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Existence of Exact Penalty and its Stability for Inequality-Constrained Optimization Problems

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In this paper we use the penalty approach in order to study a large class of inequality-constrained minimization problems in Banach spaces. A penalty function is said to have the generalized exact penalty property if there is a penalty coefficient for which approximate solutions of the unconstrained penalized problem are close enough to approximate solutions of the corresponding constrained problem. In this paper we show that the generalized exact penalty property is stable under perturbations of cost functions, constraint functions and the right-hand side of constraints.

Keywords: Approximate solution, Ekeland's variational principle, minimization problem, penalty function

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1. Introduction

Penalty methods are an important and useful tool in constrained optimization. See, for example, [1–7, 9–15] and the references mentioned there. In this paper we use the penalty approach in order to study inequality-constrained minimization problems in Banach spaces. A penalty function is said to have the exact penalty property [3, 4, 7, 11, 12] if there is a penalty coefficient for which a solution of an unconstrained penalized problem is a solution of the corresponding constrained problem. The notion of exact penalization was introduced in [9, 13]. For a review of the literature on exact penalization see [3, 4, 7].

In this paper we will establish the exact penalty property for a large class of inequality-constrained minimization problems

$$(P) \qquad f(x) \to \min \quad \text{subject to } x \in A$$

where

$$A = \{x \in X : g_i(x) \le c_i \text{ for } i = 1, \dots, n\}.$$

Here X is a Banach space, c_i, $i = 1, \dots, n$ are real numbers, and the constraint functions g_i, $i = 1, \dots, n$ and the objective function f are lower semicontinuous and satisfy certain assumptions.

We associate with the inequality-constrained minimization problem above the corre-

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sponding family of unconstrained minimization problems

$$f(z) + \gamma \sum_{i=1}^{n} \max\{g_i(z) - c_i, 0\} \to \min, \quad z \in X$$

where $\gamma > 0$ is a penalty. In this paper we establish the existence of a penalty coefficient for which approximate solutions of the unconstrained penalized problem are close enough to approximate solutions of the corresponding constrained problem. This novel approach in the penalty type methods was used in [14, 15]. In the present paper we obtain a generalization of the main results of [15].

Consider a minimization problem $h(z) \to \min$, $z \in X$ where $h : X \to R^1$ is a lower semicontinuous bounded from below function. If the space X is infinite-dimensional, then the existence of solutions of the problem is not guaranteed and in this situation we consider δ-approximate solutions. Namely, $x \in X$ is a δ-approximate solution of the problem $h(z) \to \min$, $z \in X$, where $\delta > 0$, if $h(x) \le \inf\{h(z) : z \in X\} + \delta$.

In [15] and in this paper we are interested in approximate solutions of the unconstrained penalized problem and in approximate solutions of the corresponding constrained problem. Under certain assumptions which hold for a large class of problems we show the existence of a constant $\Lambda_0 > 0$ such that the following property holds:

For each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ which depends only on ϵ such that if x is a $\delta(\epsilon)$-approximate solution of the unconstrained penalized problem whose penalty coefficient is larger than Λ_0, then there exists an ϵ-approximate solution y of the corresponding constrained problem such that $||y - x|| \le \epsilon$.

This property implies that any exact solution of the unconstrained penalized problem whose penalty coefficient is larger than Λ_0, is an exact solution of the corresponding constrained problem. Indeed, let x be a solution of the unconstrained penalized problem whose penalty coefficient is larger than Λ_0. Then for any $\epsilon > 0$ the point x is also a $\delta(\epsilon)$-approximate solution of the same unconstrained penalized problem and in view of the property above there is an ϵ-approximate solution y_ϵ of the corresponding constrained problem such that $||x - y_\epsilon|| \le \epsilon$. Since ϵ is an arbitrary positive number we can easily deduce that x is an exact solution of the corresponding constrained problem. Therefore our results also includes the classical penalty result as a special case.

It should be mentioned that if one uses methods in order to solve optimization problems these methods usually provide only approximate solutions of the problems. Therefore our results are important and useful even when optimization problems have exact solutions. Note that exact penalty results in the classical sense for convex minimization problems on finite-dimensional spaces were obtained in [6, 9] and our results are their extensions.

In [15] we established the existence of exact penalty for the problem (P) with convex constraint functions g_i, $i = 1, \dots, n$ and assuming that the objective function f belongs to a large class of functions. This class of functions includes the set of all convex bounded from below semicontinuous functions $f : X \to R^1$ which satisfy the growth condition $\lim_{||x|| \to \infty} f(x) = \infty$ and the set of all functions f on X which satisfy the growth condition above and which are Lipschitzian on all bounded subsets of X.

In this paper we study the stability of the generalized exact penalty property under perturbations of the functions f and $g_1, \dots, g_n$ and of the parameters $c_1, \dots, c_n$. The stability of the generalized exact penalty property is crucial in practice. One reason is that in practice we deal with a problem which consists a perturbation of the problem we wish to consider. Another reason is that the computations introduce numerical errors.

In this paper we continue to study the problem (P) with constraint functions $g_1, \dots, g_n$ and an objective function f as in [15]. More precisely, we consider a family of constrained minimization problems of type (P) with an objective function close to f and with constraint functions close to $g_1, \dots, g_n$ in a certain natural sense. We show that all the constrained minimization problems belonging to this family possess the generalized exact penalty property with the same penalty coefficient which depends only on f, $g_1, \dots, g_n$, $c_1, \dots, c_n$. It should be mentioned that for a general natural number n we suppose that constraint functions of any problem from our family are convex while for $n = 1$ constraint functions are not assumed to be necessarily convex.

2. Preliminaries and the main result

In this paper we use the convention that $\lambda \cdot \infty = \infty$ for all $\lambda \in (0, \infty)$, $\lambda + \infty = \infty$ and $\max\{\lambda, \infty\} = \infty$ for any real number λ and that supremum over empty set is $-\infty$.

Let $(X, ||\cdot||)$ be a Banach space. For each $x \in X$ and each $r > 0$ set

$$B(x, r) = \{y \in X : ||x - y|| \le r\}, \qquad B^o(x, r) = \{y \in X : ||x - y|| < r\}.$$

For each function $f : X \to R^1 \cup \{\infty\}$ and each nonempty set $A \subset X$ put

$$\mathrm{dom}(f) = \{x \in X : f(x) < \infty\}, \qquad \inf(f) = \inf\{f(z) : z \in X\},$$

$$\inf(f; A) = \inf\{f(z) : z \in A\}.$$

For each $x \in X$ and each $B \subset X$ set

$$d(x, B) = \inf\{||x - y|| : y \in B\}. \tag{1}$$

Let n be a natural number. For each $\kappa \in (0, 1)$ denote by Ω_κ the set of all $\gamma = (\gamma_1, \dots, \gamma_n) \in R^n$ such that

$$\kappa \le \min\{\gamma_i : i = 1, \dots, n\} \quad \text{and} \quad \max\{\gamma_i : i = 1, \dots, n\} = 1. \tag{2}$$

Assume that $\phi : X \to R^1$ satisfies

$$\lim_{||x|| \to \infty} \phi(x) = \infty \quad \text{and} \quad \inf(\phi) > -\infty. \tag{3}$$

In this paper we will consider problems of type (P) with objective functions f which satisfy $f(x) \ge \phi(x)$ for all $x \in X$.

Let $\bar{c}_0 \in R^1$, $\bar{c} = (\bar{c}_1 \dots, \bar{c}_n) \in R^n$ and let $f_i : X \to R^1 \cup \{\infty\}$, $i = 1, \dots, n$ be convex lower semicontinuous functions and put

$$A = \{x \in X : f_i(x) \le \bar{c}_i \ \text{ for all } i = 1, \dots, n\}. \tag{4}$$

In this paper we will consider a family of problems of type (P) with constraint functions close to $f_1, \dots, f_n$ in a certain natural sense. We assume that $\theta \in X$ satisfies

$$f_i(\theta) < \bar{c}_i, \quad i = 1, \dots, n. \tag{5}$$

Let us now describe the collection of objective functions (denoted by $\mathcal{A}$) which corresponds to our family of constrained minimization problems.

By (3) there is a real number M_0 such that

$$||\theta|| + 4 < M_0,$$

$$\phi(z) > \bar{c}_0 + 4 \ \text{ for all } z \in X \text{ satisfying } ||z|| \geq M_0 - 4. \tag{6}$$

Let X_0 be a nonempty convex subset of $B^o(0, M_0)$ such that $\theta \in X_0$.

Assume that a function $h : B^o(0, M_0) \times X_0 \to R^1 \cup \{\infty\}$ satisfies the following assumptions:

(A1) $h(z, y)$ is finite for all $y, z \in X_0$ and $h(y, y) = 0$ for each $y \in X_0$;
(A2) for each $y \in X_0$ the function $h(\cdot, y) : B^o(0, M_0) \to R^1 \cup \{\infty\}$ is convex;
(A3) for each $z \in X_0$

$$\sup\{h(z, y) : y \in X_0\} < \infty.$$

Let $M_1 > 0$. We denote by $\mathcal{A}$ a set of all lower semicontinuous functions $f : X \to R^1 \cup \{\infty\}$ such that

$$f(x) \geq \phi(x) \ \text{ for all } x \in X, \tag{7}$$

$$f(\theta) \leq \bar{c}_0,$$

$$B^o(0, M_0) \cap \text{dom}(f) \subset X_0$$

and that the following assumption holds:

(A4) for each $y \in \text{dom}(f) \cap B^o(0, M_0)$ there exists a neighborhood V of y in X such that $V \subset B^o(0, M_0)$ and that

$$f(z) - f(y) \leq M_1 h(z, y) \ \text{ for all } z \in V.$$

Below we consider two examples of the function h and the set $\mathcal{A}$.

Example 2.1. Assume that a function $f_0 : X \to R^1$ is Lipschitz on bounded subsets of X, $f_0(\theta) \leq \bar{c}_0$ and that $f_0(x) \geq \phi(x)$ for all $x \in X$. Then there exists $L_0 > 0$ such that

$$|f_0(z_1) - f_0(z_2)| \leq L_0 ||z_1 - z_2|| \ \text{ for all } z_1, z_2 \in B(0, M_0).$$

Let $L_1 > 0$ and set

$$M_1 = L_0 + L_1, \ h(z, y) = ||z - y||, \quad z, y \in X,$$

$$X_0 = B^o(0, M_0).$$

It is not difficult to see that the set $\mathcal{A}$ contains all lower semicontinuous functions $f : X \to R^1 \cup \{\infty\}$ such that $f(x) \geq \phi(x)$ for all $x \in X$, $f(\theta) \leq \bar{c}_0$, $f(z)$ is finite for all $z \in B^o(0, M_0)$ and that

$$|(f - f_0)(z_1) - (f - f_0)(z_2)| \leq L_1 ||z_1 - z_2|| \ \text{ for all } z_1, z_2 \in B^o(M_0).$$

Example 2.2. Assume that $f_0 : X \to R^1 \cup \{\infty\}$ is a lower semicontinuous convex function such that $f_0(x) \geq \phi(x)$ for all $x \in X$ and $f_0(\theta) \leq \bar{c}_0$. Let $L_1 > 0$ and set

$$X_0 = \mathrm{dom}(f_0) \cap B^o(0, M_0).$$

Clearly, $\theta \in X_0$. Put $M_1 = 1$.

For each $z \in B^o(0, M_0)$ and each $y \in X_0$ define

$$h(z, y) = \sup \{\xi(z) - \xi(y) : \ \xi \in \mathcal{B}\} + L_1||z - y||, \tag{8}$$

where $\mathcal{B}$ is the set of all convex functions $\xi : B^o(0, M_0) \to R^1 \cup \{\infty\}$ such that

$$|\xi(v) - f_0(v)| \leq L_1 \ \text{ for all } v \in \mathrm{dom}(f_0) \cap B^o(0, M_0).$$

It is not difficult to see that the function $h(\cdot, \cdot)$ is well defined and that the assumptions (A1), (A2) and (A3) hold.

Assume that $f : X \to R^1 \cup \{\infty\}$ is a lower semicontinuous function such that

$$f(\theta) \leq \bar{c}_0, \ f(x) \geq \phi(x) \ \text{ for all } x \in X, \tag{9}$$

$$\mathrm{dom}(f) \cap B^o(0, M_0) = \mathrm{dom}(f_0) \cap B^o(0, M_0)$$

and that there exists a convex function $g : B^o(0, M_0) \to R^1 \cup \{\infty\}$ such that

$$\mathrm{dom}(f) \cap B^0(0, M_0) = \mathrm{dom}(g) \cap B^o(0, M_0), \tag{10}$$

$$|f_0(z) - g(z)| \leq L_1 \ \text{ for all } z \in \mathrm{dom}(f_0) \cap B^o(0, M_0), \tag{11}$$

$$|(f - g)(z_1) - (f - g)(z_2)| \leq L||z_1 - z_2|| \ \text{ for all } z_1, z_2 \in \mathrm{dom}(f_0) \cap B^o(0, M_0).$$

We show that $f \in \mathcal{A}$. In order to meet this goal it is sufficient to show that (A4) holds.

Let $y \in B^o(0, M_0) \cap$ dom(f) and $z \in B^o(0, M_0)$. If $f(z) = \infty$, then $f_0(z) = \infty$ and

$$h(z, y) \geq f_0(z) - f_0(y) = \infty = f(z) - f(y)$$

by definition.

If $f(z) < \infty$, then it follows from (9) and (10) that $f_0(z) < \infty$, $g(z) < \infty$, $f_0(y) < \infty$ and $g(y) < \infty$. Together with (8)–(12) this implies that

$$\begin{aligned} f(z) - f(y) &= g(z) - g(y) + [(f - g)(z) - (f - g)(y)] \\ &\leq g(z) - g(y) + L_1||z - y|| \leq h(z, y). \end{aligned}$$

Thus $f(z) - f(y) \leq h(z, y)$ in both cases, (A4) holds and $f \in \mathcal{A}$.

Let us now describe the collections of constraint functions which correspond to our family of constrained minimization problems.

For each $i \in \{1, \ldots, n\}$ and each $\epsilon > 0$ denote by $\mathcal{V}(i, \epsilon)$ the set of all convex lower semicontinuous functions $g : X \to R^1 \cup \{\infty\}$ such that

$$\mathrm{dom}(g) \cap B^o(0, M_0) = \mathrm{dom}(f_i) \cap B^o(0, M_0), \tag{12}$$

$$|f_i(x) - g(x)| \leq \epsilon \ \text{ for all } x \in \mathrm{dom}(f_i) \cap B^o(0, M_0). \tag{13}$$

We will establish the following result.

Theorem 2.3. *Let $\kappa \in (0,1)$. Then there exist $\Lambda_0 > 0$ and $\Delta_0 \geq 1$ such that for each $\epsilon > 0$ there exists $\delta \in (0, \epsilon)$ for which the following assertion holds:*

If $g_0 \in \mathcal{A}$, $g_i \in \mathcal{V}(i, \Delta_0^{-1})$, $i = 1, \dots, n$, $\gamma \in \Omega_\kappa$, $\lambda \geq \Lambda_0$, $c = (c_1, \dots, c_n) \in R^n$ satisfies

$$|\bar{c}_i - c_i| \leq \Delta_0^{-1}, \quad i = 1, \dots, n$$

and if $x \in X$ satisfies

$$g_0(x) + \sum_{i=1}^{n} \lambda\gamma_i \max\{g_i(x) - c_i, 0\}$$
$$\leq \inf\left\{ g_0(z) + \sum_{i=1}^{n} \lambda\gamma_i \max\{g_i(z) - c_i, 0\} : z \in X \right\} + \delta,$$

then there is $y \in X$ such that

$$||y - x|| \leq \epsilon, \quad g_i(y) \leq c_i, \quad i = 1, \dots, n,$$

$$g_0(y) \leq \inf\{g_0(z) : z \in X \text{ and } g_i(z) \leq c_i, \ i = 1, \dots, n\} + \epsilon.$$

Theorem 2.3 implies the following result.

Corollary 2.4. *Let $\kappa \in (0,1)$ and let $\Lambda_0 > 0$ and $\Delta_0 \geq 1$ be as guaranteed by Theorem 2.3. Then for each $g_0 \in \mathcal{A}$, each $g_i \in \mathcal{V}(i, \Delta_0^{-1})$, $i = 1, \dots, n$, each $\gamma \in \Omega_\kappa$, each $\lambda \geq \Lambda_0$, each $c = (c_1, \dots, c_n) \in R^n$ which satisfies $|\bar{c}_i - c_i| \leq \Delta_0^{-1}$, $i = 1, \dots, n$ and each sequence $\{x_i\}_{i=1}^\infty \subset X$ which satisfies*

$$\lim_{j\to\infty} \left[g_0(x_j) + \sum_{i=1}^{n} \lambda\gamma_i \max\{g_i(x_j) - c_i, 0\} \right]$$
$$= \inf\left\{ g_0(z) + \sum_{i=1}^{n} \lambda\gamma_i \max\{g_i(z) - c_i, 0\} : z \in X \right\}$$

there is a sequence $\{y_i\}_{i=1}^\infty \subset \{z \in X : g_i(z) \leq c_i, \ i = 1, \dots, n\}$ such that

$$\lim_{j\to\infty} g_0(y_j) = \inf\{g_0(z) : z \in X \text{ and } g_i(z) \leq c_i, \ i = 1, \dots, n\},$$

$$\lim_{i\to\infty} ||x_i - y_i|| = 0.$$

3. Proof of Theorem 2.3

We show that there is $\Lambda_0 \geq 1$ such that the following property holds:

(P1) For each $\epsilon \in (0,1)$ there is $\delta \in (0, \epsilon)$ such that for each $g_0 \in \mathcal{A}$, each $g_i \in \mathcal{V}(i, \Lambda_0^{-1})$, $i = 1, \dots, n$, each $c = (c_1, \dots, c_n) \in R^n$ satisfying

$$|\bar{c}_i - c_i| \leq \Lambda_0^{-1}, \ i = 1, \dots, n,$$

each $\gamma \in \Omega_\kappa$, each $\lambda \geq \Lambda_0$, each $x \in X$ which satisfies

$$g_0(x) + \sum_{i=1}^{n} \lambda\gamma_i \max\{g_i(x) - c_i, 0\}$$
$$\leq \inf\left\{g_0(z) + \sum_{i=1}^{n} \lambda\gamma_i \max\{g_i(z) - c_i, 0\} : z \in X\right\} + \delta$$

there is $y \in X$ such that

$$||y - x|| \leq \epsilon, \; g_i(y_i) \leq c_i \; \text{ for all } i = 1, \dots, n,$$

$$g_0(y) + \sum_{i=1}^{n} \lambda\gamma_i \max\{g_i(y) - c_i, 0\} \leq g_0(x) + \sum_{i=1}^{n} \lambda\gamma_i \max\{g_i(x) - c_i, 0\}.$$

It is clear that Theorem 2.3 easily follows from the property (P1).

Assume that there is no $\Lambda_0 \geq 1$ such that (P1) holds. Then for each natural number k there exist

$$\epsilon_k \in (0,1), \quad g_0^{(k)} \in \mathcal{A}, \quad g_i^{(k)} \in \mathcal{V}(i, k^{-1}), \quad i = 1, \dots, n, \tag{14}$$

$c^{(k)} = (c_1^{(k)}, \dots, c_n^{(k)}) \in R^n$ satisfying

$$\left|c_i^{(k)} - \bar{c}_i\right| \leq 1/k, \quad i = 1, \dots, n, \tag{15}$$

$$\gamma^{(k)} = \left(\gamma_1^{(k)}, \dots, \gamma_n^{(k)}\right) \in \Omega_\kappa, \quad \lambda_k \geq k \tag{16}$$

and $x_k \in X$ which satisfies

$$g_0^{(k)}(x_k) + \sum_{i=1}^{n} \lambda_k\gamma_i^{(k)} \max\left\{g_i^{(k)}(x_k) - c_i^{(k)}, 0\right\} \tag{17}$$
$$\leq \inf\left\{g_0^{(k)}(z) + \lambda_k \sum_{i=1}^{n} \gamma_i^{(k)} \max\left\{g_i^{(k)}(z) - c_i^{(k)}, 0\right\} : z \in X\right\} + 2^{-1}\epsilon_k k^{-2}$$

and

$$\Bigg\{ y \in B(x_k, \epsilon_k) : g_i^{(k)}(y) \leq c_i \text{ for all } i = 1, \dots, n \text{ and}$$
$$g_0^{(k)}(y) + \sum_{i=1}^{n} \lambda_k\gamma_i^{(k)} \max\left\{g_i^{(k)}(y) - c_i^{(k)}, 0\right\} \tag{18}$$
$$\leq g_0^{(k)}(x_k) + \sum_{i=1}^{n} \lambda_k\gamma_i^{(k)} \max\left\{g_i(x_k) - c_i^{(k)}, 0\right\} \Bigg\} = \emptyset.$$

For any integer $k \geq 1$ set

$$\psi_k(z) = g_0^{(k)}(z) + \sum_{i=1}^{n} \lambda_k\gamma_i^{(k)} \max\left\{g_i^{(k)}(z) - c_i^{(k)}, 0\right\}, \quad z \in X. \tag{19}$$

Clearly, for any natural number k the function ψ_k is lower semicontinuous and

$$\psi_k(z) \geq \phi(z) \quad \text{for all } z \in X. \tag{20}$$

Let k be a natural number. It follows from (17), (19) and Ekeland's variational principle [8] that there is $y_k \in X$ such that

$$\psi_k(y_k) \leq \psi_k(x_k), \tag{21}$$

$$||y_k - x_k|| \leq (2k)^{-1}\epsilon_k, \tag{22}$$

$$\psi_k(y_k) \leq \psi_k(z) + k^{-1}||z - y_k|| \quad \text{for all } z \in X. \tag{23}$$

By (18), (19), (21) and (22) for all natural numbers k

$$y_k \notin \left\{ z \in X : g_i^{(k)}(z) \leq c_i^{(k)} \text{ for all } i = 1, \dots, n \right\}. \tag{24}$$

For each natural number k we set

$$I_{1k} = \left\{ i \in \{1, \dots, n\} : g_i^{(k)}(y_k) > c_i^{(k)} \right\}, \tag{25}$$

$$I_{2k} = \left\{ i \in \{1, \dots, n\} : g_i^{(k)}(y_k) = c_i^{(k)} \right\},$$

$$I_{3k} = \left\{ i \in \{1, \dots, n\} : g_i^{(k)}(y_k) < c_i^{(k)} \right\},$$

In view of (24) and (25),

$$I_{1k} \neq \emptyset \quad \text{for all integers } k \geq 1. \tag{26}$$

Extracting a subsequence and re-indexing we may assume without loss of generality that

$$I_{1k} = I_{11}, \quad I_{2k} = I_{21}, \quad I_{3k} = I_{31} \quad \text{for all natural numbers } k. \tag{27}$$

By (5) there exists a natural number k_0 such that

$$8k_0^{-1} < \min \left\{ \bar{c}_i - f_i(\theta) : i = 1, \dots, n \right\}. \tag{28}$$

Assume that an integer $k \geq k_0$. In view of (14), (15), (5), (6) and (28) for all integers $i = 1, \dots, n$

$$c_i^{(k)} - g_i^{(k)}(\theta) \geq -k^{-1} + \bar{c}_i - f_i(\theta) - k^{-1} \geq 8k_0^{-1} - 3k^{-1} > 0. \tag{29}$$

By (14), the definition of $\mathcal{A}$, (29), (19), (17), (14), (21) and (25)–(27),

$$\begin{aligned}
\bar{c}_0 &\geq g_0^{(k)}(\theta) \geq \inf \left\{ g_0^{(k)}(z) : z \in X \text{ and } g_i^{(k)}(z) \leq c_i^{(k)} \text{ for all integers } i = 1, \dots, n \right\} \\
&= \inf \left\{ \psi_k(z) : z \in X \text{ and } g_i^{(k)}(z) \leq c_i^{(k)} \text{ for all integers } i = 1, \dots, n \right\} \\
&\geq \inf(\psi_k) \geq \psi_k(x_k) - 1 \geq \psi_k(y_k) - 1 \\
&= g_0^{(k)}(y_k) + \sum_{i \in I_{11}} \lambda_k \gamma_i^{(k)} \left(g_i^{(k)}(y_k) - c_i^{(k)} \right) - 1.
\end{aligned} \tag{30}$$

Relations (25)–(27) and (30) imply that

$$g_0^{(k)}(y_k) \le \bar{c}_0 + 1 \quad \text{for all natural numbers } k \ge k_0. \tag{31}$$

Together with (14) and (7) this implies that

$$\phi(y_k) \le \bar{c}_0 + 1 \quad \text{for all natural numbers } k \ge k_0. \tag{32}$$

Relations (32) and (6) imply that

$$||y_k|| \le M_0 - 4 \quad \text{for all natural numbers } k \ge k_0. \tag{33}$$

Let $k \ge k_0$ be an integer. By (A4), (14), (31) and (33) there exists a neighborhood V_k of y_k in X such that

$$V_k \subset B^o(0, M_0),$$

$$g_0^{(k)}(z) - g_0^{(k)}(y_k) \le M_1 h(z, y_k) \quad \text{for all } z \in V_k. \tag{34}$$

In view of (30), (25)–(27), (16), (14), (7), (3) and (2) for each $i \in I_{11}$ and each integer $k \ge k_0$,

$$\begin{aligned} 0 < g_i^{(k)}(y_k) - c_i^{(k)} &\le \left[1 + \bar{c}_0 - \inf\left(g_0^{(k)}\right)\right] \left(\gamma_i^{k)}\right)^{-1} k^{-1} \\ &\le [1 + \bar{c}_0 - \inf(\phi)] k^{-1} \kappa^{-1}. \end{aligned} \tag{35}$$

Let $k \ge k_0$ be an integer. Since the functions $g_i^{(k)}$, $i = 1, \ldots, n$ are lower semicontinuous it follows from (25)–(27) that there exist a positive number $r_k < 1$ such that for each $y \in B(y_k, r_k)$

$$g_i^{(k)}(y) > c_i^{(k)} \quad \text{for each } i \in I_{11}. \tag{36}$$

It follows from (19), (25)–(27), (36), (21), (17), (30), (31) and (23) that for each $z \in B(y_k, r_k) \cap \operatorname{dom}\left(g_0^{(k)}\right)$

$$\begin{aligned} &\sum_{i \in I_{11}} \lambda_k \gamma_i^{(k)} \left(g_i^{(k)}(z) - c_i^{(k)}\right) + \sum_{i \in I_{21} \cup I_{31}} \lambda_k \gamma_i^{(k)} \max\left\{g_i^{(k)}(z) - c_i^{(k)}, 0\right\} \\ &\quad - \sum_{i \in I_{11}} \lambda_k \gamma_i^{(k)} \left(g_i^{(k)}(y_k) - c_i^{(k)}\right) - \sum_{i \in I_{21} \cup I_{31}} \lambda_k \gamma_i^{(k)} \max\left\{g_i^{(k)}(y_k) - c_i^{(k)}, 0\right\} \\ &= \psi_k(z) - \psi_k(y_k) - g_0^{(k)}(z) + g_0^{(k)}(y_k) \ge -k^{-1}||z - y_k|| + g_0^{(k)}(y_k) - g_0^{(k)}(z). \end{aligned}$$

This relation implies that for each $z \in B(y_k, r_k)$

$$\begin{aligned} &\sum_{i \in I_{11}} \gamma_i^{(k)} g_i^{(k)}(z) + \sum_{i \in I_{21} \cup I_{31}} \gamma_i^{(k)} \max\left\{g_i^{(k)}(z) - c_i^{(k)}, 0\right\} \\ &\quad - \sum_{i \in I_{11}} \gamma_i^{(k)} g_i^{(k)}(y_k) - \sum_{i \in I_{21} \cup I_{31}} \gamma_i^{(k)} \max\left\{g_i^{(k)}(y_k) - c_i^{(k)}, 0\right\} \\ &\quad + \lambda_k^{-1} \left(g_0^{(k)}(z) - g_0^{(k)}(y_k)\right) \ge -\lambda_k^{-1} k^{-1} ||y_k - z||. \end{aligned} \tag{37}$$

In view of (37) and (34) for each $z \in B(y_k, r_k) \cap V_k$

$$\sum_{i\in I_{11}} \gamma_i^{(k)} g_i^{(k)}(z) + \sum_{i\in I_{21}\cup I_{31}} \gamma_i^{(k)} \max\left\{g_i^{(k)}(z) - c_i^{(k)}, 0\right\} + \lambda_k^{-1} M_1 h(z, y_k) + \lambda_k^{-1} k^{-1} ||y_k - z|| \geq \sum_{i\in I_{11}} \gamma_i^{(k)} g_i^{(k)}(y_k) + \sum_{i\in I_{21}\cup I_{31}} \gamma_i^{(k)} \max\left\{g_i^{(k)}(y_k) - c_i^{(k)}, 0\right\}. \tag{38}$$

By (14) the functions $g_i^{(k)}$, $i = 1, \ldots, n$ are convex. It follows from (A2), (31), (14), (33) and (7) that the function

$$\sum_{i\in I_{11}} \gamma_i^{(k)} g_i^{(k)}(z) + \sum_{i\in I_{21}\cup I_{31}} \gamma_i^{(k)} \max\left\{g_i^{(k)}(z) - c_i^{(k)}, 0\right\} + \lambda_k^{-1} M_1 h(z, y_k) + \lambda_k^{-1} k^{-1} ||z - y_k||, \quad z \in B^o(0, M_0)$$

is convex. Combined with (A1) this implies that (38) is true for all $z \in B^o(0, M_0)$.

Extracting a subsequence and re-indexing we may assume without loss of generality that for each $i \in \{1, \ldots, n\}$ there is

$$\gamma_i = \lim_{k\to\infty} \gamma_i^{(k)} \in [0, 1]. \tag{39}$$

Clearly,

$$\gamma = (\gamma_1, \ldots, \gamma_n) \in \Omega_\kappa.$$

Let

$$z \in B^0(0, M_0) \cap [\cap_{i=1}^n \mathrm{dom}(f_i)] \cap X_0. \tag{40}$$

Relations (16) and (33) imply that

$$\lim_{k\to\infty} \lambda_k^{-1} k^{-1} ||z - y_k|| = 0. \tag{41}$$

In view of (16), (A3), (40), (33), (31), (14) and (7),

$$\lim_{k\to\infty} \lambda_k^{-1} M_1 h(z, y_k) = 0. \tag{42}$$

By (40), (14), (25), (13), (15), (41), (42), (38) which holds for z, (25), (27), (39) and (35),

$$\begin{aligned}
&\sum_{i\in I_{11}} \gamma_i f_i(z) + \sum_{i\in I_{21}\cup I_{31}} \gamma_i \max\{f_i(z) - \bar{c}_i, 0\} \\
&= \lim_{k\to\infty} \Bigg[\sum_{i\in I_{11}} \gamma_i^{(k)} g_i^{(k)}(z) + \sum_{i\in I_{21}\cup I_{31}} \gamma_i^{(k)} \max\left\{g_i^{(k)}(z) - c_i^{(k)}, 0\right\} \\
&\qquad + \lambda_k^{-1} k^{-1} ||z - y_k|| + \lambda_k^{-1} M_1 h(z, y_k) \Bigg] \\
&\geq \limsup_{k\to\infty} \sum_{i\in I_{11}} \gamma_i^{(k)} g_i^{(k)}(y_k) = \sum_{i\in I_{11}} \lim_{k\to\infty} \left(\gamma_i^{(k)} g_i^{(k)}(y_k)\right) = \sum_{i\in I_{11}} \gamma_i \bar{c}_i.
\end{aligned}$$

Therefore we have shown that for each $z \in X$ satisfying (40) we have

$$\sum_{i\in I_{11}} \gamma_i f_i(z) + \sum_{i\in I_{21}\cup I_{31}} \gamma_i \max\{f_i(z) - \bar{c}_i, 0\} \geq \sum_{i\in I_{11}} \gamma_i \bar{c}_i.$$

Combined with (5), (6) the inclusions $\theta \in X_0$ and $\gamma \in \Omega_\kappa$, (5) and (25)-(27) this implies that

$$\sum_{i\in I_{11}} \gamma_i \bar{c}_i \leq \sum_{i\in I_{11}} \gamma_i f_i(\theta) + \sum_{i\in I_{21}\cup I_{31}} \gamma_i \max\{f_i(\theta) - \bar{c}_i, 0\}$$
$$= \sum_{i\in I_{11}} \gamma_i f_i(\theta) < \sum_{i\in I_{11}} \gamma_i \bar{c}_i.$$

The contradiction we have reached proves that there exists $\Lambda_0 \geq 1$ such that the property (P1) holds. This completes the proof of Theorem 2.3.

4. An extension of Theorem 2.3 for minimization problems with one constraint function

In this section we assume that $n = 1$ and use the notation and definitions from Section 2. In this case

$$\bar{c} \in R^1. \tag{43}$$

Set

$$f = f_1. \tag{44}$$

In this case we also have that

$$A = \{x \in X : f(x) \leq \bar{c}\}, \quad f(\theta) < \bar{c}. \tag{45}$$

For each $\epsilon > 0$ denote by $\mathcal{U}(\epsilon)$ the set of all lower semicontinuous functions $g : X \to R^1 \cup \{\infty\}$ for which there exists a convex function $h : X \to R^1 \cup \{\infty\}$ such that

$$\mathrm{dom}(g) \cap B^o(0, M_0) = \mathrm{dom}(f) \cap B^o(0, M_0) = \mathrm{dom}(h) \cap B^o(0, M_0), \tag{46}$$

$$|f(x) - h(x)| \leq \epsilon \quad \text{for all } x \in \mathrm{dom}(f) \cap B^o(0, M_0), \tag{47}$$

$$|(h - g)(z_1) - (h - g)(z_2)| \leq \epsilon ||z_1 - z_2|| \quad \text{for all } z_1, z_2 \in \mathrm{dom}(f) \cap B^o(0, M_0), \tag{48}$$

$$|h(z) - g(z)| \leq \epsilon \quad \text{for all } z \in \mathrm{dom}(f) \cap B^o(0, M_0). \tag{49}$$

We will establish the following result.

Theorem 4.1. *There exist $\Lambda_0 > 0$ and $\Delta_0 \geq 1$ such that for each $\epsilon > 0$ there exists $\delta \in (0, \epsilon)$ for which the following assertion holds:*

If $g_0 \in \mathcal{A}$, $g \in \mathcal{U}(\Delta_0^{-1})$, $\lambda \geq \Lambda_0$, $c \in R^1$ satisfies $|\bar{c} - c| \leq \Delta_0^{-1}$ and if $x \in X$ satisfies

$$g_0(x) + \lambda \max\{g(x) - c, 0\} \leq \inf\{g_0(z) + \lambda \max\{g(z) - c, 0\} : z \in X\} + \delta,$$

then there is $y \in X$ such that

$$||y - x|| \leq \epsilon, \qquad g(y) \leq c,$$

$$g_0(y) \leq \inf\{g_0(z) : z \in X \text{ and } g(z) \leq c\} + \epsilon.$$

Since the set $\mathcal{U}(\epsilon)$ is larger than the set $\mathcal{V}(1,\epsilon)$ Theorem 4.1 is a generalization of Theorem 2.3 in the case $n = 1$.

Theorem 4.1 implies the following result.

Corollary 4.2. *Let $\Lambda_0 > 0$ and $\Delta_0 \geq 1$ be as guaranteed by Theorem 4.1. Then for each $g_0 \in \mathcal{A}$, each $g \in \mathcal{U}(\Delta_0^{-1})$, each $\lambda \geq \Lambda_0$, each $c \in R^1$ which satisfies $|\bar{c} - c| \leq \Delta_0^{-1}$, and each sequence $\{x_i\}_{i=1}^{\infty} \subset X$ which satisfies*

$$\lim_{j\to\infty} [g_0(x_j) + \lambda \max\{g(x_j) - c, 0\}] = \inf\{g_0(z) + \lambda \max\{g(z) - c, 0\} : z \in X\}$$

there is a sequence $\{y_i\}_{i=1}^{\infty} \subset \{z \in X : g(z) \leq c\}$ such that

$$\lim_{j\to\infty} g_0(y_j) = \inf\{g_0(z) : z \in X \text{ and } g(z) \leq c\} \quad \text{and} \quad \lim_{i\to\infty} ||x_i - y_i|| = 0.$$

5. Proof of Theorem 4.1

We show that there is $\Lambda_0 \geq 1$ such that the following property holds:

(P2) For each $\epsilon \in (0, 1)$ there is $\delta \in (0, \epsilon)$ such that for each $g_0 \in \mathcal{A}$, each $g \in \mathcal{U}(\Lambda_0^{-1})$, each $c \in R^1$ satisfying $|\bar{c} - c| \leq \Lambda_0^{-1}$, each $\lambda \geq \Lambda_0$ and each $x \in X$ which satisfies

$$g_0(x) + \lambda \max\{g(x) - c, 0\} \leq \inf\{g_0(z) + \lambda \max\{g(z) - c, 0\} : z \in X\} + \delta$$

there is $y \in X$ such that

$$||y - x|| \leq \epsilon, \qquad g(y) \leq c,$$

$$g_0(y) + \lambda \max\{g(y) - c, 0\} \leq g_0(x) + \lambda \max\{g(x) - c, 0\}.$$

It is clear that Theorem 4.1 easily follows from the property (P2).

Assume that there is no $\Lambda_0 \geq 1$ such that (P2) holds. Then for each natural number k there exist

$$\epsilon_k \in (0, 1), \qquad g_0^{(k)} \in \mathcal{A}, \qquad g^{(k)} \in \mathcal{U}(k^{-1}), \tag{50}$$

$c^{(k)} \in R^1$ satisfying

$$|c_k - \bar{c}| \leq 1/k, \tag{51}$$

$$\lambda_k \geq k \tag{52}$$

and $x_k \in X$ which satisfies

$$\begin{aligned} &g_0^{(k)}(x_k) + \lambda_k \max\left\{g^{(k)}(x_k) - c_k, 0\right\} \\ &\leq \inf\left\{g_0^{(k)}(z) + \lambda_k \max\left\{g^{(k)}(z) - c_k, 0\right\} : z \in X\right\} + 2^{-1}\epsilon_k k^{-2} \end{aligned} \tag{53}$$

and

$$\begin{aligned} &\Big\{y \in B(x_k, \epsilon_k) : g^{(k)}(y) \leq c_k \text{ and} \\ &g_0^{(k)}(y) + \lambda_k \max\left\{g^{(k)}(y) - c_k, 0\right\} \leq g_0^{(k)}(x_k) + \lambda_k \max\left\{g^{(k)}(x_k) - c_k, 0\right\}\Big\} = \emptyset. \end{aligned} \tag{54}$$

For any integer $k \geq 1$ set

$$\psi_k(z) = g_0^{(k)}(z) + \lambda_k \max\left\{g^{(k)}(z) - c_k, 0\right\}, \quad z \in X. \tag{55}$$

Clearly, for any natural number k the function ψ_k is lower semicontinuous and

$$\psi_k(z) \geq \phi(z) \text{ for all } z \in X. \tag{56}$$

Let k be a natural number. It follows from (53), (55), (56) and Ekeland's variational principle [8] that there is $y_k \in X$ such that

$$\psi_k(y_k) \leq \psi_k(x_k), \tag{57}$$

$$||y_k - x_k|| \leq (2k)^{-1}\epsilon_k, \tag{58}$$

$$\psi_k(y_k) \leq \psi_k(z) + k^{-1}||z - y_k|| \text{ for all } z \in X. \tag{59}$$

By (54), (55), (57) and (58) for all natural numbers k

$$g^{(k)}(y_k) > c_k. \tag{60}$$

By (52) there exists a natural number k_0 such that

$$8k_0^{-1} < \bar{c} - f(\theta). \tag{61}$$

Assume that an integer $k \geq k_0$. In view of (51), (50), (54), (52), (55) and (61),

$$c_k - g^{(k)}(\theta) \geq -k^{-1} + \bar{c} - f(\theta) - k^{-1} \geq 8k_0^{-1} - 2k^{-1} > 0. \tag{62}$$

By (50), the definition of $\mathcal{A}$, (55), (53), (50), (57) and (60),

$$\begin{aligned}\bar{c}_0 \geq g_0^{(k)}(\theta) &\geq \inf\left\{g_0^{(k)}(z) : z \in X \text{ and } g^{(k)}(z) \leq c_k\right\} \\ &= \inf\left\{\psi_k(z) : z \in X \text{ and } g^{(k)}(z) \leq c_k\right\} \\ &\geq \inf(\psi_k) \geq \psi_k(x_k) - 1 \geq \psi_k(y_k) - 1 = g_0^{(k)}(y_k) + \lambda_k\left(g^{(k)}(y_k) - c_k\right) - 1. \end{aligned} \tag{63}$$

Relations (60) and (63) imply that

$$g_0^{(k)}(y_k) \leq \bar{c}_0 + 1 \text{ for all natural numbers } k \geq k_0. \tag{64}$$

Together with (50) and (7) this implies that

$$\phi(y_k) \leq \bar{c}_0 + 1 \text{ for all natural numbers } k \geq k_0.$$

Combined with (6) this implies that

$$||y_k|| \leq M_0 - 4 \text{ for all natural numbers } k \geq k_0. \tag{65}$$

Let $k \geq k_0$ be an integer. By (A4), (50), (65), (64) and (7) there exists a neighborhood V_k of y_k in X such that

$$V_k \subset B^o(0, M_0),$$

$$g_0^{(k)}(z) - g_0^{(k)}(y_k) \leq M_1 h(z, y_k) \text{ for all } z \in V_k. \tag{66}$$

In view of (60), (63), (50) and (7) for each integer $k \geq k_0$,

$$0 < g^{(k)}(y_k) - c_k \leq \left[1 + \bar{c}_0 - \inf\left(g_0^{(k)}\right)\right] \lambda_k^{-1} \leq [1 + \bar{c}_0 - \inf(\phi)]k^{-1}. \tag{67}$$

Let $k \geq k_0$ be an integer. Since the function $g^{(k)}$ is lower semicontinuous it follows from (60) that there exists a positive number $r_k < 1$ such that

$$B(y_k, r_k) \subset V_k,$$

$$g^{(k)}(y) > c_k \ \text{ for each } y \in B(y_k, r_k). \tag{68}$$

It follows from (68), (65) and (64) that for each $z \in B(y_k, r_k) \cap \mathrm{dom}\left(g_0^{(k)}\right)$

$$\begin{aligned}\lambda_k(g^{(k)}(z) - c_k) - \lambda_k(g^{(k)}(y_k) - c_k) &= \psi_k(z) - \psi_k(y_k) - g_0^{(k)}(z) + g_0^{(k)}(y_k) \\ &\geq \ - k^{-1}||z - y_k|| + g_0^{(k)}(y_k) - g_0^{(k)}(z)\end{aligned}$$

and

$$g^{(k)}(z) - g^{(k)}(y_k) \geq \lambda_k^{-1}\left(g_0^{(k)}(y_k) - g_0^{(k)}(z)\right) - \lambda_k^{-1}k^{-1}||y_k - z||. \tag{69}$$

By (50) and the definition of $\mathcal{U}(k^{-1})$ there exists a convex function $h_k : X \to R^1 \cup \{\infty\}$ such that

$$\mathrm{dom}\left(g^{(k)}\right) \cap B^o(0, M_0) = \mathrm{dom}(f) \cap B^o(0, M_0) = \mathrm{dom}(h_k) \cap B^o(0, M_0), \tag{70}$$

$$|f(x) - h_k(x)| \leq 1/k \ \text{ for all } x \in \mathrm{dom}(f) \cap B^o(0, M_0), \tag{71}$$

$$\begin{aligned}&\left|\left(h_k - g^{(k)}\right)(z_1) - \left(h_k - g^{(k)}\right)(z_2)\right| \\ &\leq ||z_1 - z_2||/k \ \text{ for all } z_1, z_2 \in \mathrm{dom}(f) \cap B^o(0, M_0),\end{aligned} \tag{72}$$

$$\left|h_k(z) - g^{(k)}(z)\right| \leq 1/k \ \text{ for all } z \in \mathrm{dom}(f) \cap B^o(0, M_0). \tag{73}$$

It follows from (68), (66), (70), (67), (72) and (65) that for each $z \in B(y_k, r_k) \cap \mathrm{dom}\left(g_0^{(k)}\right) \cap \mathrm{dom}(f)$,

$$\begin{aligned}h_k(z) - h_k(y_k) &= g^{(k)}(z) - g^{(k)}(y_k) + \left(\left(h_k - g^{(k)}\right)(z) - \left(h_k - g^{(k)}\right)(y_k)\right) \\ &\geq g^{(k)}(z) - g^{(k)}(y_k) - ||z - y_k||/k \\ &\geq \ - ||z - y_k||/k + \lambda_k^{-1}\left(g_0^{(k)}(y_k) - g_0^{(k)}(z)\right) - \lambda_k^{-1}k^{-1}||y_k - z||.\end{aligned}$$

Combined with (66) and (68) this implies that for each $z \in B(y_k, r_k) \cap \mathrm{dom}(f)$,

$$\begin{aligned}0 &\leq h_k(z) - h_k(y_k) + ||z - y_k||\left(1/k + (\lambda_k k)^{-1}\right) + \lambda_k^{-1}\left(g_0^{(k)}(z) - g_0^{(k)}(y_k)\right) \\ &\leq h_k(z) - h_k(y_k) + ||z - y_k||\left(1/k + (\lambda_k k)^{-1}\right) + \lambda_k^{-1}M_1 h(z, y_k).\end{aligned}$$

Combined with (70), (66) and (68) this implies that for all $z \in B(y_k, r_k)$,

$$h_k(y_k) \leq h_k(z) + ||z - y_k||\left(1/k + (\lambda_k k)^{-1}\right) + \lambda_k^{-1}M_1 h(z, y_k). \tag{74}$$

It follows from (A2), (65), (64), (50) and (7) that the function

$$z \to h_k(z) + \lambda_k^{-1} M_1 h(z, y_k) + \left(\lambda_k^{-1} k^{-1} + k^{-1}\right) ||z - y_k||, \quad z \in B^o(0, M_0)$$

is convex. Combined with (A1), (7), (64), (65), (50) and (67) this implies that (74) is true for all $z \in B^0(0, M_0)$.

By (65) and (52) for all $z \in B^0(0, M_0)$

$$\lim_{k\to\infty} \left(k^{-1} + \lambda_k^{-1} k^{-1}\right) ||z - y_k|| = 0. \tag{75}$$

In view of (52), (65), (A3), (64), (50) and (7) for all $z \in X_0$

$$\lim_{k\to\infty} \lambda_k^{-1} M_1 h(z, y_k) = 0. \tag{76}$$

By (75), (76), (71), (74) which holds for all $z \in X_0 \cap \mathrm{dom}(f)$, (65), (67), (70), (73), (60) and (51),

$$\begin{aligned} f(z) = & \lim_{k\to\infty} \left[h_k(z) + \lambda_k^{-1} M_1 h(z, y_k) + \left(\lambda_k^{-1} k^{-1} + k^{-1}\right) ||z - y_k||\right] \\ \geq & \limsup_{k\to\infty} h_k(y_k) = \limsup_{k\to\infty} g^{(k)}(y_k) \geq \lim_{k\to\infty} c_k = \bar{c}. \end{aligned}$$

Since $\theta \in X_0 \cap \mathrm{dom}(f)$ (see (45)) we conclude that $f(\theta) \geq \bar{c}$. This contradicts (45). The contradiction we have reached proves that there exists $\Lambda_0 \geq 1$ such that the property (P2) holds. This completes the proof of Theorem 4.1.

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