $v \in X$, and $k, h > 0$ such that $z - hv, z + kv \in \Omega$ and $h + k \leq t$. Using Lemma 2.8 (with $a := z - hv$, and then with $a := z$), we can choose $0 < h^* < h$ such that $d^+ f(z - h^* v; v) \geq (f(z) - f(z - hv))/h$ and $0 < k^* < k$ such that $d^+ f(z + k^* v; v) \leq (f(z + kv) - f(z))/k$. Therefore,

$$\begin{aligned} &\left(\frac{f(z) - f(z - hv)}{h} - \frac{f(z + kv) - f(z)}{k}\right)^+ \\ \leq\ &\left(d^+ f(z - h^* v; v) - d^+ f(z - h^* v + (h^* + k^*)v; v)\right)^+ \leq \omega_f^*(t), \end{aligned}$$

which, together with (6), implies $\omega_f \leq \omega_f^*$.

To prove the second inequality, suppose that $t > 0$, $x \in \Omega$, $v \in X$ is a unit vector, and $0 < \tau \leq t$ such that $x + \tau v \in \Omega$ is given. Choose sequences $\lambda_n \searrow 0$, $\mu_n \searrow 0$ such that

$$\lim_{n\to\infty} (f(x + \lambda_n v) - f(x))/\lambda_n = d^+ f(x; v),$$
$$\lim_{n\to\infty} (f(x + \tau v + \mu_n v) - f(x + \tau v))/\mu_n = d^+ f(x + \tau v; v).$$

Then, using (6), we obtain

$$\left(d^+ f(x; v) - \frac{f(x + \tau v) - f(x)}{\tau}\right)^+$$
$$= \lim_{n\to\infty} \left(\frac{f(x + \lambda_n v) - f(x)}{\lambda_n} - \frac{f(x + \tau v) - f(x + \lambda_n v)}{\tau - \lambda_n}\right)^+ \leq \omega_f(t)$$

and

$$\left(\frac{f(x + \tau v) - f(x)}{\tau} - d^+ f(x + \tau v; v)\right)^+$$
$$= \lim_{n\to\infty} \left(\frac{f(x + \tau v) - f(x + \mu_n v)}{\tau - \mu_n} - \frac{f(x + \tau v + \mu_n v) - f(x + \tau v)}{\mu_n}\right)^+ \leq \omega_f(t).$$

Consequently, $(d^+ f(x; v) - d^+ f(x + \tau v; v))^+ \leq 2\,\omega_f(t)$, which implies $\omega_f^* \leq 2\,\omega_f$. □

Proposition 3.5. *Let f be a continuous real valued function on an open convex subset Ω of a Banach space X. Then the following holds:*

(*i*) *f is semiconvex if and only if $\lim_{t\to 0+} \omega_f(t) = 0$.*

(*ii*) *If f is semiconvex, then there exists a concave $\tilde{\omega} \in \mathcal{M}$ such that $\omega_f \leq \tilde{\omega} \leq 4\omega_f$.*

Proof. The implication "⇒" of (*i*) is obvious. So, suppose that $\lim_{t\to 0+} \omega_f(t) = 0$. Choose $t_0 > 0$ with $\omega_f(t_0) < \infty$ and set $\varphi(t) := \min(\omega_f(t), \omega_f(t_0))$. Then f is clearly semiconvex with modulus φ on each convex open set $\Omega^* \subset \Omega$ with $\operatorname{diam}(\Omega^*) < t_0$. So, f is locally Lipschitz on Ω by Remark 2.13(ii), and Lemma 3.4 implies $\omega_f \leq \omega_f^* \leq 2\omega_f$. Consequently, $\lim_{t\to 0+} \omega_f^*(t) = 0$. By Lemma 3.3, $\omega_f^* \in \mathcal{M}$ and there exists a concave $\tilde{\omega} \in \mathcal{M}$ such that $\omega_f \leq \omega_f^* \leq \tilde{\omega} \leq 2\omega_f^* \leq 4\omega_f$. Now, both (*i*) and (*ii*) immediately follow. □

Corollary 3.6. *Let f be a semiconvex function with modulus $\varphi \in \mathcal{M}$ on an open convex subset Ω of a Banach space X. Then f is semiconvex with a concave modulus $\omega \in \mathcal{M}$ for which $\omega \leq 4\varphi$.*

Proof. Since $\omega_f \leq \varphi$, the statement follows from Proposition 3.5. □

Corollary 3.7. *Let f be a continuous function on an open convex subset Ω of a Banach space X. Then f is semiconvex (equivalently: f is strongly paraconvex) if and only if f is uniformly approximately convex on Ω.*

Proof. Obviously, the function f is uniformly approximately convex on Ω if and only if $\lim_{t\to 0+} \omega_f(t) = 0$. So, the assertion follows from Proposition 3.5(*i*). □

Remark 3.8. Note that [27, Theorem 4] (which works also with nonconvex Ω), implies that f is uniformly approximately convex on a convex open Ω if and only if f is strongly paraconvex (with possibly infinite α). So, Corollary 3.7 implies that if f is strongly $\alpha(\cdot)$-paraconvex on Ω (with possibly infinite α), then it is strongly $\tilde{\alpha}(\cdot)$-paraconvex with some finite $\tilde{\alpha}$.

4. Subdifferentiability characterizations of semiconvex functions

First, recall the notion of the *Fréchet subgradient* which was (possibly) first defined (under another name) in [3].

Definition 4.1. Let f be a real valued function defined on an open subset G of a Banach space X. We say that $x^* \in X^*$ is a *Fréchet subgradient* of f at $x \in G$ provided

$$\liminf_{h \to 0} \frac{f(x+h) - f(x) - \langle h, x^* \rangle}{\|h\|} \geq 0.$$

Then, we will write $x^* \in \partial^F f(x)$, and we will say that f is *Fréchet subdifferentiable at x* when $\partial^F f(x) \neq \emptyset$.

Remark 4.2. If f is Lipschitz with constant K and $x^* \in X^*$ is a Fréchet subgradient of f at x, then it is easy to see that $\|x^*\| \leq K$.

This natural notion and mainly the symmetrically defined Fréchet supergradient found many applications (see, e.g., [7], [32], or [34]).

It is well-known that semiconvex functions (see [2] and [7] for functions on $\mathbb{R}^n$), strongly paraconvex functions, and also approximately convex functions (see [21, Theorem 3.6, Proposition 3.2, and Corollary 3.5]) are Fréchet subdifferentiable at all points.

Moreover, Rolewicz [28] (generalizing a result of [14], where the case of $\alpha(t) = Ct^\beta$ is considered) proved that each strongly paraconvex function f is everywhere subdifferentiable in a *stronger quantitative global sense*, which depends on the modulus of f. Rolewicz works with "$\alpha(\cdot)$-subgradient" (whose meaning in [28] is slightly different from that in [29]). We will use the notation "$[\alpha]$-subgradient", and we will extend the definition for functions *defined on arbitrary sets*. The usefulness of this generalization will be shown in [11] via Proposition 5.12 below.

Definition 4.3. Let f be a real valued function defined on an arbitrary nonempty subset A of a Banach space X, and let $\alpha \in \mathcal{M}$ with $\lim_{t \to 0+} \alpha(t)/t = 0$. We say that $x^* \in X^*$ is an $[\alpha]$-*subgradient* of f at $x \in A$ provided

$$\langle h, x^* \rangle - (f(x+h) - f(x)) \leq \alpha(\|h\|) \quad \text{whenever } h \neq 0 \text{ and } x+h \in A. \tag{8}$$

Then we will write $x^* \in \partial^\alpha_A f(x)$.

We will say that f is $[\alpha]$-subdifferentiable at $x \in A$ (resp. $[\alpha]$-subdifferentiable on A) if $\partial^\alpha_A f(x) \neq \emptyset$ (resp. $\partial^\alpha_A f(y) \neq \emptyset$ for each $y \in A$).

Remark 4.4. Obviously, if (in Definition 4.3) A is open, then each $[\alpha]$-subgradient of f at x is a Fréchet subgradient of f at x.

An immediate consequence of [28, Theorem 3] (cf. [29, Proposition 2]) is the following.

Theorem R. *Let f be a strongly $\alpha(\cdot)$-paraconvex function defined on an open convex subset Ω of a Banach space X. Then f is $[\alpha]$-subdifferentiable at each point $x \in \Omega$.*

Moreover, an $x^* \in X^*$ is an $[\alpha]$-subgradient of f at x if and only if x^* belongs to the Clarke subdifferential $\partial^C f(x)$.

By Remark 2.11(a(i)) we obtain (see also [7, Proposition 3.3.1 and Proposition 3.3.4] for $X = \mathbb{R}^n$):

Corollary 4.5. *Let $\omega \in \mathcal{M}$ and let f be a semiconvex function with modulus ω on a nonempty open convex subset Ω of a Banach space X. Then f is $[\alpha]$-subdifferentiable at each point $x \in \Omega$, where $\alpha(t) := t\omega(t)$.*

We will show (Theorem 4.16) that the converse of the first part of Theorem R (resp. Corollary 4.5) almost (up multiplication by 2 of the modulus) holds, so we obtain a subdifferentiability characterization of functions which are strongly $(C\alpha(\cdot))$-paraconvex (resp. semiconvex with modulus $C\omega$) for some $C > 0$.

In [33] and [18], the notion of a uniformly almost superdifferentiable function on a convex open subset of a Banach space was defined and applied to differentiability of distance functions in uniformly Fréchet differentiable spaces. It turns out that this notion coincides with the notion of a semiconcave function. (For the proof, it is sufficient to apply Theorem 4.12 to $-f$.) Since we deal with semiconvex functions, we define the dual notion, using the term "uniform Fréchet subdifferentiability", which better corresponds to the modern terminology.

Definition 4.6. Let f be a real valued function defined on a nonempty open convex subset Ω of a Banach space X. We say that f is *uniformly Fréchet subdifferentiable* provided there exists $g : \Omega \to X^*$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\langle h, g(x) \rangle - (f(x+h) - f(x)) \leq \varepsilon \|h\| \tag{9}$$

whenever $x, x+h \in \Omega$ and $\|h\| \leq \delta$.

Remark 4.7. (i) Due to an oversight, the formulation of [33, Definition 2] of "almost uniform superdifferentiability" is confusing, but it is used correctly in [33] and [18].

(ii) The idea of uniform Fréchet subdifferentiability was probably first used (for $f : \mathbb{R}^n \to \mathbb{R}$) by E. A. Nurminskii [23]; cf. Corollary 4.15 below.

Obviously, if there exists $\alpha \in \mathcal{M}$ with $\lim_{t \to 0+} \alpha(t)/t = 0$ such that f is $[\alpha]$-subdifferentiable at each point $x \in \Omega$, then f is uniformly Fréchet subdifferentiable on Ω. We will show (Theorem 4.12) that also the converse of this assertion is true, so obtaining another subdifferential characterization of semiconvex (strongly paraconvex) functions. We will work with the following auxiliary modulus.

Definition 4.8. Let f be a real valued function defined on an arbitrary nonempty subset

A of a Banach space X. Then we set $\overline{\omega}_f(0) := 0$ and, for $t > 0$,

$$\overline{\omega}_f(t) := \sup_{x \in A} \inf_{x^* \in X^*} \sup \left\{ \frac{(\langle h, x^* \rangle - (f(x+h) - f(x)))^+}{\|h\|} : 0 < \|h\| \le t,\ x + h \in A \right\}. \tag{10}$$

Remark 4.9. (i) If $\omega \in \mathcal{M}$, $\alpha(t) := t\omega(t)$, and f is $[\alpha]$-subdifferentiable on A, then clearly $\overline{\omega}_f \le \omega$.

(ii) If A is convex open and f is uniformly Fréchet subdifferentiable on A, then clearly $\lim_{t \to 0} \overline{\omega}_f(t) = 0$.

Lemma 4.10. *Let f be a real valued function defined on a nonempty open convex subset Ω of a Banach space X. Then the following holds:*

(*i*) *If f is continuous and $\omega_f \in \mathcal{M}$ (i.e., f is semiconvex on Ω), then f is $[\alpha]$-subdifferentiable on Ω, where $\alpha(t) := t\omega_f(t)$. In particular, $\overline{\omega}_f \le \omega_f$.*

(*ii*) $\omega_f \le 2\overline{\omega}_f$.

Proof. The statement (*i*) follows from Corollary 4.5 and Remark 4.9(i).

To prove (*ii*), fix an arbitrary $t > 0$. Suppose that $z \in \Omega$, a unit vector $v \in X$, and $k, h > 0$ such that $z - hv, z + kv \in \Omega$ and $h + k \le t$ are given. By (6), it is sufficient to prove that

$$\frac{f(z) - f(z - hv)}{h} - \frac{f(z + kv) - f(z)}{k} \le 2\overline{\omega}_f(t). \tag{11}$$

To this end, choose an arbitrary $\varepsilon > 0$ and find $x^* \in X^*$ such that

$$\sup \left\{ \frac{\langle s, x^* \rangle - (f(z+s) - f(z))}{\|s\|} : 0 < \|s\| \le t,\ z + s \in A \right\} \le \overline{\omega}_f(t) + \varepsilon.$$

Applying this inequality to $s = kv$ and then to $s = -hv$, we obtain

$$\langle v, x^* \rangle - \frac{f(z + kv) - f(z)}{k} \le \overline{\omega}_f(t) + \varepsilon, \qquad -\langle v, x^* \rangle - \frac{f(z - hv) - f(z)}{h} \le \overline{\omega}_f(t) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, adding these inequalities, we obtain (11). □

Corollary 4.11. *A continuous function on Ω is semiconvex if and only if $\overline{\omega}_f \in \mathcal{M}$.*

Now we easily obtain results on subdifferentiability characterizations of semiconvex and strongly paraconvex functions.

Theorem 4.12. *Let f be a real valued function defined on a nonempty open convex subset Ω of a Banach space X. Then the following assertions are equivalent.*

(*i*) *f is semiconvex on Ω.*

(*ii*) *f is strongly paraconvex on Ω.*

(*iii*) *There exists $\alpha \in \mathcal{M}$ with $\lim_{t \to 0+} \alpha(t)/t = 0$ such that f is $[\alpha]$-subdifferentiable on Ω.*

(*iv*) *f is uniformly Fréchet subdifferentiable on Ω.*

(*v*) *f has the following property:*

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in \Omega \; \exists x^* \in X^* : \\ \langle y - x, x^* \rangle - (f(y) - f(x)) \leq \varepsilon \|y - x\| \quad \text{if } y \in \Omega \text{ and } \|y - x\| \leq \delta.$$

Proof. We have $(i) \Leftrightarrow (ii)$ by Remark 2.11. Further, $(ii) \implies (iii)$ by Theorem R, and $(iii) \implies (iv)$ is obvious (as already mentioned above). Since condition (v) is clearly equivalent to $\lim_{t\to 0} \overline{\omega}_f(t) = 0$, Remark 4.9(ii) gives $(iv) \implies (v)$. If (v) holds, then $\lim_{t\to 0} \omega_f(t) = 0$ by Lemma 4.10(ii). Consequently, f is clearly approximately convex (even uniformly approximately convex) on Ω. Further, (v) clearly implies that f is lower semicontinuous on Ω. So, a result of [21] (see Remark 2.13(ii)) implies that f is continuous on Ω. Therefore, f is semiconvex by Proposition 3.5(i). □

Remark 4.13. Theorem 4.12 ($(iv) \Leftrightarrow (ii)$) and Theorem R easily imply that, in Definition 4.6, we can choose $g(x)$ to be any element of the Clarke subdifferential $\partial^C f(x)$.

Corollary 4.14. *Let X be a Banach space, $a \in X$, and φ be a real valued function defined an a neigbourhood of a. Then φ is uniformly upper subdifferentiable around the point a in the sense of* [20] *if and only if φ is semiconcave (i.e., $-\varphi$ is semiconvex) on an open convex neigbourhood of a.*

Proof. By [20, Definition 3.1] (cf. [19] for a formally different definition) φ is uniformly upper subdifferentiable around a if and only if there exists an open convex neighbourhood Ω of a such that $f := -\varphi$ is uniformly Fréchet subdifferentiable (in the sense of Definition 4.6) on Ω. So, the statement follows from Theorem 4.12. □

Corollary 4.15. *A function $f : \mathbb{R}^n \to \mathbb{R}$ is weakly convex in Nurminskii's sense if and only if f is locally semiconvex on $\mathbb{R}^n$.*

Proof. Recall (see [16, 1.11.2]) that f is weakly convex if and only if f is continuous and there exists a function $r : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that

$$r(x, u)/\|x - u\| \to 0 \;\text{ as } u \to x \text{ uniformly relative to } x \in B, \tag{12}$$

for each closed bounded $B \subset \mathbb{R}^n$, and the set

$$G(x) := \{x^* \in (\mathbb{R}^n)^* : f(u) - f(x) - \langle u - x, x^* \rangle + r(x, u) \geq 0 \;\text{ for each } u \in \mathbb{R}^n\} \tag{13}$$

is nonempty for any $x \in \mathbb{R}^n$.

So, if f is weakly convex, then f is clearly uniformly Fréchet subdifferentiable (in the sense of Definition 4.6) on each open ball in $\mathbb{R}^n$. Consequently, Theorem 4.12 implies that f is locally semiconvex on $\mathbb{R}^n$.

Now suppose that f is locally semiconvex on $\mathbb{R}^n$. Then the compactness argument of [35, Remark 2.6] easily implies that f is uniformly approximately convex on each open ball. So, Corollary 3.7 gives that f is semiconvex (and so uniformly Fréchet subdifferentiable by Theorem 4.12) on each open ball. Choose, for each $x \in \mathbb{R}^n$, a functional $g(x) \in \partial^C f(x)$. Setting

$$r(x, u) := \big(\langle u - x, g(x) \rangle - (f(u) - f(x))\big)^+,$$

and using Remark 4.13, we easily obtain that (12) holds for each closed bounded B. Since cleary $g(x) \in G(x)$ (where G is as in (13)), we obtain that f is weakly convex. □

Theorem 4.16. *Let f be a continuous function defined on a nonempty open convex subset Ω of a Banach space X. Let $\omega \in \mathcal{M}$, and $\alpha(t) := t\omega(t)$. Then the following assertions are equivalent.*

(i) *f is semiconvex with modulus $C_1\omega$ for some $C_1 > 0$.*
(ii) *f is strongly $C_2\alpha(\cdot)$-paraconvex for some $C_2 > 0$.*
(iii) *f is $[C_3\alpha]$-subdifferentiable on Ω for some $C_3 > 0$.*

Proof. We have (i) $\Leftrightarrow$ (ii) by Remark 2.11. Further, (ii) $\implies$ (iii) by Theorem R. If (iii) holds, then $\overline{\omega}_f \leq C_3\omega$ by Remark 4.9(i), and consequently $\omega_f \leq 2C_3\omega$ by Lemma 4.10(ii). Setting $C_1 := 2C_3$, we obtain (i). □

5. Representations and extensions of semiconvex functions

The following well-known fact (see, e.g., [7, Proposition 2.1.5]) follows easily from the definition of semiconvexity.

Lemma 5.1. *Let X be a Banach space and $\Omega \subset X$ be convex and open. Let $\{u_\alpha\}_{\alpha \in A}$ be a family of functions defined in Ω and semiconvex with the same modulus $\omega \in \mathcal{M}$. Then the function $u := \sup_{\alpha \in A} u_\alpha$ is also semiconvex on Ω with modulus ω provided $u(x) < \infty,\ x \in \Omega$.*

The following well-known fact (see, e.g., [7, Proposition 2.1.2]) follows also immediately from Lemma 3.4.

Lemma 5.2. *Let X be a Banach space, $\Omega \subset X$ convex open, and $f : \Omega \to \mathbb{R}$ Fréchet differentiable such that f' is uniformly continuous on Ω with modulus $\omega \in \mathcal{M}$. Then f is semiconvex on Ω with modulus ω.*

Lemma 5.3. *Let $(X, \|\cdot\|) = (X, n)$ be a Banach space such that n' is uniformly continuous on S_X with minimal modulus of continuity $\omega_1(t) \leq C_1 t^\beta$, $(0 < \beta \leq 1)$. Let $\varphi \in \mathcal{M}$ be concave. For $x, s, t \in [0, \infty)$, set $\psi(x) := \int_0^x \varphi$,*

$$\lambda(s) := \sup_{r \geq s} \varphi(r)\, \omega_1(2s/r),\ \ \tilde{\eta}(s) := 3\varphi(s) + \lambda(s), \quad \text{and} \quad \eta(t) := \sup_{0 \leq s \leq t} \tilde{\eta}(s). \tag{14}$$

Then the function

$$h(x) := \psi(\|x\|), \quad x \in X,$$

has the following properties:

(i) *h is Fréchet differentiable on X and*

$$\omega^c_{h'}(t) := \sup_{\|x-y\| \leq t} \|h'(x) - h'(y)\| \leq \eta(t) \leq \infty \ \ \text{for } t \geq 0.$$

(ii) *If φ is bounded, then h' is uniformly continuous on X and $\omega^c_{h'}$ is bounded. Moreover, there exists $D_1 > 0$ such that*

$$\omega^c_{h'}(t) \leq D_1 \cdot \left(\varphi(\sqrt{t}) + \omega_1(2\sqrt{t})\right), \quad t > 0.$$

(iii) *If φ is bounded and $\varphi(t) \leq C_2 t,\ t \geq 0$, then*

$$\omega^c_{h'}(t) \leq D_2 t^\beta, \quad t \geq 0, \quad \text{for some } D_2 > 0.$$

(*iv*) *If* $\beta = 1$, *then*

$$\omega^c_{h'}(t) \le D_3\varphi(t), \quad t \ge 0, \quad \text{for some } D_3 > 0.$$

(*v*) *If* $0 < \alpha \le \beta$ *and* $\varphi(t) \le C_3 t^\alpha$, $t \ge 0$, *then*

$$\omega^c_{h'}(t) \le D_4 t^\alpha, \quad t \ge 0, \quad \text{for some } D_4 > 0.$$

Proof. Since the case $\varphi \equiv 0$ is trivial, we can supose that φ is positive on $(0,\infty)$. To prove (i), first observe that $h'(x) = \psi'(\|x\|) \cdot n'(x) = \varphi(\|x\|) \cdot n'(x)$ for $x \ne 0$, and clearly $h'(0) = 0$, since n is Lipschitz and $\psi'_+(0) = 0$. The Lipschitzness of n with constant 1 implies that $\|n'(x)\| \le 1$ for $x \ne 0$ and the positive homogeneity of n gives

$$n'(x) = n'\left(\frac{x}{\|x\|}\right) \quad \text{whenever } x \ne 0. \tag{15}$$

To estimate $\omega^c_{h'}(t)$, consider arbitrary points $x_1, x_2 \in X$ with $0 < \|x_1 - x_2\| \le t$ and $\|x_1\| \le \|x_2\|$. If $x_1 = 0$, then

$$\begin{aligned}\|h'(x_1) - h'(x_2)\| &= \|h'(x_2)\| = \varphi(\|x_2\|)\,\|n'(x_2)\| \le \varphi(\|x_2\|) \\ &= \varphi(\|x_1 - x_2\|) \le \varphi(t) \le \tilde{\eta}(t) \le \eta(t).\end{aligned}$$

If $x_1 \ne 0$, then $x_2 \ne 0$. Consequently, we obtain

$$\begin{aligned}\|h'(x_1) - h'(x_2)\| &= \|\varphi(\|x_2\|)\,n'(x_2) - \varphi(\|x_1\|)\,n'(x_1)\| \\ &= \|\varphi(\|x_2\|)\,(n'(x_2) - n'(x_1)) + n'(x_1)\,(\varphi(\|x_2\|) - \varphi(\|x_1\|))\| \\ &\le (\varphi(\|x_2\|) - \varphi(\|x_1\|)) \cdot \|n'(x_1)\| + \varphi(\|x_2\|) \cdot \|n'(x_2) - n'(x_1)\| \\ &\le \varphi(\|x_2 - x_1\|) + \varphi(\|x_2\|) \cdot \|n'(x_2) - n'(x_1)\|.\end{aligned} \tag{16}$$

The last inequality holds, since φ is subadditive (see Remark 2.4), and therefore

$$\varphi(\|x_2\|) - \varphi(\|x_1\|) \le \varphi(\|x_2\| - \|x_1\|) \le \varphi(\|x_2 - x_1\|).$$

If $\|x_2\| \le \|x_1 - x_2\|$, then the estimate (16) shows that

$$\|h'(x_1) - h'(x_2)\| \le \varphi(\|x_1 - x_2\|) + 2\varphi(\|x_1 - x_2\|) \le 3\varphi(t) \le \tilde{\eta}(t) \le \eta(t).$$

Otherwise $\|x_2\| > \|x_1 - x_2\|$, and thus, since by [17, Lemma 5.1]

$$\left\|\frac{a}{\|a\|} - \frac{b}{\|b\|}\right\| \le \frac{2}{\|a\|}\|a - b\| \quad \text{for all } a, b \in X \setminus \{0\},$$

by (16) and (15) we obtain

$$\begin{aligned}\|h'(x_1) - h'(x_2)\| &\le \varphi(\|x_1 - x_2\|) + \varphi(\|x_2\|) \cdot \left\|n'\left(\frac{x_1}{\|x_1\|}\right) - n'\left(\frac{x_2}{\|x_2\|}\right)\right\| \\ &\le \varphi(\|x_1 - x_2\|) + \varphi(\|x_2\|)\,\omega_1\left(\left\|\frac{x_1}{\|x_1\|} - \frac{x_2}{\|x_2\|}\right\|\right) \\ &\le \varphi(\|x_1 - x_2\|) + \varphi(\|x_2\|)\,\omega_1\left(\frac{2}{\|x_2\|}\|x_1 - x_2\|\right).\end{aligned} \tag{17}$$

Using (17), and the definiton of $\lambda(s)$ (with $s := \|x_1 - x_2\|$ and $r := \|x_2\|$), we obtain that

$$\|h'(x_1) - h'(x_2)\| \leq \varphi(\|x_1 - x_2\|) + \lambda(\|x_1 - x_2\|) \leq \tilde{\eta}(\|x_1 - x_2\|) \leq \eta(t),$$

which proves (i).

To prove (ii), first observe that ω_1 is bounded, since clearly $\omega_1(t) \leq \omega_1(2)$, $t \geq 0$. Since also φ is bounded, it is easy to see that the functions λ, $\tilde{\eta}$, η are bounded as well.

Now consider an arbitrary $s \in (0,1)$ and $r \geq s$. Denote $A := \sup_{t\geq 0} \varphi(t)$.

If $r \geq \sqrt{s}$, then

$$\varphi(r)\omega_1(2s/r) \leq A \cdot \omega_1(2\sqrt{s}).$$

If $s \leq r < \sqrt{s}$, then

$$\varphi(r)\omega_1(2s/r) \leq \varphi(\sqrt{s})\omega_1(2).$$

Thus, for some $D^* > 0$, $\lambda(s) \leq D^*(\varphi(\sqrt{s}) + \omega_1(2\sqrt{s}))$ $(s \in (0,1))$. Consequently (since φ, ω_1 are nondecreasing, and $t < \sqrt{t}$, $t \in (0,1)$), for $t \in (0,1)$ and each $D_1 \geq D^* + 3$,

$$\eta(t) \leq D_1 \cdot \left(\varphi(\sqrt{t}) + \omega_1(2\sqrt{t})\right). \tag{18}$$

Since η is bounded and $\varphi(\sqrt{t}) \geq \varphi(1) > 0$ for $t \geq 1$, we can clearly choose $D_1 \geq D^* + 3$ so big that (18) holds for all $t > 0$. Since $\omega^c_{h'} \leq \eta$ by (i), we obtain the inequality of (ii).

To prove (iii), consider once more an arbitrary $s \in (0,1)$ and $r \geq s$. If $s \leq r \leq 1$, we obtain

$$\varphi(r)\omega_1(2s/r) \leq C_1C_2 r 2^\beta s^\beta r^{-\beta} \leq C_1C_2 2^\beta s^\beta.$$

For $r \geq 1$, setting $A := \sup_{t\geq 0} \varphi(t)$, we have

$$\varphi(r)\omega_1(2s/r) \leq AC_1 2^\beta s^\beta.$$

So, for some $\tilde{D} > 0$ we have, for each $s \in (0,1)$, $\lambda(s) \leq \tilde{D}s^\beta$, and consequently $\tilde{\eta}(s) \leq (\tilde{D} + 3C_2)s^\beta$. Since both φ and ω_1 are bounded, we have that $\tilde{\eta}$ is bounded. So, since $s^\beta \geq 1$ for $s \geq 1$, we can choose D_2 so big that $\tilde{\eta}(s) \leq D_2 s^\beta$ for each $s > 0$. Consequently, we obtain

$$\omega^c_{h'}(t) \leq \eta(t) \leq D_2 t^\beta \quad \text{for each } t > 0.$$

To prove (iv), observe that $\varphi(t)/t$ is nonincreasing (since φ is concave) on $(0,\infty)$. Consequently, if $s > 0$ and $r \geq s$, then

$$\varphi(r)\omega_1(2s/r) \leq \varphi(r)C_1(2s/r) = 2C_1 s(\varphi(r)/r) \leq 2C_1 s(\varphi(s)/s) = 2C_1\varphi(s).$$

So, we easily obtain that

$$\omega^c_{h'}(t) \leq \eta(t) \leq (2C_1 + 3)\varphi(t) \quad \text{for each } t > 0.$$

For part (v), note that we have

$$\begin{aligned}\lambda(s) &= \sup_{r\geq s} \varphi(r)\omega_1(2s/r) \leq \sup_{r\geq s} C_3 r^\alpha C_1 2^\beta s^\beta r^{-\beta} \\ &= 2^\beta C_3 C_1 \sup_{r\geq s} r^{\alpha-\beta} s^\beta = 2^\beta C_3 C_1 s^\alpha,\end{aligned}$$

and the conclusion easily follows. □

Lemma 5.4. *Let X be a superreflexive Banach space which admits an equivalent norm with modulus of smoothness of power type $p \geq 1+\beta$, where $0 < \beta \leq 1$. Let $\emptyset \neq A \subset X$ and a concave $\varphi \in \mathcal{M}$ be given. Suppose that at least one of the following conditions holds:*

(*i*) φ *is bounded.*
(*ii*) $\beta = 1$.
(*iii*) $\varphi(t) \leq Ct^\alpha,\ t \geq 0$, *for some* $0 < \alpha \leq \beta$ *and* $C > 0$.
(*iv*) A *is bounded.*
(*v*) φ *is bounded and* $\varphi(t) \leq Ct,\ t \geq 0$, *for some* $C > 0$.
(*vi*) A *is bounded and* $\varphi(t) \leq Ct,\ t \geq 0$, *for some* $C > 0$.

Then there exist $\eta \in \mathcal{M}$ *and a function* g *on* X *such that the following properties hold:*

(*a*) $g(0) = 0$, $g'(0) = 0$, *and* g' *exists and is uniformly continuous on* X *with modulus* η.
(*b*) *If* φ *is bounded or* A *is bounded, then* g *is Lipschitz.*
(*c*) *If* $f : A \to \mathbb{R}$, $a \in A$, *and* $x^* \in X^*$ *is a* $[t\varphi(t)]$*-subgradient of* f *at* a, *then*

$$g_a(x) := f(a) + \langle x - a, x^* \rangle - g(x-a) \leq f(x), \quad x \in A.$$

Moreover, in cases (*ii*), (*iii*), (*v*) *and* (*vi*), *we can choose* η *in a special form, namely:*

- *in case* (*ii*), *we can set* $\eta(t) = D\varphi(t)$ *for some* $D > 0$;
- *in case* (*iii*), *we can set* $\eta(t) = Dt^\alpha$ *for some* $D > 0$, *and*
- *in cases* (*v*) *and* (*vi*), *we can set* $\eta(t) = Dt^\beta$ *for some* $D > 0$.

Proof. Let n be an equivalent norm on X such that n' is uniformly continuous on S_X with minimal modulus of continuity $\omega_1(t) \leq C_1 t^\beta$ (see Lemma 2.6). We will give the proof in two steps.

Step 1. Define ψ and h as in Lemma 5.3 and set $g := 2h$. Then clearly $g(0) = 0$ and $g'(0) = 0$ (cf. the beginning of the proof of Lemma 5.3). Further, if φ is bounded, ψ is Lipschitz, and therefore $g(x) = 2\psi(\|x\|)$ is Lipschitz as well.

For each $\tau > 0$, concavity of φ implies $\varphi(t) \geq t \cdot (\varphi(\tau)/\tau)$ for $0 \leq t \leq \tau$, and therefore $\psi(\tau) = \int_0^\tau \varphi \geq (1/2)\tau\varphi(\tau)$. So, $g(x) \geq \|x\| \cdot \varphi(\|x\|)$.

To prove property (*c*), let $f : A \to \mathbb{R}$, $a \in A$, and $x^* \in X^*$ be a $[t\varphi(t)]$-subgradient of f at a. By (8), for each $x \in A$, we have

$$\langle x - a, x^* \rangle - (f(x) - f(a)) \leq \varphi(\|x-a\|)\|x-a\| \leq g(x-a),$$

which is equivalent to $g_a(x) \leq f(x),\ x \in A$. So (*c*) is proved.

Now suppose that (*i*) (resp. (*ii*); resp. (*iii*); resp. (*v*)) holds. Then, Lemma 5.3(*ii*) (resp. (*iv*), resp. (*v*); resp. (*iii*)) gives that h', and thus also g', is uniformly continuous with a modulus η (resp. with a modulus of the form $\eta(t) = D\varphi(t)$, resp. with a modulus of the form $\eta(t) = Dt^\alpha$; resp. with a modulus of the form $\eta(t) = Dt^\beta$). So, we have proved all assertions of the lemma in the case when A is unbounded, or if (*i*) or (*v*) holds. (Otherwise we have not proved condition (*b*).)

Step 2. Now suppose that A is bounded and one of conditions (*ii*), (*iii*), (*iv*), (*vi*) holds. Set

$$\tilde{\varphi}(t) := \min(\varphi(t), \varphi(\operatorname{diam} A)), \quad t \geq 0.$$

Then $\tilde{\varphi}$ is bounded, concave, and

$$\varphi(t) \leq \tilde{\varphi}(t) \quad \text{whenever } 0 \leq t \leq \operatorname{diam}(A). \tag{19}$$

Let $\tilde{\psi}$, $\tilde{h}$ be the functions which correspond to $\tilde{\varphi}$ as in Lemma 5.3, and define $g := 2\tilde{h}$.

Note that $\tilde{\varphi}$ has the same properties as φ in cases (ii) and (iii). Further, in case (iv) (resp. (vi)), $\tilde{\varphi}$ has the same property as φ in the condition (i) (resp. (v)). Consequently, *Step 1* implies that g has all the desired properties, with the only exception:

We do not know that the condition (c) holds, but we know that (c) holds, if we replace "$[t\varphi(t)]$-subgradient" by "$[t\tilde{\varphi}(t)]$-subgradient".

(Note that (b) holds since $\tilde{\varphi}$ is bounded.)

However, (19) implies that $x^* \in X^*$ is a $[t\tilde{\varphi}(t)]$-subgradient of f at a whenever x^* is a $[t\varphi(t)]$-subgradient of f at a. So, property (c) holds. □

Using Lemma 5.4, we easily obtain several versions of results on representations and extensions of semiconvex functions.

Theorem 5.5. *Let X be a superreflexive Banach space, $\Omega \subset X$ be an open convex bounded set and f be a real valued function on Ω. Then the following assertions are equivalent.*

- (i) *f is semiconvex on Ω.*
- (ii) *There exists a family $\{g_\alpha \colon \alpha \in A\}$ of Fréchet differentiable functions on Ω such that $f(x) = \max\{g_\alpha(x) : \alpha \in A\}$, $x \in \Omega$, and the derivatives g'_α, $\alpha \in A$, are equally uniformly continuous on Ω.*
- (iii) *There exists a family $\{g_\alpha \colon \alpha \in A\}$ of Fréchet differentiable functions on Ω such that $f(x) = \sup\{g_\alpha(x) \colon \alpha \in A\}$, $x \in \Omega$, and the derivatives g'_α, $\alpha \in A$, are equally uniformly continuous on Ω.*

Moreover, if we add in (i) the statement that f is Lipschitz and in (ii) and (iii) the statement that g_α, $\alpha \in A$, are equi-Lipschitz, we obtain assertions $(i)^*$, $(ii)^*$, $(iii)^*$, which are also pairwise equivalent.

Proof. Let (i) hold. By Corollary 3.6 we can choose a concave $\varphi \in \mathcal{M}$ which is a modulus of semiconvexity of f. By Corollary 4.5, we can choose for each $a \in \Omega$ a $[t\varphi(t)]$-subgradient x^*_a of f at a. Since Lemma 5.4(iv) (with $A := \Omega$) is satisfied, we can choose $\eta \in \mathcal{M}$ and a C^1 Lipschitz function g on X such that $g(0) = 0$, $g'(0) = 0$, g' is uniformly continuous with modulus η, and, for each $a \in \Omega$,

$$g_a(x) := f(a) + \langle x - a, x^*_a \rangle - g(x - a) \leq f(x), \quad x \in \Omega. \tag{20}$$

Now we denote $A := \Omega$. Since $g_\alpha(\alpha) = f(\alpha)$ for each $\alpha \in \Omega$, we have $f(x) = \max\{g_\alpha(x) : \alpha \in A\}$, $x \in \Omega$, by (20). Further, $g'_\alpha(x) = x^*_\alpha - g'(x - \alpha)$ for each $\alpha \in A$ and $x \in X$, which clearly implies that

$$g'_\alpha, \quad \alpha \in A, \quad \text{are equally uniformly continuous on } X \text{ with modulus } \eta. \tag{21}$$

So, $(i) \Rightarrow (ii)$ is proved. Now suppose that even $(i)^*$ holds and K_1 is a Lipschitz constant of f. Then, in the above proof, we have $\|x^*_\alpha\| \leq K_1$, $\alpha \in A$ (see Remark 4.2). Let K_2 be a Lipschitz constant of g. Then clearly $\|g'_\alpha(x)\| \le K_1 + K_2,\ \alpha \in A, x \in \Omega$, which implies that

$$g_\alpha,\quad \alpha \in A,\quad \text{are equi-Lipschitz on } X. \tag{22}$$

So, we have proved that $(i)^* \Rightarrow (ii)^*$. The implications $(ii) \Rightarrow (iii)$ and $(ii)^* \Rightarrow (iii)^*$ are trivial.

Now suppose that (iii) holds and $\eta \in \mathcal{M}$ be such that $g'_\alpha,\ \alpha \in A$, are uniformly continuous with modulus η on Ω. Then all $g_\alpha,\ \alpha \in A$, are semiconvex on Ω with modulus η by Lemma 5.2, and consequently also f is semiconvex on Ω with modulus η by Lemma 5.1. Moreover, if $g_\alpha,\ \alpha \in A$, are equi-Lipschitz, then f is clearly Lipschitz. So, we have proved the implications $(iii) \Rightarrow (i)$ and $(iii)^* \Rightarrow (i)^*$. □

Remark 5.6. A local version of Theorem 5.5 is essentially contained in the proof of [22, Theorem 25]. (This theorem is not correct as stated, but it is correct, if we replace the assumption of approximate convexity of f around x_0 by the stronger assumption of semiconvexity of f on a neighbourhood of x_0. Indeed, this stronger assumption is necessary and sufficient for the existence of μ in the proof of [22, Lemma 23].)

The proof of Theorem 5.5 easily gives the following extension theorem.

Theorem 5.7. *Let X be a superreflexive Banach space, $\Omega \subset X$ be an open convex bounded set and f be a Lipschitz semiconvex function on Ω. Then there exists a Lipschitz semiconvex function F on X such that $F|_\Omega = f$.*

Proof. In the proof of Theorem 5.5 (the implication $(i)^* \Rightarrow (ii)^*$) we have obtained a family $\{g_\alpha \colon \alpha \in A\}$ of Fréchet differentiable functions on X such that $f(x) = \max\{g_\alpha(x) : \alpha \in A\},\ x \in \Omega$, and moreover (21) and (22) hold. Put $F(x) := \sup\{g_\alpha(x) : \alpha \in A\},\ x \in X$. Then F is clearly an extension of f. Further, F is Lipschitz by (22) and semiconvex by (21), Lemma 5.2, and by Lemma 5.1. □

In an analogous way, we prove the main theorems of the article, which contain similar "quantitative" results on special superreflexive spaces.

Theorem 5.8. *Let $0 < \beta \le 1$ and let X be a superreflexive Banach space which admits an equivalent norm with modulus of smoothness of power type $1+\beta$ (e.g., $X = L^p(\mu)$ with $p \ge 1+\beta$). Let $\Omega \subset X$ be an open convex set, f be a real valued function on Ω, and let $\varphi \in \mathcal{M}$ be given. Suppose that at least one of the following conditions holds:*

(*a*) $\beta = 1$.
(*b*) $\varphi(t) = t^\alpha,\ t \ge 0$, *for some* $0 < \alpha \le \beta$.

Then the following assertions are equivalent.

(*i*) *There is a $C > 0$ such that f is semiconvex on Ω with modulus $C\varphi$.*
(*ii*) *There exists a $C_1 > 0$ and a family $\{g_a \colon a \in A\}$ of Fréchet differentiable functions on Ω such that $f(x) = \max\{g_a(x) : a \in A\},\ x \in \Omega$, and the derivatives $g'_a,\ a \in A$, are equally uniformly continuous on Ω with modulus $C_1\varphi$.*
(*iii*) *There exists a $C_2 > 0$ and a family $\{g_a \colon a \in A\}$ of Fréchet differentiable functions on Ω such that $f(x) = \sup\{g_a(x) : a \in A\},\ x \in \Omega$, and the derivatives $g'_a,\ a \in A$, are equally uniformly continuous on Ω with modulus $C_2\varphi$.*

Moreover, if we add in (i) *the statement that* f *is Lipschitz and in* (ii) *and* (iii) *the statement that* g_a, $a \in A$, *are equi-Lipschitz, we obtain assertions* $(i)^*$, $(ii)^*$, $(iii)^*$, *which are pairwise equivalent, whenever* Ω *is bounded.*

Proof. The proof is almost the same as that of Theorem 5.5; the only difference is in the proofs of implications $(i) \Rightarrow (ii)$ and $(i)^* \Rightarrow (ii)^*$. Suppose that (i) holds. By Corollary 3.6, find a concave $\tilde{\varphi}$ such that $\tilde{\varphi} \leq 4C\varphi$ and f is semiconvex on Ω with modulus $\tilde{\varphi}$. So, we can proceed as in the proof of Theorem 5.5, now with $\tilde{\varphi}$ instead of φ, and using the fact that either condition (ii) or condition (iii) of Lemma 5.4 is satisfied. In this way, we obtain η and g such that $\eta = D\tilde{\varphi} \leq 4CD\varphi$ and (21) holds; so (ii) holds with $C_1 = 4CD$. If Ω is bounded and $(i)^*$ holds, then Remark 4.2 and Lemma 5.4(b) yield (22), and therefore $(ii)^*$ holds. □

Proceeding quite similarly as in the proof Theorem 5.8, and using Lemma 5.4(vi), we obtain the following less precise result.

Proposition 5.9. *Let* X *be a superreflexive Banach space which admits an equivalent norm with modulus of smoothness of power type* $1+\beta$, *where* $0 < \beta \leq 1$. *Let* $\Omega \subset X$ *be a bounded open convex set and* f *be a real valued function on* Ω *which is semiconvex with linear modulus. Then there exists a* $C > 0$ *and a family* $\{g_a \colon a \in A\}$ *of Fréchet differentiable functions on* Ω *such that* $f(x) = \max\{g_a(x) : a \in A\}$, $x \in \Omega$, *and the derivatives* g_a', $a \in A$, *are equally uniformly continuous on* Ω *with modulus* $\eta(t) = Ct^\beta$.

Theorem 5.10. *Let* X *be a superreflexive Banach space which admits an equivalent norm with modulus of smoothness of power type* $1+\beta$, *where* $0 < \beta \leq 1$ *(e.g.,* $X = L^p(\mu)$ *with* $p \geq 1+\beta$*). Let* $\varphi \in \mathcal{M}$, $\Omega \subset X$ *be an open convex bounded set and* f *be a Lipschitz function which is semiconvex on* Ω *with modulus* φ. *Suppose that at least one of the following conditions holds:*

(a) $\beta = 1$.

(b) $\varphi(t) = Ct^\alpha$, $t \geq 0$, *for some* $0 < \alpha \leq \beta$ *and* $C > 0$.

Then there exists a $E > 0$ *and a Lipschitz function* F *on* X *which is semiconvex with modulus* $E\varphi$ *such that* $F|_\Omega = f$.

Proof. In the proof of Theorem 5.8 (the implication $(i)^* \Rightarrow (ii)^*$) we have obtained a family $\{g_a \colon a \in A\}$ of Fréchet differentiable functions on X such that $f(x) = \max\{g_a(x) : a \in A\}$, $x \in \Omega$, and moreover (21) and (22) (where $\eta \leq E\varphi$ with $E := 4CD$) hold. Put $F(x) := \sup\{g_a(x) : a \in A\}$, $x \in X$. Then F is clearly an extension of f. Further, F is Lipschitz by (22) and semiconvex with modulus $E\varphi$ by (21), Lemma 5.2, and Lemma 5.1. □

Remark 5.11. By slightly modifying the corresponding proofs, we can obtain analogues of Theorems 5.5, 5.7 and 5.10, in which it is not supposed that Ω is bounded, but it is supposed that f is semiconvex with a bounded modulus of semiconvexity. Since the case of bounded moduli does not seem to be sufficiently interesting, we state the following theorem only (whose proof is based on Lemma 5.4(i)):

Let X *be a superreflexive Banach space,* $\Omega \subset X$ *be an open convex set and* f *be a real valued function on* Ω *which is semiconvex with a bounded modulus. Then the assertion* *(ii) of Theorem 5.5 holds. Moreover, if f is also Lipschitz, then f admits a Lipschitz extension to all X, which is semiconvex with a bounded modulus.*

Finally, we prove a result on extensions of $[t\varphi(t)]$-subdifferentiable functions from arbitrary bounded sets, which will be applied in [11].

Proposition 5.12. *Let X be a superreflexive Banach space which admits an equivalent norm with modulus of smoothness of power type $1+\beta$, where $0<\beta\le 1$. Let $\emptyset \ne A \subset X$ be a bounded set, let $\varphi \in \mathcal{M}$ be concave, and $K \ge 0$. Let f be a Lipschitz function on A which has at each point $a \in A$ a $[t\varphi(t)]$-subgradient $h_a \in X^*$ with $\|h_a\| \le K$. Then there exists a Lipschitz semiconvex function F on X such that $F|_A = f$. Moreover, if*

(*i*) $\beta = 1$ *or*

(*ii*) $\varphi(t) = Ct^\alpha$ *for some* $C > 0$ *and* $0 < \alpha \le \beta$,

then F is semiconvex with modulus $D\varphi$ for some $D > 0$.

Proof. By Lemma 5.4(*ii*) and (*iii*), there exists $\eta \in \mathcal{M}$ (which can be chosen in the form $\eta = D\varphi$, $D > 0$ if (*i*) or (*ii*) holds) and a Lipschitz C^1 function g on X such that $g(0) = 0$, g' is uniformly continuous with modulus η, and, for each $a \in A$,

$$g_a(x) := f(a) + \langle x-a, h_a\rangle - g(x-a) \le f(x), \quad x \in A. \tag{23}$$

Similarly as in the proof of Theorem 5.5, we obtain that (21) and (22) hold. Set $F(x) := \sup\{g_a(x) \colon a \in A\}$, $x \in X$. By (23) we obtain that $F(x) = f(x)$ for $x \in A$, thus (22) implies that F is Lipschitz on X. Using (21), Lemma 5.2, and Lemma 5.1, we obtain that F is semiconvex with modulus η, which completes the proof. □

Remark 5.13. If we suppose in Proposition 5.12 that $\varphi(t) = Ct$, then the above proof (now with the application of Lemma 5.4(*vi*)) gives that F can be found to be semiconvex with modulus $\eta(t) = Dt^\beta$.

Acknowledgements. The authors thank the anonymous referee for helpful remarks.

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