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Strong Convergence Theorems by Hybrid Methods for Maximal Monotone Operators and Relatively Nonexpansive Mappings in Banach Spaces

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In this paper, we prove strong convergence theorems for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of a relatively nonexpansive mapping in a Banach space by using two hybrid methods. Using these results, we obtain new convergence results for resolvents of maximal monotone operators and relatively nonexpansive mappings in Banach spaces.

1. Introduction

Let E be a real Banach space and let E^* be the dual space of E. Let A be a maximal monotone operator from E to E^*. Then we know the problem of finding a point $u \in E$ satisfying

$$0 \in Au.$$

Such a problem contains numerous problems in physics, optimization and economics. A well-known method to solve this problem is called the proximal point algorithm: $x_1 \in E$ and

$$x_{n+1} = J_{r_n} x_n, \quad n = 1, 2, \dots,$$

where $\{r_n\} \subset (0, \infty)$ and J_{r_n} are the resolvents of A.

Many researchers have studied this algorithm in a Hilbert space, see, for instance, [2, 3, 5, 10, 12, 16, 20, 22] and in a Banach space, see, for instance, [7, 8].

A mapping S of C into E is called nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

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We denote by $F(S)$ the set of fixed points of S.

There are some methods for approximation of fixed points of a nonexpansive mapping; see, for instance, [4, 11, 18, 21, 27]. In particular, in 2003 Nakajo–Takahashi [15] proved the following strong convergence theorem by using the hybrid method:

Theorem 1.1 (Nakajo and Takahashi [15]). *Let C be a nonempty closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ u_{n+1} = P_{C_n \cap Q_n} x, \quad n \in \mathbb{N}, \end{cases}$$

where $P_{C_n \cap Q_n}$ is the metric projection from C onto $C_n \cap Q_n$ and $\{\alpha_n\}$ is chosen so that $0 \leq \alpha_n \leq a < 1$. Then, $\{x_n\}$ converges strongly to $P_{F(T)}x$, where $P_{F(T)}$ is the metric projection from C onto $F(T)$.

Let us call the hybrid method in Theorem 1.1 the normal hybrid method. Very recently, Takahashi–Takeuchi–Kubota [25] proved the following theorem by using another hybrid method called the shrinking projection method.

Theorem 1.2 (Takahashi, Takeuchi and Kubota [25]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)Tu_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}} x_0, \quad n \in \mathbb{N}, \end{cases}$$

where $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then, $\{u_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

In this paper, by using the normal hybrid method and the shrinking projection method, we study two strong convergence theorems for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of a relatively nonexpansive mapping in a Banach space. Using these results, we obtain new convergence results for resolvents of maximal monotone operators and relatively nonexpansive mappings in Banach spaces.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of positive integers and real numbers, respectively. Let E be a Banach space and let E^* be the topological dual of E. For all $x \in E$ and $x^* \in E^*$, we denote the value of x^* at x by $\langle x, x^* \rangle$. Then, the duality mapping J on E is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. By the Hahn-Banach theorem, $J(x)$ is nonempty; see [23] for more details. We denote the strong convergence and the weak convergence of a sequence $\{x_n\}$ to x in E by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. We also denote the weak* convergence of a sequence $\{x_n^*\}$ to x^* in E^* by $x_n^* \overset{*}{\rightharpoonup} x^*$. A Banach space E is said to be strictly convex if $\frac{\|x+y\|}{2} < 1$ for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is also said to be uniformly convex if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that $\frac{\|x+y\|}{2} \leq 1 - \delta$ for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \epsilon$. The space E is said to be smooth if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all $x, y \in S(E) = \{z \in E : \|z\| = 1\}$. It is also said to be uniformly smooth if the limit exists uniformly in $x, y \in S(E)$. We know that if E is smooth, strictly convex and reflexive, then the duality mapping J is single-valued, one-to-one and onto; see [23, 24] for more details.

Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed convex subset of E. Throughout this paper, define the function ϕ by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2, \quad \forall y, x \in E.$$

Following Alber [1], the generalized projection Π_C from E onto C is defined by

$$\Pi_C(x) = \arg\min_{y \in C} \phi(y, x), \quad \forall x \in E.$$

If E is a Hilbert space, then $\phi(y, x) = \|y - x\|^2$ and Π_C is the metric projection of H onto C. We know the following lemmas for generalized projections.

Lemma 2.1 (Alber [1], Kamimura and Takahashi [6]). *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. Then*

$$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y), \quad \forall x \in C \text{ and } y \in E.$$

Lemma 2.2 (Alber [1], Kamimura and Takahashi [6]). *Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space, let $x \in E$ and let $z \in C$. Then*

$$z = \Pi_C x \iff \langle y - z, Jx - Jz\rangle \leq 0, \quad \forall y \in C.$$

Let E be a smooth, strictly convex and reflexive Banach space, and let A be a set-valued mapping from E to E^* with graph $G(A) = \{(x, x^*) : x^* \in Ax\}$, domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \cup\{Az : z \in D(A)\}$. We denote a set-valued operator A from E to E^* by $A \subset E \times E^*$. A is said to be monotone if

$$\langle x - y, x^* - y^*\rangle \geq 0, \quad \forall (x, x^*), (y, y^*) \in A.$$

A monotone operator $A \subset E \times E^*$ is said to be maximal monotone if its graph is not properly contained in the graph of any other monotone operator. We know that if A is a maximal monotone operator, then $A^{-1}0 = \{z \in D(A) : 0 \in Az\}$ is closed and convex; see [23, 24] for more details. The following theorem is well-known.

Theorem 2.3 (Rockafellar [19]). *Let E be a smooth, strictly convex and reflexive Banach space and let $A \subset E \times E^*$ be a monotone operator. Then A is maximal if and only if $R(J + rA) = E^*$ for all $r > 0$.*

Let E be a smooth, strictly convex and reflexive Banach space, let C be a nonempty closed convex subset of E and let $A \subset E \times E^*$ be a monotone operator satisfying

$$D(A) \subset C \subset J^{-1}\left(\cap_{r>0} R(J + rA)\right).$$

Then we can define the resolvent $J_r : C \to D(A)$ of A by

$$J_r x = \{z \in D(A) : Jx \in Jz + rAz\}, \quad \forall x \in C.$$

We know that $J_r x$ consists of one point. For all $r > 0$, the Yosida approximation $A_r : C \to E^*$ is defined by $A_r x = \frac{Jx - JJ_r x}{r}$ for all $x \in C$. We also know the following lemma; see, for instance, [9].

Lemma 2.4. *Let E be a smooth, strictly convex and reflexive Banach space, let C be a nonempty closed convex subset of E and let $A \subset E \times E^*$ be a monotone operator satisfying*

$$D(A) \subset C \subset J^{-1}\left(\cap_{r>0} R(J + rA)\right).$$

Let $r > 0$ and let J_r and A_r be the resolvent and the Yosida approximation of A, respectively. Then, the following hold:

(1) $\phi(u, J_r x) + \phi(J_r x, x) \le \phi(u, x)$ *for all $x \in C$ and $u \in A^{-1}0$;*
(2) $(J_r x, A_r x) \in A$ *for all $x \in C$;*
(3) $F(J_r) = A^{-1}0$.

Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E, let T be a mapping from C into itself. We denoted by $F(T)$ the set of fixed points of T. A point $p \in C$ is said to be an asymptotic fixed point of T [17] if there exists $\{x_n\}$ in C which converges weakly to p and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote the set of all asymptotic fixed points of T by $\hat{F}(T)$. Following Matsushita and Takahashi [13], a mapping $T : C \to C$ is said to be relatively nonexpansive if the following conditions are satisfied:

(1) $F(T)$ is nonempty;
(2) $\phi(u, Tx) \le \phi(u, x)$, $\forall u \in F(T)$, $x \in C$;
(3) $\hat{F}(T) = F(T)$.

The following lemma is due to Matsushita and Takahashi [13].

Lemma 2.5 (Matsushita and Takahashi [13]). *Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E, and let T be a relatively nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex.*

We also know the following lemma.

Lemma 2.6 (Kamimura and Takahashi [6]). *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_n \phi(x_n, y_n) = 0$, then $\lim_n \|x_n - y_n\| = 0$.*

3. Convergence theorem by the normal hybrid method

In this section, we prove a strong convergence theorem for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of a relatively nonexpansive mapping in a Banach space by using the normal hybrid method.

Theorem 3.1. *Let E be a uniformly smooth and uniformly convex Banach space, and let C be a nonempty closed convex subset of E. Let $A \subset E \times E^*$ be a maximal monotone operator satisfying*

$$D(A) \subset C \subset J^{-1}\left(\cap_{r>0} R(J + rA)\right)$$

and let $J_r = (J + rA)^{-1}J$ for all $r > 0$. Let S be a relatively nonexpansive mapping from C into itself such that $F(S) \cap A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in C$ and

$$\begin{cases} u_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JSJ_{r_n}x_n), \\ H_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x \end{cases}$$

for every $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E, $\{\alpha_n\} \subset [0, 1)$ satisfies $\liminf_{n\to\infty}(1-\alpha_n) > 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then, $\{x_n\}$ converges strongly to $\Pi_{F(S)\cap A^{-1}0}x$, where $\Pi_{F(S)\cap A^{-1}0}$ is the generalized projection of E onto $F(S) \cap A^{-1}0$.

Proof. We first show that $H_n \cap W_n$ is closed and convex. It is obvious that H_n is closed and W_n is closed and convex. Since

$$\begin{aligned} &\phi(z, u_n) \leq \phi(z, x_n) \\ \iff &\|u_n\|^2 - \|x_n\|^2 - 2\langle z, Ju_n - Jx_n\rangle \geq 0, \end{aligned}$$

H_n is convex. So, $H_n \cap W_n$ is a closed convex subset of E for all $n \in \mathbb{N} \cup \{0\}$.

Let $u \in F(S) \cap A^{-1}0$. Put $y_n = J_{r_n}x_n$ for all $n \in \mathbb{N}$. Since J_{r_n} and S are relatively nonexpansive, we have

$$\begin{aligned} \phi(u, u_n) &= \phi(u, J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JSy_n)) \\ &= \|u\|^2 - 2\langle u, \alpha_n Jx_n + (1-\alpha_n)JSy_n\rangle + \|\alpha_n Jx_n + (1-\alpha_n)JSy_n\|^2 \\ &\leq \|u\|^2 - 2\alpha_n\langle u, Jx_n\rangle - 2(1-\alpha_n)\langle u, JSy_n\rangle + \alpha_n\|x_n\|^2 + (1-\alpha_n)\|Sy_n\|^2 \\ &= \alpha_n\phi(u, x_n) + (1-\alpha_n)\phi(u, Sy_n) \\ &= \alpha_n\phi(u, x_n) + (1-\alpha_n)\phi(u, SJ_{r_n}x_n) \\ &\leq \phi(u, x_n). \end{aligned}$$

Hence, we have $u \in H_n$. This implies that

$$F(S) \cap A^{-1}0 \subset H_n, \quad \forall n \in \mathbb{N} \cup \{0\}.$$

Next we show by induction that $F(S) \cap A^{-1}0 \subset H_n \cap W_n$ for all $n \in \mathbb{N} \cup \{0\}$. From $W_0 = C$, we have

$$F(S) \cap A^{-1}0 \subset H_0 \cap W_0.$$

Suppose that $F(S) \cap A^{-1}0 \subset H_k \cap W_k$ for some $k \in \mathbb{N} \cup \{0\}$. Then there exists $x_{k+1} \in H_k \cap W_k$ such that

$$x_{k+1} = \Pi_{H_k \cap W_k} x.$$

From the definition of x_{k+1}, we have, for all $z \in H_k \cap W_k$,

$$\langle x_{k+1} - z, Jx - Jx_{k+1} \rangle \geq 0.$$

Since $F(S) \cap A^{-1}0 \subset H_k \cap W_k$, we have

$$\langle x_{k+1} - z, Jx - Jx_{k+1} \rangle \geq 0, \quad \forall z \in F(S) \cap A^{-1}0$$

and hence $z \in W_{k+1}$. So, we have

$$F(S) \cap A^{-1}0 \subset W_{k+1}.$$

Therefore we have

$$F(S) \cap A^{-1}0 \subset H_{k+1} \cap W_{k+1}.$$

So, we have that $F(S) \cap A^{-1}0 \subset H_n \cap W_n$ for all $n \in \mathbb{N} \cup \{0\}$. This means that $\{x_n\}$ is well-defined.

From the definition of W_n, we have $x_n = \Pi_{W_n} x$. Using $x_n = \Pi_{W_n} x$, we have

$$\phi(x_n, x) = \phi(\Pi_{W_n} x, x) \leq \phi(u, x) - \phi(u, \Pi_{W_n} x) \leq \phi(u, x)$$

for all $u \in F(S) \cap A^{-1}0 \subset W_n$. Then, $\{\phi(x_n, x)\}$ is bounded. Therefore, $\{x_n\}$ and $\{J_{r_n} x_n\} = \{y_n\}$ are bounded.

Since $x_{n+1} = \Pi_{H_n \cap W_n} x \in H_n \cap W_n \subset W_n$ and $x_n = \Pi_{W_n} x$, we have

$$\phi(x_n, x) \leq \phi(x_{n+1}, x), \quad \forall n \in \mathbb{N} \cup \{0\}.$$

Thus $\{\phi(x_n, x)\}$ is nondecreasing. So, the limit of $\{\phi(x_n, x)\}$ exists. From $x_n = \Pi_{W_n} x$,

$$\begin{aligned} \phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{W_n} x) \\ &\leq \phi(x_{n+1}, x) - \phi(\Pi_{W_n} x, x) \\ &= \phi(x_{n+1}, x) - \phi(x_n, x) \end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$. This means that $\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0$. From $x_{n+1} = \Pi_{H_n \cap W_n} x \subset H_n$, we have

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n), \quad \forall n \in \mathbb{N} \cup \{0\}.$$

Therefore, we have

$$\lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0.$$

Since $\lim_{n\to\infty} \phi(x_{n+1}, x_n) = \lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0$ and E is uniformly convex and smooth, we have from Lemma 2.6 that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|x_{n+1} - u_n\| = 0.$$

So, we have

$$\lim_{n\to\infty} \|x_n - u_n\| = 0.$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|Jx_{n+1} - Jx_n\| = \lim_{n\to\infty} \|Jx_{n+1} - Ju_n\| = \lim_{n\to\infty} \|Jx_n - Ju_n\| = 0. \tag{1}$$

On the other hand, we have

$$\begin{aligned}\|Jx_{n+1} - Ju_n\| &= \|Jx_{n+1} - (\alpha_n Jx_n + (1-\alpha_n)JSy_n)\| \\ &= \|\alpha_n(Jx_{n+1} - Jx_n) + (1-\alpha_n)(Jx_{n+1} - JSy_n)\| \\ &\geq (1-\alpha_n)\|Jx_{n+1} - JSy_n\| - \alpha_n\|Jx_{n+1} - Jx_n\|.\end{aligned}$$

Therefore we have

$$\begin{aligned}\|Jx_{n+1} - JSy_n\| &\leq \frac{1}{1-\alpha_n}(\|Jx_{n+1} - Ju_n\| + \alpha_n\|Jx_{n+1} - Jx_n\|) \\ &\leq \frac{1}{1-\alpha_n}(\|Jx_{n+1} - Ju_n\| + \|Jx_{n+1} - Jx_n\|).\end{aligned}$$

From (1) and $\liminf_{n\to\infty}(1-\alpha_n) > 0$, we have

$$\lim_{n\to\infty} \|Jx_{n+1} - JSy_n\| = 0.$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|x_{n+1} - Sy_n\| = 0.$$

From

$$\|x_n - Sy_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - Sy_n\|,$$

we have

$$\lim_{n\to\infty} \|x_n - Sy_n\| = 0.$$

Using $y_n = J_{r_n}x_n$ and Lemma 2.4, we have

$$\begin{aligned}\phi(y_n, x_n) = \phi(J_{r_n}x_n, x_n) &\leq \phi(u, x_n) - \phi(u, J_{r_n}x_n) \\ &= \phi(u, x_n) - \phi(u, y_n).\end{aligned}$$

From $\phi(u, u_n) \leq \alpha_n\phi(u, x_n) + (1-\alpha_n)\phi(u, y_n)$, we have

$$\phi(u, y_n) \geq \frac{\phi(u, u_n) - \alpha_n\phi(u, x_n)}{1-\alpha_n}$$

and

$$\begin{aligned}\phi(y_n, x_n) &\leq \phi(u, x_n) - \phi(u, y_n) \\ &\leq \phi(u, x_n) - \frac{\phi(u, u_n) - \alpha_n\phi(u, x_n)}{1-\alpha_n} \\ &= \frac{\phi(u, x_n) - \phi(u, u_n)}{1-\alpha_n}.\end{aligned} \tag{2}$$

Since

$$\begin{aligned}\phi(u,x_n)-\phi(u,u_n) &= \|x_n\|^2-\|u_n\|^2-2\langle u, Jx_n-Ju_n\rangle \\ &\le \left|\|x_n\|^2-\|u_n\|^2\right|+2|\langle u, Jx_n-Ju_n\rangle| \\ &\le |\|x_n\|-\|u_n\||\,(\|x_n\|+\|u_n\|)+2\|u\|\|Jx_n-Ju_n\| \\ &\le \|x_n-u_n\|\,(\|x_n\|+\|u_n\|)+2\|u\|\|Jx_n-Ju_n\|,\end{aligned}$$

and $\liminf_{n\to\infty}(1-\alpha_n)>0$, we have from (2)

$$\lim_{n\to\infty}\phi(y_n,x_n)=0.$$

Since E is uniformly convex and smooth, we have from Lemma 2.6 that

$$\lim_{n\to\infty}\|y_n-x_n\|=0. \tag{3}$$

From $\lim_{n\to\infty}\|x_n-Sy_n\|=0$, we have

$$\lim_{n\to\infty}\|y_n-Sy_n\|=0.$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup\hat{x}$. From $\lim_{n\to\infty}\|x_n-y_n\|=0$, we have $y_{n_k}\rightharpoonup\hat{x}$. Since S is relatively nonexpansive, we have $\hat{x}\in\hat{F}(S)=F(S)$. Next, we show $\hat{x}\in A^{-1}0$. Since J is uniformly norm-to-norm continuous on bounded sets, from (3) we have

$$\lim_{n\to\infty}\|Jx_n-Jy_n\|=0.$$

From $r_n\ge a$, we have

$$\lim_{n\to\infty}\frac{\|Jx_n-Jy_n\|}{r_n}=0.$$

Therefore, we have

$$\lim_{n\to\infty}\|A_{r_n}x_n\|=\lim_{n\to\infty}\frac{1}{r_n}\|Jx_n-Jy_n\|=0.$$

For $(z,z^*)\in A$, from the monotonicity of A, we have

$$\langle z-y_n, z^*-A_{r_n}x_n\rangle\ge 0$$

for all $n\in\mathbb{N}$.

Replacing n by n_k and letting $k\to\infty$, we have

$$\langle z-\hat{x}, z^*\rangle\ge 0.$$

From the maximality of A, we have $\hat{x}\in A^{-1}0$.

Let $w=\Pi_{F(S)\cap A^{-1}0}x$. From $x_{n+1}=\Pi_{H_n\cap W_n}x$ and $w\in F(S)\cap A^{-1}0\subset H_n\cap W_n$, we have

$$\phi(x_{n+1},x)\le\phi(w,x).$$

Since the norm is weakly lower semicontinuous, we have

$$\begin{aligned}\phi(\hat{x},x) &= \|\hat{x}\|^2 - 2\langle \hat{x}, Jx\rangle + \|x\|^2 \\ &\leq \liminf_{k\to\infty} \left(\|x_{n_k}\|^2 - 2\langle x_{n_k}, Jx\rangle + \|x\|^2\right) \\ &= \liminf_{k\to\infty} \phi(x_{n_k}, x) \\ &\leq \limsup_{k\to\infty} \phi(x_{n_k}, x) \\ &\leq \phi(w,x).\end{aligned}$$

From the definition of $\Pi_{F(S)\cap A^{-1}0}$, we have $\hat{x} = w$. Hence $\lim_{k\to\infty} \phi(x_{n_k}, x) = \phi(w,x)$. Therefore we have

$$\begin{aligned}0 &= \lim_{k\to\infty} (\phi(x_{n_k}, x) - \phi(w,x)) \\ &= \lim_{k\to\infty} (\|x_{n_k}\|^2 - \|w\|^2 - 2\langle x_{n_k} - w, Jx\rangle) \\ &= \lim_{k\to\infty} (\|x_{n_k}\|^2 - \|w\|^2).\end{aligned}$$

Since E has the Kadec-Klee property, we have that $x_{n_k} \to w = \Pi_{F(S)\cap A^{-1}0}x$. Therefore, $\{x_n\}$ converges strongly to $\Pi_{F(S)\cap A^{-1}0}x$. □

As direct consequences of Theorem 3.1, we can obtain the following corollaries.

Corollary 3.2. *Let E be a uniformly smooth and uniformly convex Banach space, let $A \subset E \times E^*$ be a maximal monotone operator with $A^{-1}0 \neq \emptyset$ and let $J_r = (J + rA)^{-1}J$ for all $r > 0$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in C$ and*

$$\begin{cases} u_n = J_{r_n} x_n, \\ H_n = \{z \in E : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ W_n = \{z \in E : \langle x_n - z, Jx - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x \end{cases}$$

for every $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then, $\{x_n\}$ converges strongly to $\Pi_{A^{-1}0}x$, where $\Pi_{A^{-1}0}$ is the generalized projection of E onto $A^{-1}0$.

Proof. Putting $S = I$, $C = E$ and $\alpha_n = 0$ in Theorem 3.1, we obtain Corollary 3.2. □

Let E be a Banach space and let $f : E \to (-\infty, \infty]$ be a proper lower semicontinuous convex function. Define the subdifferential of f as follows:

$$\partial f(x) = \{x^* \in E^* : f(y) \geq \langle y - x, x^*\rangle + f(x),\ \forall y \in E\}$$

for each $x \in E$. Then, we know that ∂f is a maximal monotone operator; see [23] for more details.

Corollary 3.3 (Matsushita and Takahashi [13]). *Let E be a uniformly smooth and uniformly convex Banach space, let C be a nonempty closed convex subset of E, and let S be a relatively nonexpansive mapping from C into itself such that $F(S) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in C$ and*

$$\begin{cases} u_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JSx_n), \\ H_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x \end{cases}$$

for every $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E, $\{\alpha_n\} \subset [0,1)$ satisfies $\liminf_{n\to\infty}(1-\alpha_n) > 0$. Then, $\{x_n\}$ converges strongly to $\Pi_{F(S)}x$, where $\Pi_{F(S)}$ is the generalized projection of E onto $F(S)$.

Proof. Set $A = \partial i_C$ in Theorem 3.1, where i_C is the indicator function of C, i.e.,

$$i_C = \begin{cases} 0 & x \in C, \\ \infty & \text{otherwise.} \end{cases}$$

Then, we have that A is a maximal monotone operator and $J_r = \Pi_C$ for $r > 0$, in fact, for any $x \in E$ and $r > 0$, we have from Lemma 2.2 that

$$\begin{aligned} & z = J_r x \\ \iff & Jz + r\partial i_C(z) \ni Jx \\ \iff & Jx - Jz \in r\partial i_C(z) \\ \iff & i_C(y) \geq \left\langle y - z, \frac{Jx - Jz}{r} \right\rangle + i_C(z), \quad \forall y \in E \\ \iff & 0 \geq \langle y - z, Jx - Jz \rangle, \quad \forall y \in C \\ \iff & z = \arg\min_{y \in C} \phi(y, x) \\ \iff & z = \Pi_C x. \end{aligned}$$

So, from Theorem 3.1, we obtain Corollary 3.3. □

4. Convergence theorem by the shrinking projection method

In this section, we prove a strong convergence theorem for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of a relatively nonexpansive mapping in a Banach space by using the shrinking projection method.

Theorem 4.1. *Let E be a uniformly smooth and uniformly convex Banach space, and let C be a nonempty closed convex subset of E. Let $A \subset E \times E^*$ be a maximal monotone operator satisfying*

$$D(A) \subset C \subset J^{-1}\left(\cap_{r>0} R(J + rA)\right)$$

and let $J_r = (J + rA)^{-1}J$ for all $r > 0$. Let S be a relatively nonexpansive mapping from C into itself such that $F(S) \cap A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in C$,

$H_0 = C$ *and*

$$\begin{cases} u_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JSJ_{r_n}x_n), \\ H_{n+1} = \{z \in H_n : \phi(z, u_n) \le \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{H_{n+1}}x \end{cases}$$

for every $n \in \mathbb{N} \cup \{0\}$, *where* J *is the duality mapping on* E, $\{\alpha_n\} \subset [0,1)$ *satisfies* $\liminf_{n\to\infty}(1-\alpha_n) > 0$ *and* $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$. *Then,* $\{x_n\}$ *converges strongly to* $\Pi_{F(S)\cap A^{-1}0}x$, *where* $\Pi_{F(S)\cap A^{-1}0}$ *is the generalized projection of* E *onto* $F(S) \cap A^{-1}0$.

Proof. Putting $y_n = J_{r_n}x_n$ for all $n \in \mathbb{N}$, we know that J_{r_n} are relatively nonexpansive.

We first show that H_n is closed and convex. It is obvious that H_n is closed. Since

$$\begin{aligned} & \phi(z, u_n) \le \phi(z, x_n) \\ \Longleftrightarrow \quad & \|u_n\|^2 - \|x_n\|^2 - 2\langle z, Ju_n - Jx_n\rangle \ge 0, \end{aligned}$$

we also have that H_n is convex. So, H_n is a closed convex subset of E for all $n \in \mathbb{N} \cup \{0\}$.

Next we show by induction that $A^{-1}0 \cap F(S) \subset H_n$ for all $n \in \mathbb{N} \cup \{0\}$. From $H_0 = C$, we have

$$F(S) \cap A^{-1}0 \subset H_0.$$

Suppose that $F(S) \cap A^{-1}0 \subset H_k$ for some $k \in \mathbb{N} \cup \{0\}$. Then let $u \in F(S) \cap A^{-1}0 \subset H_k$. Since J_{r_k} and S are relatively nonexpansive, we have

$$\begin{aligned} \phi(u, u_n) &= \phi(u, J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JSy_n)) \\ &= \|u\|^2 - 2\langle u, \alpha_n Jx_n + (1-\alpha_n)JSy_n\rangle + \|\alpha_n Jx_n + (1-\alpha_n)JSy_n\|^2 \\ &\le \|u\|^2 - 2\alpha_n\langle u, Jx_n\rangle - 2(1-\alpha_n)\langle u, JSy_n\rangle + \alpha_n\|x_n\|^2 + (1-\alpha_n)\|Sy_n\|^2 \\ &= \alpha_n\phi(u, x_n) + (1-\alpha_n)\phi(u, Sy_n) \\ &= \alpha_n\phi(u, x_n) + (1-\alpha_n)\phi(u, SJ_{r_n}x_n) \\ &\le \phi(u, x_n). \end{aligned}$$

Hence, we have $u \in H_{k+1}$. So, we have that

$$F(S) \cap A^{-1}0 \subset H_n, \quad \forall n \in \mathbb{N} \cup \{0\}.$$

This means that $\{x_n\}$ is well-defined.

From the definition of x_n and Lemma 2.1, we have

$$\phi(x_n, x) = \phi(\Pi_{H_n}x, x) \le \phi(u, x) - \phi(u, \Pi_{H_n}x) \le \phi(u, x)$$

for all $u \in F(S) \cap A^{-1}0 \subset H_n$. Then, $\{\phi(x_n, x)\}$ is bounded. Therefore, $\{x_n\}$ and $\{J_{r_n}x_n\} = \{y_n\}$ are bounded.

From $H_{n+1} \subset H_n$ and $x_n = \Pi_{H_n}x$, we have

$$\phi(x_n, x) \le \phi(x_{n+1}, x), \quad \forall n \in \mathbb{N} \cup \{0\}.$$

Thus $\{\phi(x_n, x)\}$ is nondecreasing. So, the limit of $\{\phi(x_n, x)\}$ exists. Since

$$\begin{aligned} \phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{H_n}x) \\ &\le \phi(x_{n+1}, x) - \phi(\Pi_{H_n}x, x) \\ &= \phi(x_{n+1}, x) - \phi(x_n, x) \end{aligned}$$

for all $n \in \mathbb{N}$, we have $\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0$. From $x_{n+1} = \Pi_{H_{n+1}} x \in H_{n+1}$, we also have

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n), \quad \forall n \in \mathbb{N} \cup \{0\}.$$

Therefore, we have

$$\lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0.$$

Since $\lim_{n\to\infty} \phi(x_{n+1}, x_n) = \lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0$ and E is uniformly convex and smooth, we have from Lemma 2.6 that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|x_{n+1} - u_n\| = 0.$$

So, we have

$$\lim_{n\to\infty} \|x_n - u_n\| = 0.$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|Jx_{n+1} - Jx_n\| = \lim_{n\to\infty} \|Jx_{n+1} - Ju_n\| = \lim_{n\to\infty} \|Jx_n - Ju_n\| = 0. \tag{4}$$

On the other hand, we have

$$\begin{aligned}
\|Jx_{n+1} - Ju_n\| &= \|Jx_{n+1} - (\alpha_n Jx_n + (1-\alpha_n)JSy_n)\| \\
&= \|\alpha_n(Jx_{n+1} - Jx_n) + (1-\alpha_n)(Jx_{n+1} - JSy_n)\| \\
&\geq (1-\alpha_n)\|Jx_{n+1} - JSy_n\| - \alpha_n\|Jx_{n+1} - Jx_n\|.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\|Jx_{n+1} - JSy_n\| &\leq \frac{1}{1-\alpha_n}(\|Jx_{n+1} - Ju_n\| + \alpha_n\|Jx_{n+1} - Jx_n\|) \\
&\leq \frac{1}{1-\alpha_n}(\|Jx_{n+1} - Ju_n\| + \|Jx_{n+1} - Jx_n\|).
\end{aligned}$$

From (4) and $\liminf_{n\to\infty}(1-\alpha_n) > 0$, we have

$$\lim_{n\to\infty} \|Jx_{n+1} - JSy_n\| = 0.$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|x_{n+1} - Sy_n\| = 0.$$

From

$$\|x_n - Sy_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - Sy_n\|,$$

we have

$$\lim_{n\to\infty} \|x_n - Sy_n\| = 0. \tag{5}$$

Using $y_n = J_{r_n} x_n$ and Lemma 2.4, we have

$$\begin{aligned}
\phi(y_n, x_n) = \phi(J_{r_n} x_n, x_n) &\leq \phi(u, x_n) - \phi(u, J_{r_n} x_n) \\
&= \phi(u, x_n) - \phi(u, y_n).
\end{aligned}$$

From $\phi(u,u_n) \le \alpha_n\phi(u,x_n)+(1-\alpha_n)\phi(u,y_n)$, we have

$$\phi(u,y_n) \ge \frac{\phi(u,u_n)-\alpha_n\phi(u,x_n)}{1-\alpha_n}$$

and

$$\begin{aligned}\phi(y_n,x_n) &\le \phi(u,x_n)-\phi(u,y_n)\\ &\le \phi(u,x_n)-\frac{\phi(u,u_n)-\alpha_n\phi(u,x_n)}{1-\alpha_n}\\ &= \frac{\phi(u,x_n)-\phi(u,u_n)}{1-\alpha_n}.\end{aligned} \tag{6}$$

Since

$$\begin{aligned}\phi(u,x_n)-\phi(u,u_n) &= \|x_n\|^2-\|u_n\|^2-2\langle u, Jx_n-Ju_n\rangle\\ &\le \left|\|x_n\|^2-\|u_n\|^2\right|+2|\langle u, Jx_n-Ju_n\rangle|\\ &\le |\|x_n\|-\|u_n\||\,(\|x_n\|+\|u_n\|)+2\|u\|\|Jx_n-Ju_n\|\\ &\le \|x_n-u_n\|\,(\|x_n\|+\|u_n\|)+2\|u\|\|Jx_n-Ju_n\|,\end{aligned}$$

and $\liminf_{n\to\infty}(1-\alpha_n)>0$, we have from (6) that

$$\lim_{n\to\infty}\phi(y_n,x_n)=0.$$

Since E is uniformly convex and smooth, we have from Lemma 2.6

$$\lim_{n\to\infty}\|y_n-x_n\|=0. \tag{7}$$

From (5) and (7), we have

$$\lim_{n\to\infty}\|y_n-Sy_n\|=0.$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup \hat{x}$. From $\lim_{n\to\infty}\|x_n-y_n\|=0$, we have $y_{n_k}\rightharpoonup\hat{x}$. Since S is relatively nonexpansive, we have $\hat{x}\in\hat{F}(S)=F(S)$. Next, we show $\hat{x}\in A^{-1}0$. Since J is uniformly norm-to-norm continuous on bounded sets, from (7) we have

$$\lim_{n\to\infty}\|Jx_n-Jy_n\|=0.$$

From $r_n\ge a$, we have

$$\lim_{n\to\infty}\frac{\|Jx_n-Jy_n\|}{r_n}=0.$$

Therefore, we have

$$\lim_{n\to\infty}\|A_{r_n}x_n\|=\lim_{n\to\infty}\frac{1}{r_n}\|Jx_n-Jy_n\|=0.$$

For $(z,z^*)\in A$, from the monotonicity of A, we have

$$\langle z-y_n, z^*-A_{r_n}x_n\rangle \ge 0$$

for all $n \in \mathbb{N}$.

Replacing n by n_k and letting $k \to \infty$, we have

$$\langle z - \hat{x}, z^* \rangle \geq 0.$$

From the maximality of A, we have $\hat{x} \in A^{-1}0$.

Let $w = \Pi_{F(S)\cap A^{-1}0}x$. From $x_{n+1} = \Pi_{H_{n+1}}x$ and $w \in F(S) \cap A^{-1}0 \subset H_{n+1}$, we have

$$\phi(x_{n+1}, x) \leq \phi(w, x).$$

Since the norm is weakly lower semicontinuous, we have

$$\begin{aligned}
\phi(\hat{x}, x) &= \|\hat{x}\|^2 - 2\langle \hat{x}, Jx\rangle + \|x\|^2 \\
&\leq \liminf_{k\to\infty} \left(\|x_{n_k}\|^2 - 2\langle x_{n_k}, Jx\rangle + \|x\|^2\right) \\
&= \liminf_{k\to\infty} \phi(x_{n_k}, x) \\
&\leq \limsup_{k\to\infty} \phi(x_{n_k}, x) \\
&\leq \phi(w, x).
\end{aligned}$$

From the definition of $\Pi_{F(S)\cap A^{-1}0}$, we have $\hat{x} = w$. Hence $\lim_{k\to\infty} \phi(x_{n_k}, x) = \phi(w, x)$. Therefore we have

$$\begin{aligned}
0 &= \lim_{k\to\infty} (\phi(x_{n_k}, x) - \phi(w, x)) \\
&= \lim_{k\to\infty} (\|x_{n_k}\|^2 - \|w\|^2 - 2\langle x_{n_k} - w, Jx\rangle) \\
&= \lim_{k\to\infty} (\|x_{n_k}\|^2 - \|w\|^2).
\end{aligned}$$

Since E has the Kadec-Klee property, we have that $x_{n_k} \to w = \Pi_{F(S)\cap A^{-1}0}x$. Therefore, $\{x_n\}$ converges strongly to $\Pi_{F(S)\cap A^{-1}0}x$. □

As direct consequences of Theorem 4.1, we can obtain the following corollaries.

Corollary 4.2. *Let E be a uniformly smooth and uniformly convex Banach space. Let $A \subset E \times E^*$ be a maximal monotone operator with $A^{-1}0 \neq \emptyset$ and let $J_r = (J + rA)^{-1}J$ for all $r > 0$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in E$, $H_0 = E$ and*

$$\begin{cases}
u_n = J_{r_n}x_n, \\
H_{n+1} = \{z \in H_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\
x_{n+1} = \Pi_{H_{n+1}}x
\end{cases}$$

for every $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then, $\{x_n\}$ converges strongly to $\Pi_{A^{-1}0}x$.

Proof. Putting $S = I$, $C = H_0 = E$ and $\alpha_n = 0$ in Theorem 4.1, we obtain Corollary 4.2. □

Corollary 4.3. *Let E be a uniformly smooth and uniformly convex Banach space, let C be a nonempty closed convex subset of E, and let S be a relatively nonexpansive mapping from C into itself such that $F(S) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in C$ and*

$$\begin{cases} u_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JSx_n), \\ H_{n+1} = \{z \in H_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{H_{n+1}} x \end{cases}$$

for every $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E, $\{\alpha_n\} \subset [0,1)$ satisfies $\liminf_{n\to\infty}(1-\alpha_n) > 0$. Then, $\{x_n\}$ converges strongly to $\Pi_{F(S)}x$, where $\Pi_{F(S)}$ is the generalized projection of E onto $F(S)$.

Proof. Set $A = \partial i_C$ in Theorem 4.1, where i_C is the indicator function of C. So, we obtain Corollary 4.3. □

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