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Morphisms of Normal Decomposition Systems

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A normal decomposition (ND) system is an algebraic structure connected with a decomposition statement for vectors of a linear space and with a variational inequality related to the decomposition. E.g., the Singular Value Decomposition for complex matrices and the trace inequality of von Neumann provide an example of an ND system. In this paper, we study morphisms and homomorphisms of ND systems. Applications to singular values of matrices are given.

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1. Preliminaries and motivation

Let V be a finite-dimensional real linear space with inner product $\langle \cdot, \cdot \rangle$, and let G be a closed subgroup of the orthogonal group $O(V)$ acting on V. The *group majorization* w.r.t. G, abbreviated as G-majorization and written as $\preceq_G$, is the G-invariant preordering on V defined by

$$y \preceq_G x \ \text{ iff } y \in \operatorname{conv} Gx,$$

where $x, y \in V$ and $C_G(x) = \operatorname{conv} Gx$ stands for the convex hull of the set $Gx = \{gx : g \in G\}$. We write $y \equiv_G x$ if $y = gx$ for some $g \in G$.

It is known that

$$y \preceq_G x \ \text{ iff } m(z, y) \leq m(z, x) \ \text{ for } z \in V,$$

where $m(z, v) = \max_{g \in G} \langle z, gv \rangle$ for $z \in V$ is the *support function* of $C_G(v)$ for $v \in V$ [24, Section 13].

The group majorization $\preceq_G$ on V is called a *group induced cone (GIC) ordering* if there exists a nonempty closed convex cone $D \subset V$ such that

(A1) $V = GD$, i.e., $V = \bigcup_{g \in G} gD = \bigcup_{x \in D} Gx$,
(A2) $\langle x, gy \rangle \leq \langle x, y \rangle$ for $x, y \in D$ and $g \in G$

(see [7, 8, 25]). Indeed GIC orderings cover many orderings of practical interest [7, 8]. If (A1) and (A2) are met, we say that the structure (V, G, D) is an *Eaton system* (for short, *E-system*) [26, 27]. In this event, the support function $m(\cdot, \cdot)$ on $D \times D$ reduces to the inner product $\langle \cdot, \cdot \rangle$.

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Each E-system (V, G, D) induces the *normal map* $(\cdot)_\downarrow : V \to D$ defined by

$$\{x_\downarrow\} = D \cap Gx \quad \text{for } x \in V \tag{1}$$

[18, p. 14]. Following Lewis [14, 15], we call the triple $(V, G, (\cdot)_\downarrow)$ a *normal decomposition (ND) system.* Conversely, each ND system induces E-system (V, G, D) with $D = V_\downarrow$ (see [14, Theorem 2.4], [15, p. 817]).

For later use, we record here two basic equivalences. Namely, for $x, y \in V$,

$$y \preceq_G x \quad \text{iff } y_\downarrow \preceq_G x_\downarrow. \tag{2}$$

Furthermore, if $x \in D$ and $y \in V$, then

$$y \preceq_G x \quad \text{iff } \langle z, gy \rangle \le \langle z, x \rangle \quad \text{for } z \in D \text{ and } g \in G. \tag{3}$$

Before presenting two basic examples of E-systems, we now give motivation and summary of the paper. Given two E-systems $\mathcal{E} = (V, G, D)$ and $\mathcal{F} = (W, H, E)$, we are interested in linear operators $K : V \to W$ such that

$$KD \subset E \quad \text{and} \quad Kx \preceq_H Kx_\downarrow \quad \text{for } x \in V. \tag{4}$$

We call them *morphisms* of $\mathcal{E}$ and $\mathcal{F}$. Throughout the paper $Kx_\downarrow$ means $K(x_\downarrow)$, that is the operation $(\cdot)_\downarrow$ has a higher priority than linear operators.

A nonlinear analog of (4) for real (nonnegative) functions has been studied by Iwasa [12] in the context of convolution inequalities with applications in statistics.

In matrix theory, the inequality in (4) can be interpreted in terms of eigenvalues and singular values of matrices via the E-systems demonstrated in Examples 1.1 and 1.2.

In the present paper we consider two problems. Firstly, in Section 2 we find conditions implying (4). The key idea of our method is using the dual operator K^* of K. In Theorem 2.1 we show that K is a morphism if and only if K^* is so. By employing some easily checkable classes of morphisms K^*, we get some morphisms K (see Theorem 2.2). Secondly, in Section 3 we study conditions under which a morphism preserves the structure of E-system (see Theorem 3.5). Such operators are called *homomorphisms*.

In order to illustrate the above-mentioned notions, we now present two examples related to the eigenvalues of Hermitian matrices and to the singular values of complex matrices. We use the following notation. Let $z_{[1]} \ge z_{[2]} \ge \ldots \ge z_{[n]}$ denote the entries of $z = (z_1, z_2, \ldots, z_n)^T \in \mathbb{R}^n$ in nonincreasing order, where $(\cdot)^T$ stands for the transpose. For $x, y \in \mathbb{R}^n$, if $\sum_{j=1}^i y_{[j]} \le \sum_{j=1}^i x_{[j]}$, $i = 1, \ldots, n$, then we write $y \prec_w x$ and say that y is *weakly majorized* by x [17, p. 10]. If, in addition, $\sum_{j=1}^n y_j = \sum_{j=1}^n x_j$, we write $y \prec x$ and say that y is *majorized* by x [17, p. 7]. The orderings $\prec$ and $\prec_w$ on $\mathbb{R}^n$ are called *majorization* and *weak majorization*, respectively. It is known that $\prec$ is GIC ordering induced by the group of $n \times n$ permutation matrices [7, p. 16]. Likewise, the group of $n \times n$ generalized permutation matrices induces $\prec_w$ on $\mathbb{R}^n_+$ [7, p. 16].

By $\mathbb{M}_n(\mathbb{C})$ we denote the space of $n \times n$ complex matrices. The symbols $\mathbb{H}_n$ and $\mathbb{D}_n(\mathbb{R})$ stand for the spaces of $n \times n$ Hermitian matrices and real diagonal matrices, respectively. By $\mathbb{U}_n$ and $\mathbb{O}_n$ we mean the groups of $n \times n$ unitary and orthogonal matrices, respectively. We denote by $\operatorname{diag} z$ the diagonal matrix with the entries of a vector $z \in \mathbb{R}^n$ on the main diagonal.

Example 1.1 (Cf. [7, p. 17], [14, pp. 943-944]). Take $V = \mathbb{H}_n$ with inner product defined by

$$\langle X, Y \rangle = \operatorname{Re} \operatorname{tr} XY \quad \text{for } X, Y \in \mathbb{H}_n,$$

and let G be the group of operators

$$X \to UXU^* \quad \text{for } X \in \mathbb{H}_n,$$

with U running over the unitary group $\mathbb{U}_n$. Here U^* means the conjugate transpose of U. It is known that (V, G, D) is an E-system for

$$D = \{\operatorname{diag}(z_1, \ldots, z_n) \in \mathbb{D}_n : z_1 \geq \ldots \geq z_n\}.$$

In fact, (A1) is the Spectral Theorem, and (A2) is the trace inequality of von Neumann (cf. [7, p. 17]).

Furthermore

$$X_\downarrow = \operatorname{diag} \lambda(X) \quad \text{for } X \in \mathbb{H}_n, \tag{5}$$

where $\lambda(X)$ stands for the vector of eigenvalues of a matrix $X \in \mathbb{H}_n$ arranged in nonincreasing order. In addition,

$$Y \preceq_G X \quad \text{iff } \lambda(Y) \prec \lambda(X) \quad \text{for } X, Y \in \mathbb{H}_n \tag{6}$$

(see (2) and [7, p. 17]). So, $\preceq_G$ on $\mathbb{D}_n(\mathbb{R})$ may be identified with the classical majorization $\prec$ on $\mathbb{R}^n$.

Example 1.2 (Cf. [7, pp. 17-18], [14, pp. 944-945]). Let V be the space $\mathbb{M}_n(\mathbb{C})$ with real inner product given by

$$\langle X, Y \rangle = \operatorname{Re} \operatorname{tr} XY^* \quad \text{for } X, Y \in \mathbb{M}_n(\mathbb{C}),$$

where $(\cdot)^*$ denotes conjugate transpose. Let G be the group of all linear operators

$$X \to U_1 X U_2 \quad \text{for } X \in \mathbb{M}_n(\mathbb{C}),$$

where U_1 and U_2 vary over the unitary group $\mathbb{U}_n$. Take

$$D = \{\operatorname{diag}(z_1, \ldots, z_n) \in \mathbb{D}_n : z_1 \geq \ldots \geq z_n \geq 0\}.$$

Here (A1) is the Singular Values Decomposition Theorem [17, p. 498], and (A2) is the trace inequality of von Neumann [17, p. 514]. Therefore (V, G, D) is an E-system.

In addition,

$$X_\downarrow = \operatorname{diag} s(X) \quad \text{for } X \in \mathbb{M}_n(\mathbb{C}), \tag{7}$$

where $s(X)$ stands for the vector of singular values of a matrix $X \in \mathbb{M}_n(\mathbb{C})$ (i.e., eigenvalues of $(X^*X)^{1/2}$) arranged in nonincreasing order. Moreover,

$$Y \preceq_G X \quad \text{iff } s(Y) \prec_w s(X) \quad \text{for } X, Y \in \mathbb{M}_n(\mathbb{C}) \tag{8}$$

(see (2) and [7, pp. 17–18]).

It is well known that

$$s(A \circ X) \prec_w s(A) \circ s(X) \quad \text{for } A, X \in \mathbb{M}_n(\mathbb{C}), \tag{9}$$

where $\circ$ stands for the Hadamard (entrywise) product of matrices and of vectors in $\mathbb{R}^n$ (see [13, p. 168]). Define $K : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_n(\mathbb{C})$ by

$$KX = A \circ X \quad \text{for } X \in \mathbb{M}_n(\mathbb{C}),$$

where A is a diagonal matrix with decreasingly ordered positive diagonal entries. Let $(W, H, E) = (V, G, D)$. Then $KD \subset E$ and inequality (9) is of form (4).

A similar result is the following inequality for the conventional product:

$$s(AX) \prec_w s(A) \circ s(X) \quad \text{for } A, X \in \mathbb{M}_n(\mathbb{C}) \tag{10}$$

(see [13, p. 168]). Taking

$$KX = AX \quad \text{for } X \in \mathbb{M}_n(\mathbb{C}),$$

with A as above, we conclude from (10) that (4) is satisfied.

2. Morphisms of E-systems

Unless otherwise stated, throughout this section V and W are finite-dimensional real inner product spaces, and G and H are closed subgroups of the orthogonal groups $O(V)$ and $O(W)$, respectively. These assumptions will not be repeated in our theorems, corollaries, etc.

In this section we introduce *morphisms* of Eaton systems and study their properties. We begin with a motivation for studying such a class of operators between two E-systems.

Assume $\mathcal{E} = (V, G, D)$ and $\mathcal{F} = (W, H, E)$ are E-systems, and $(V, G, (\cdot)_\downarrow)$ and $(W, H, (\cdot)_\downarrow)$ are related ND systems. In [23] the author characterized linear maps $K : V \to W$ preserving the normal maps of $\mathcal{E}$ and $\mathcal{F}$ in the sense that

$$(Kx)_\downarrow = Kx_\downarrow \quad \text{for } x \in V. \tag{11}$$

A direct consequence of (11) is the inclusion $KD \subset E$.

Following Lewis [14, p. 931], we say that the ND systems $(V, G, (\cdot)_\downarrow)$ and $(W, H, (\cdot)_\downarrow)$ are *isomorphic* if there exist an inner product space isomorphism $K : V \to W$ and a group isomorphism $\varphi : G \to H$ such that

$$(Kx)_\downarrow = Kx_\downarrow \quad \text{and} \quad Kgx = \varphi(g)Kx \quad \text{for } x \in V \text{ and } g \in G \tag{12}$$

(cf. [23, Theorem 3.1, Theorem 3.9]).

We say that the linear operator K is an $\mathcal{E}, \mathcal{F}$-*morphism* if

$$KD \subset E \quad \text{and} \quad (Kx)_\downarrow \preceq_H Kx_\downarrow \quad \text{for } x \in V. \tag{13}$$

This definition can be restated in other forms. By (A1) applied to $\mathcal{E}$, the inequality in (13) can be equivalently rewritten in the form

$$Kgx \preceq_H Kx \quad \text{for } x \in D \text{ and } g \in G. \tag{14}$$

It is not hard to verify by a convexity argument that (14) is equivalent to

$$y \preceq_G x \ \text{ implies } Ky \preceq_H Kx \quad \text{for } x \in D \text{ and } y \in V, \tag{15}$$

which can be restated as

$$y \preceq_G x \text{ implies } Ky \preceq_H Kx_\downarrow \text{ for } x, y \in V. \tag{16}$$

Clearly, (15) asserts that

$$KC_G(x) \subset C_H(Kx) \text{ for } x \in D, \tag{17}$$

which is equivalent to

$$KC_G(x) \subset C_H(Kx_\downarrow) \text{ for } x \in V. \tag{18}$$

An important subclass of $\mathcal{E}, \mathcal{F}$-morphisms are *simple morphisms* formed by linear operators $K : V \to W$ satisfying the following requirement:

$$KD \subset E \quad \text{and} \quad Kx \equiv_H Kx_\downarrow \text{ for } x \in V, \tag{19}$$

or equivalently,

$$KD \subset E \quad \text{and} \quad Kgx \equiv_H Kx \text{ for } x \in D \text{ and } g \in G. \tag{20}$$

The second part of (20) is equivalent to the condition:

$$\text{for } x \in D \text{ and } g \in G \text{ there exists } h \in H \text{ such that } Kgx = hKx \tag{21}$$

(cf. (12)). To see the difference between (11) and (19), consult [23, Theorem 3.1]. The set of all simple morphisms of E-systems $\mathcal{E}$ and $\mathcal{F}$ is denoted by $\mathrm{SMor}(\mathcal{E}, \mathcal{F})$.

If G and H are finite, then condition (21) says that the linear operator $K : V \to W$ preserves the group majorizations $\preceq_G$ and $\preceq_H$ in the sense that for $x, y \in V$,

$$y \preceq_G x \text{ implies } Ky \preceq_H Kx \tag{22}$$

(see [20, Theorems 2.1 and 2.2], [18, Theorem 3.1]). For $V = W = \mathbb{R}^n$ and $G = H =$ the group of $n \times n$ permutation matrices, such operators have been studied extensively in [2, 6]. For the matrix group majorizations described in Examples 1.1 and 1.2, a characterization of linear operators satisfying (21) is given in a paper by Li and Pierce [16].

Hereinafter $K^* : W \to V$ is the *dual operator* of K defined by

$$\langle Kx, y \rangle = \langle x, K^* y \rangle \text{ for } x \in V \text{ and } y \in W.$$

The set of $\mathcal{E}, \mathcal{F}$-morphisms, denoted by $\mathrm{Mor}(\mathcal{E}, \mathcal{F})$, is a *closed convex cone*:

(1) $K \in \mathrm{Mor}(\mathcal{E}, \mathcal{F})$ and $t \geq 0$ imply $tK \in \mathrm{Mor}(\mathcal{E}, \mathcal{F})$,
(2) $K_1, K_2 \in \mathrm{Mor}(\mathcal{E}, \mathcal{F})$ implies $K_1 + K_2 \in \mathrm{Mor}(\mathcal{E}, \mathcal{F})$,
(3) $K_i \in \mathrm{Mor}(\mathcal{E}, \mathcal{F}), i = 1, 2, \ldots,$ implies $\lim_{i \to \infty} K_i \in \mathrm{Mor}(\mathcal{E}, \mathcal{F})$.

In addition, $\mathrm{Mor}\,\mathcal{E} = \mathrm{Mor}(\mathcal{E}, \mathcal{E})$ is a *selfadjoint semigroup*:

(4) $\mathrm{id}_V \in \mathrm{Mor}\,\mathcal{E}$,
(5) $K_1, K_2 \in \mathrm{Mor}\,\mathcal{E}$ implies $K_2 K_1 \in \mathrm{Mor}\,\mathcal{E}$,
(6) $K \in \mathrm{Mor}\,\mathcal{E}$ implies $K^* \in \mathrm{Mor}\,\mathcal{E}$.

The proof of properties **(1)**–**(5)** is straightforward by using (3) and (13)–(16). Property **(6)** follows from Theorem 2.1 which says that

$$K \text{ is an } \mathcal{E},\mathcal{F}\text{-morphism iff } K^* \text{ is an } \mathcal{F},\mathcal{E}\text{-morphism.}$$

In addition, this result induces some special classes of morphisms (see Theorem 2.2).

Theorem 2.1. *Assume $\mathcal{E} = (V, G, D)$ and $\mathcal{F} = (W, H, E)$ are E-systems, and $K : V \to W$ is a linear operator. The following two conditions are equivalent:*

$$KD \subset E \quad \text{and} \quad Kgx \preceq_H Kx \quad \text{for } x \in D \text{ and } g \in G. \tag{23}$$

$$K^*E \subset D \quad \text{and} \quad K^*hz \preceq_G K^*z \quad \text{for } z \in E \text{ and } h \in H. \tag{24}$$

Proof. By duality, it is enough to prove (23) $\Rightarrow$ (24). By using (3) and $KD \subset E$, we rewrite the inequality of (23) in the form

$$\langle z, hKgx\rangle \le \langle z, Kx\rangle \quad \text{for } x \in D,\ g \in G,\ z \in E \text{ and } h \in H. \tag{25}$$

Substituting $h = \mathrm{id}_W$ (the identity operator on W) into (25), we get

$$\langle K^*z, x - gx\rangle \ge 0 \quad \text{for } x \in D,\ g \in G \text{ and } z \in E. \tag{26}$$

The set $\{x - gx : x \in D, g \in G\}$ is a generator of the cone

$$\operatorname{dual} D = \{v \in V : \langle v, z\rangle \ge 0 \text{ for } z \in D\}$$

(see [19, Lemma 3.2]). Therefore (26) means

$$K^*z \in \operatorname{dual}(\operatorname{dual} D) = D.$$

Thus we obtain the inclusion $K^*E \subset D$.

On the other hand, (25) is equivalent to

$$\langle g^*K^*h^*z, x\rangle \le \langle K^*z, x\rangle \quad \text{for } x \in D,\ g \in G,\ z \in E \text{ and } h \in H. \tag{27}$$

Remind that $G^* = G$ and $H^* = H$, since G and H are groups consisting of orthogonal operators. It now follows from (3) that (27) gives the inequality of (24). This completes the proof of (24). □

Combining Theorem 2.1 and (13)–(18), one sees that

$$K \in \operatorname{Mor}(\mathcal{E}, \mathcal{F}) \text{ implies } K^*K \in \operatorname{Mor}(\mathcal{E}) \text{ and } KK^* \in \operatorname{Mor}(\mathcal{F}).$$

Assume that $\mathcal{F} = (W, H, E)$ is an E-system. If the triple $\mathcal{F}_0 = (W_0, H_0, E_0)$ is an E-system, where W_0 is a subspace of W, $E_0 \subset W_0$ is a closed convex subcone of E, and H_0 is a closed subgroup of H such that $H_0W_0 \subset W_0$, then $\mathcal{F}_0$ is called a *subsystem* of $\mathcal{F}$ (cf. [14, pp. 933, 937]).

In the next theorem we show how to construct morphisms for given subsystem in $\mathcal{F}$.

Theorem 2.2. *Assume $\mathcal{E} = (V, G, D)$ and $\mathcal{F} = (W, H, E)$ are E-systems, and $K : V \to W$ is a linear operator. Suppose that $\mathcal{F}_0 = (W_0, H_0, E_0)$ is a subsystem of $\mathcal{F}$ such that $KV \subset W_0$.*

If $K^* \in \mathrm{Mor}\,(\mathcal{F}_0, \mathcal{E})$, i.e., $K^*E_0 \subset D$ and

$$K^*h_0 z \preceq_G K^*z \quad \text{for } z \in E_0 \text{ and } h_0 \in H_0, \tag{28}$$

then $K \in \mathrm{Mor}\,(\mathcal{E}, \mathcal{F}_0)$, i.e., $KD \subset E_0$ and

$$Kgx \preceq_{H_0} Kx \quad \text{for } x \in D \text{ and } g \in G. \tag{29}$$

In particular, if $K^* \in \mathrm{SMor}\,(\mathcal{F}_0, \mathcal{E})$, i.e., $K^*E_0 \subset D$ and

$$\text{for } z \in E_0 \text{ and } h_0 \in H_0 \text{ there exists } g \in G \text{ such that } K^*h_0 z = gK^*z, \tag{30}$$

then $K \in \mathrm{Mor}\,(\mathcal{E}, \mathcal{F}_0)$.

E.g., if $\mathcal{E} = \mathcal{F}$ and the restriction of K^* to W_0 is the identity, then (29) *holds.*

Proof. Define $K_0 : V \to W_0$ by $K_0 x = Kx$ for $x \in V$. The restriction of $K^* : W \to V$ to W_0 is K_0^*. Applying Theorem 2.1 to K_0, $\mathcal{F}_0$ and $\preceq_{H_0}$ in place of K, $\mathcal{F}$ and $\preceq_H$, respectively, we get (29) from (28).

To see the next part of Theorem 2.2, observe that (30) implies (28). In consequence, (29) holds.

If, in addition, the restriction of K^* to W_0 is the identity, then (30) holds trivially for $g = h_0$, because $\mathcal{E} = \mathcal{F}$ and $H_0 \subset H = G$. □

3. Homomorphisms of E-systems

Group induced cone orderings, Eaton systems and normal decomposition systems play a unifying role in many problems of statistics, probability, matrix theory, Lie theory, convex analysis and optimization [7, 8, 9, 10, 14, 15, 18, 21, 22, 26, 27]. In this section, our aim is to provide some sufficient conditions for a triple $\mathcal{F}_0 = (W_0, H_0, E_0)$ to be an E-system. That is, we intend to present conditions guaranteeing the validity of axioms (A1) and (A2) for $\mathcal{F}_0$.

Unless otherwise stated, throughout this section, W_0 is a finite-dimensional real inner product space, H_0 is a closed subgroup of the orthogonal group $O(W_0)$, and E_0 is a closed convex cone included in W_0.

The first theorem shows a relationship between our problem and morphisms. Also, it gives a motivatation for introducing some special classes of morphisms after the proof of Theorem 3.1.

We denote $\|w\| = \langle w, w\rangle^{1/2}$ for $w \in W_0$.

Theorem 3.1. *The following three statements are mutually equivalent:*

(i) *Axiom* (A1) *is satisfied for $\mathcal{F}_0 = (W_0, H_0, E_0)$.*

(ii) *For any $w \in W_0$ there exists $e \in E_0$ such that $w \preceq_{H_0} e$ and $\|w\| = \|e\|$.*

(*iii*) *There exist a triple* $\mathcal{E} = (V, G, D)$ *(not necessarily E-system) and a linear operator* $K : V \to W_0$ *such that the following conditions* (31), (32) *and* (33) *are satisfied.*

$$W_0 = KV, \qquad E_0 = KD \quad \textit{and} \quad H_0 \textit{ is a closed subgroup of } O(W_0). \tag{31}$$

$$\textit{For } z \in V \textit{ and } x \in D, \ z \preceq_G x \textit{ implies } Kz \preceq_{H_0} Kx. \tag{32}$$

$$\begin{aligned} &\textit{For } w \in W_0 \textit{ there exist } x \in D \textit{ and } z \in V \textit{ such that} \\ &z \preceq_G x, \ w = Kz \textit{ and } \|w\| = \|Kx\|. \end{aligned} \tag{33}$$

Proof. (*i*) ⇒ (*ii*). Fix any $w \in W_0$. By (A1) for $\mathcal{F}_0$, we have $w = h_0 w_\downarrow$ for some $h_0 \in H_0$ and $w_\downarrow \in E_0$. Taking $e = w_\downarrow$, we get $e \in E_0$, $w \preceq_{H_0} e$ and $\|w\| = \|h_0 w_\downarrow\| = \|w_\downarrow\| = \|e\|$, as required.

(*ii*) ⇒ (*iii*). In order to see (*iii*), it is sufficient to put $\mathcal{E} = \mathcal{F}_0$, i.e., $V = W_0$, $G = H_0$ and $D = E_0$, and $K =$ the identity on W_0.

(*iii*) ⇒ (*i*). Take any $w \in W_0$. By (33), we obtain $w = Kz$ for some $x \in D$ and $z \in C_G(x)$ such that $\|w\| = \|Kx\|$. Denote $e = Kx$.

On the other hand, we get $Kz \preceq_{H_0} Kx$ from (32). Therefore $w \preceq_{H_0} e$. Hence $w \in C_{H_0}(e) = \operatorname{conv} H_0 e$. In other words, $w = \sum_{i=1}^m \alpha_i h_i e$ for some positive integer m, $h_i \in H_0$ and positive reals α_i summing to one. Without lost of generality, it can be assumed that there are not equal vectors among $h_i e$, $i = 1, \ldots, m$. In fact, if $h_{i_1} e = h_{i_2} e$ for some $i_1, i_2 \in \{1, \ldots, m\}$ with $i_1 \neq i_2$, then we can write $\alpha h_{i_1} e$ in place of $\alpha_{i_1} h_{i_1} e + \alpha_{i_2} h_{i_2} e$, where $\alpha = \alpha_{i_1} + \alpha_{i_2}$. So, the sum $\sum_{i=1}^m \alpha_i h_i e$ can be rewritten in form of a sum of the same type but with different vectors among $h_i e$ and with some smaller number of summands.

It now follows that

$$\|e\| = \|w\| = \|\sum_{i=1}^m \alpha_i h_i e\| \leq \sum_{i=1}^m \alpha_i \|h_i e\| = \sum_{i=1}^m \alpha_i \|e\| = \|e\|,$$

where $\|\cdot\| = \langle \cdot, \cdot \rangle^{1/2}$. This is the equality case of the triangle inequality. So, all the vectors $h_i e$ are equal. This is a contradiction, unless $m = 1$. Thus we obtain $w = h_1 e$.

We have shown that $W_0 \subset \bigcup_{h_0 \in H_0} h_0 E_0$. The reverse inclusion holds by $H_0 W_0 \subset W_0$. Thus (A1) is proved for $\mathcal{F}_0$. □

In the remainder of this section, we study axioms (A1) and (A2) for systems $\mathcal{F}_0 = (W_0, H_0, E_0)$ of form (31), where $K : V \to W_0$ is a linear operator and $\mathcal{E} = (V, G, D)$ is a triple (not necessarily E-system). According to the last theorem, it is sufficient to focus on operators satisfying (32)–(33).

In the literature (see e.g., [5, 16]), if $\prec_1$ and $\prec_2$ are orderings on V and W_0, respectively, then a linear operator $K : V \to W_0$ is said to be a *preserver* (resp., *strong preserver*) if

$$\text{for } z \in V \text{ and } x \in V, \ z \prec_1 x \text{ implies (resp., if and only if) } Kz \prec_2 Kx. \tag{34}$$

In our considerations we need *directional monotonicity* in sense of (32) (cf. [12, Definition 2.3]). According to (13) and (15), we say that K is an $\mathcal{E}, \mathcal{F}_0$*-morphism* if (32) is met. (Here we do not assume that $\mathcal{E}$ and $\mathcal{F}_0$ are E-systems.) In contrary to (34), (32) distinguishes directions in D from other directions in V. It is clear that (32) means

$$KC_G(x) \subset C_{H_0}(Kx_\downarrow) \text{ for } x \in V \tag{35}$$

(see (18)).

A related condition is of Schur-Horn-Kostant type:

$$KGx = C_{H_0}(Kx_\downarrow) \text{ for } x \in V. \tag{36}$$

The case K is the orthoprojector from V onto $W_0 \subset V$, has been studied in [27, Section 6]. See also [28] for an interpretation of (36) in Lie theory.

In light of Theorem 3.1, it is useful to work with some special morphisms. We introduce radial morphisms as follows. An $\mathcal{E}, \mathcal{F}_0$-morphism K is said to be $\mathcal{E}, \mathcal{F}_0$-*radial morphism* if (33) holds. It is readily seen that simple morphisms are radial morphisms, but not vice versa (see (19)–(20)).

We say that an $\mathcal{E}, \mathcal{F}_0$-*morphism* K is an $\mathcal{E}, \mathcal{F}_0$-*homomorphism* if

$$\begin{array}{l}\text{for } e \in E_0 \text{ and } w \in W_0,\ w \equiv_{H_0} e \text{ implies that} \\ \text{there exist } x \in D \text{ and } z \in V \text{ such that } z \preceq_G x,\ e = Kx \text{ and } w = Kz.\end{array} \tag{37}$$

For given $w \in W_0 = KV$ we denote

$$K^{-1}\{w\} = \{z \in V : Kz = w\},$$

$$D_K(w) = \{x \in D : Kz = w \text{ for some } z \preceq_G x\}$$

and

$$D_K^s(w) = \{x \in D : Kz = w \text{ for some } z \equiv_G x\}.$$

Notice that if K is an $\mathcal{E}, \mathcal{F}_0$-morphism, then

$$K \text{ is radial iff for each } w \in W_0,\ \|w\| = \|Kx\| \text{ for some } x \in D_K(w). \tag{38}$$

A characterization of radial morphisms is given in

Lemma 3.2. *Let $\mathcal{E} = (V, G, D)$ be an E-system and let K be an $\mathcal{E}, \mathcal{F}_0$-morphism. Then the following two statements are valid.*

(i) *For any $w \in W_0$,*

$$\inf_{x \in D_K(w)} \|Kx\| = \inf_{z \in K^{-1}\{w\}} \|Kz_\downarrow\|. \tag{39}$$

(ii) *The following three conditions are mutually equivalent.*

$$K \textit{ is an } \mathcal{E}, \mathcal{F}_0\textit{-radial morphism.} \tag{40}$$

$$\textit{For each } w \in W_0,\ \|w\| = \min_{x \in D_K(w)} \|Kx\|. \tag{41}$$

$$\textit{For each } w \in W_0,\ \|w\| = \min_{z \in K^{-1}\{w\}} \|Kz_\downarrow\|. \tag{42}$$

Proof. (i) Fix any $w \in W_0$. Since $(K^{-1}\{w\})_\downarrow = D_K^s(w) \subset D_K(w)$, we have

$$\inf_{x \in D_K(w)} \|Kx\| \le \inf_{x \in D_K^s(w)} \|Kx\| = \inf_{z \in K^{-1}\{w\}} \|Kz_\downarrow\|. \tag{43}$$

On the other hand, for each $x \in D_K(w)$ there exists $z_x \in V$ such that $z_x \preceq_G x$ and $Kz_x = w$. Hence $(z_x)_\downarrow \preceq_G x_\downarrow = x$, and further $K(z_x)_\downarrow \preceq_{H_0} Kx$, because $\mathcal{E}, \mathcal{F}_0$-morphisms are $\preceq_G, \preceq_{H_0}$-increasing on D. Consequently, $\|K(z_x)_\downarrow\| \le \|Kx\|$, since the function $\|\cdot\| = \langle \cdot, \cdot \rangle^{1/2}$ is convex and H_0-invariant by $H_0 \subset O(W_0)$. Therefore for each $x \in D_K(w)$, we obtain

$$\inf_{z \in K^{-1}\{w\}} \|Kz_\downarrow\| \le \|K(z_x)_\downarrow\| \le \|Kx\|.$$

Hence

$$\inf_{z \in K^{-1}\{w\}} \|Kz_\downarrow\| \le \inf_{x \in D_K(w)} \|Kx\|. \tag{44}$$

Combining (43) and (44) proves (39).

(ii) By (38) it is clear that (41) implies (40). To see the reverse implication, for any $x \in D_K(w)$ we derive

$$\|w\| = \inf_{z \in K^{-1}\{w\}} \|Kz\| \le \inf_{z \in K^{-1}\{w\}} \|Kz_\downarrow\| = \inf_{x \in D_K(w)} \|Kx\| \le \|Kx\|$$

by $Kz \preceq_{H_0} Kz_\downarrow$ and $\|Kz\| \le \|Kz_\downarrow\|$ and by (39). Therefore (40) implies (41).

Likewise, for any $x \in D_K(w)$ we have

$$\|w\| = \inf_{z \in K^{-1}\{w\}} \|Kz\| \le \|Kz_x\| \le \|K(z_x)_\downarrow\| \le \|Kx\|$$

with z_x defined after (43). Therefore (40) implies (42). The reverse implication is obvious by (38) and $(K^{-1}\{w\})_\downarrow \subset D_K(w)$.

This completes the proof of the equivalences (40) $\Leftrightarrow$ (41) $\Leftrightarrow$ (42). □

We now concentrate on axiom (A2).

Theorem 3.3. *Let $\mathcal{E} = (V, G, D)$ be an E-system and let K be an $\mathcal{E}, \mathcal{F}_0$-radial morphism.*

The following two conditions are equivalent:

(i) *Axiom* (A2) *is satisfied for $\mathcal{F}_0 = (W_0, H_0, E_0)$.*
(ii) *$K^* : W_0 \to V$ is an $\mathcal{F}_0, E$-morphism, i.e., if $K^* E_0 \subset D$ and*

$$K^* h_0 e \preceq_G K^* e \quad \text{for } h_0 \in H_0 \text{ and } e \in E_0. \tag{45}$$

Proof. $(i) \Rightarrow (ii)$. By (i) and Theorem 3.1, one sees that $\mathcal{F}_0$ is an E-system. Since K is an $\mathcal{E}, \mathcal{F}_0$-morphism, it now follows from Theorem 2.1 applied to $\mathcal{E}$ and $\mathcal{F}_0$ that $K^* : W_0 \to V$ is an $\mathcal{F}_0, \mathcal{E}$-morphism.

$(ii) \Rightarrow (i)$. Since $\mathcal{E} = (V, G, D)$ is an E-system, by (3) and (45) we get

$$\langle x, gK^* h_0 e \rangle \le \langle x, K^* e \rangle \quad \text{for } x \in D,\ g \in G,\ h_0 \in H_0 \text{ and } e \in E_0. \tag{46}$$

Substituting $g = \mathrm{id}$ into (46) yields

$$\langle Kx, h_0 e\rangle \le \langle Kx, e\rangle \quad \text{for } x \in D,\ h_0 \in H_0 \text{ and } e \in E_0.$$

Since $E_0 = KD$, the last inequality means that condition (A2) is fulfilled for $\mathcal{F}_0 = (W_0, H_0, E_0)$. □

For triples $\mathcal{F}_0 = (W_0, H_0, E_0)$ and $\mathcal{F} = (W, H, E)$, we write $\mathcal{F}_0 \subset \mathcal{F}$ if

$$W_0 \subset W, \qquad E_0 \subset E \quad \text{and} \quad H_0 \subset \{h \in H : hW_0 \subset W_0\}|_{W_0}. \tag{47}$$

Corollary 3.4. *Let $\mathcal{E} = (V, G, D)$ be an E-system and let K be an $\mathcal{E}, \mathcal{F}_0$-radial morphism.*

If there exists an E-system $\mathcal{F} = (W, H, E)$ such that $\mathcal{F}_0 \subset \mathcal{F}$, then $\mathcal{F}_0 = (W_0, H_0, E_0)$ is an E-system.

Proof. To see (A1) for $\mathcal{F}_0$, apply Theorem 3.1. The validity of (A2) for $\mathcal{F}_0$ is a simple consequence of (A2) for $\mathcal{F}$ (see (47)). □

Theorem 3.5 provides further sufficient conditions for $\mathcal{F}_0 = (W_0, H_0, E_0)$ to be an E-system. This result extends [23, Theorems 3.4 and 3.9].

Theorem 3.5. *Let $\mathcal{E} = (V, G, D)$ be an Eaton system and let K be an $\mathcal{E}, \mathcal{F}_0$-radial homomorphism.*

If $K^*KD \subset D$, then $\mathcal{F}_0 = (W_0, H_0, E_0)$ is an E-system.

Proof. By Theorem 3.1 it is easily seen that (A1) holds for $\mathcal{F}_0$. It remains to prove that (A2) is satisfied for $\mathcal{F}_0$. Take any $e_1, e_2 \in E_0$ and $h_0 \in H_0$. We have to show that $\langle e_2, h_0 e_1\rangle \le \langle e_2, e_1\rangle$. Denote $w = h_0 e_1$ and $e = e_1$. By (37), we get $e = Kx$ and $w = Kz$ for some $x \in D$ and $z \in C_G(x) = \operatorname{conv} Gx$.

On the other hand, it follows from the inclusion $K^*KD \subset D$ and from (A2) applied to $\mathcal{E} = (V, G, D)$ that $\langle K^*Ky, gx\rangle \le \langle K^*Ky, x\rangle$ for any $y \in D$ and $g \in G$. Therefore $\langle K^*Ky, z\rangle \le \langle K^*Ky, x\rangle$, because $z \in \operatorname{conv} Gx$. In consequence,

$$\langle Ky, Kz\rangle \le \langle Ky, Kx\rangle \quad \text{for any } y \in D.$$

But $e_2 \in E_0 = KD$, so $e_2 = Ky$ for some $y \in D$. Hence

$$\langle e_2, h_0 e_1\rangle = \langle Ky, Kz\rangle \le \langle Ky, Kx\rangle = \langle e_2, e_1\rangle,$$

completing the proof. □

We now study our preservance problem for partial isometries. Remind that a linear operator $K : V \to W$ is said to be a *partial isometry* if $KK^*K = K$. In this event, the operator $KK^* : W \to W$ is the orthoprojector from W onto the subspace $KK^*W = KV = W_0$.

The next theorem extends [18, Theorem 3.2] from orthoprojectors to partial isometries. It is also related to [22, Theorem 3.1] applied to the orthoprojector KK^*.

Theorem 3.6. *Let $K : V \to W$ be a partial isometry with finite-dimensional real inner product spaces V and W, and let $D \subset V$ be a closed convex cone. Denote $\mathcal{F}_0 = (W_0, H_0, E_0)$, where $W_0 = KV$, $E_0 = KD$ and H_0 is a closed subgroup of $O(W_0)$. Suppose that $\mathcal{F} = (W, H, E)$ is an E-system such that $\mathcal{F}_0 \subset \mathcal{F}$ and $KK^*E = E_0$.*

Then the following statements are mutually equivalent:

(*i*) $\mathcal{F}_0 = (W_0, H_0, E_0)$ *is an E-system.*

(*ii*) *The following inequality holds*

$$KK^*w \preceq_{H_0} KK^*w_\downarrow \quad \text{for } w \in W, \tag{48}$$

where $(\cdot)_\downarrow$ stands for the normal map of $\mathcal{F}$.

(*iii*) *The following inclusion holds*

$$W_0 \subset \bigcup_{h\in H} hE_0, \tag{49}$$

and, in addition, the operator KK^* is an $\mathcal{F}, \mathcal{F}_0$-radial homomorphism.

Proof. (*i*) ⇒ (*ii*). Since K is a partial isometry, KK^* is symmetric ($(KK^*)^* = KK^*$), and the restriction of KK^* to $W_0 = KV$ is the identity on W_0,

Now, it is sufficient to apply the last part of Theorem 2.2 to the operator $KK^* : W \to W$ and to the E-systems $\mathcal{F}$ and $\mathcal{F}$.

(*ii*) ⇒ (*iii*). We shall prove (49). For this end take any $w \in W_0$. By (A1) applied to $\mathcal{F}$ we obtain $w = hw_\downarrow$ for some $h \in H$ and $w_\downarrow \in E$. It is enough to show that $w_\downarrow \in E_0$.

Since $KK^*|_{W_0}$ is the identity on W_0, we get $KK^*w = w = hw_\downarrow$. So (48) implies $hw_\downarrow \preceq_{H_0} KK^*w_\downarrow$. Hence

$$\|w_\downarrow\| = \|hw_\downarrow\| \le \|KK^*w_\downarrow\| \le \|w_\downarrow\|.$$

In fact, the first inequality follows from the convexity and H-invariance of the norm $\|\cdot\|$. The second is due to Pythagorean Theorem:

$$\|w_\downarrow\|^2 = \|w_\downarrow - KK^*w_\downarrow\|^2 + \|KK^*w_\downarrow\|^2,$$

because KK^* is an orthogonal projector. Therefore $\|KK^*w_\downarrow\| = \|w_\downarrow\|$, which implies $KK^*w_\downarrow = w_\downarrow$. But $KK^*E = E_0$ and $w_\downarrow \in E$, so $w_\downarrow \in E_0$. Finally, we have $w = hw_\downarrow$ for some $h \in H$ and $w_\downarrow \in E_0$. This yields (49).

By (48) one sees that KK^* is an $\mathcal{F}, \mathcal{F}_0$-morphism. To see that KK^* is radial (see (33)), let $w \in W_0$. By (49), $w = he$ for some $h \in H$ and $e \in E_0 \subset W_0$. Put $z = w$ and $x = e$. Since $KK^*|_{W_0}$ is the identity on W_0, it is clear that $KK^*z = w$, $KK^*x = e$, $z \preceq_H x$ and $\|w\| = \|KK^*x\|$. Thus KK^* is an $\mathcal{F}, \mathcal{F}_0$-radial morphism.

The proof that KK^* is an $\mathcal{F}, \mathcal{F}_0$-homomorphism (see (37)), is similar and therefore omitted.

(*iii*) ⇒ (*i*). Since $\mathcal{F}_0 \subset \mathcal{F}$ and K is a partial isometry, we have $(KK^*)^*(KK^*)E = KK^*E = E_0 \subset E$ (see (47)). By Theorem 3.5, the triple (KK^*W, H_0, KK^*E) is an E-system. But $KK^*W = KV = W_0$ and $KK^*E = KD = E_0$, so $\mathcal{F}_0$ is an E-system, completing the proof. □

4. Applications to matrices

In this section, we interpret Theorems 2.2 and 3.6 for matrix system and their subsystems.

Let $\mathcal{E} = (V, G, D)$ and $\mathcal{F} = (W, H, E)$ with $W = V$, $H = G$ and $E = D$, where

$V = \mathbb{M}_n(\mathbb{C})$ = the (real) space of $n \times n$ complex matrices with the inner product $\langle A, B\rangle = \operatorname{Re}\operatorname{tr} AB^*$ for $A, B \in \mathbb{M}_n(\mathbb{C})$,

G = the group of unitary equivalences $u(\cdot)v$ with u and v running over the group $\mathbb{U}_n$ of $n \times n$ unitary matrices,

$D = \{\operatorname{diag}(s_1, \ldots, s_n) : s_1 \geq \ldots \geq s_n \geq 0\}$

(see Example 1.2). The following items constitute some subsystems $\mathcal{F}_0 = (W_0, H_0, E_0)$ of $\mathcal{F}$ [7, 8, 14, 26].

(a)

$W_0 = \mathbb{D}_n(\mathbb{R})$ = the space of $n \times n$ real diagonal matrices,

H_0 = the group of equivalences $u(\cdot)v$ with u and v running over the group $\mathbb{GP}_n$ of $n \times n$ generalized permutation matrices,

$E_0 = \{\operatorname{diag}(s_1, \ldots, s_n) : s_1 \geq \ldots \geq s_n \geq 0\}$.

(b)

$W_0 = \mathbb{D}_n(\mathbb{C})$ = the space of $n \times n$ complex diagonal matrices,

H_0 = the group of equivalences $u(\cdot)v$ with u and v running over the group $\mathbb{GP}_n(\mathbb{C})$ of $n \times n$ complex generalized permutation matrices,

$E_0 = \{\operatorname{diag}(s_1, \ldots, s_n) : s_1 \geq \ldots \geq s_n \geq 0\}$.

(c)

W_0 = the space of $n \times n$ matrices of the form $\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}$ with $k \times k$ matrix $X \in \mathbb{M}_k(\mathbb{C})$ and $1 \leq k \leq n$,

H_0 = the group of unitary similarities $u(\cdot)v$ with u and v being the matrix of the form $\begin{pmatrix} U & 0 \\ 0 & I_{n-k} \end{pmatrix}$ for some $k \times k$ unitary matrix U,

$E_0 = \{\operatorname{diag}(s_1, \ldots, s_k, 0, \ldots, 0) : s_1 \geq \ldots \geq s_k \geq 0\}$.

(d)

$W_0 = \mathbb{S}_n(\mathbb{C})$ = the space of $n \times n$ complex symmetric matrices,

H_0 = the group of unitary congruences $u(\cdot)u^T$ with u running over the group $\mathbb{U}_n$ of $n \times n$ unitary matrices,

$E_0 = \{\operatorname{diag}(s_1, \ldots, s_n) : s_1 \geq \ldots \geq s_n \geq 0\}$.

(e)

$$W_0 = \mathbb{M}_n(\mathbb{R}) = \text{the space of } n \times n \text{ real matrices},$$
$$H_0 = \text{the group of orthogonal equivalences } u(\cdot)v \text{ with } u \text{ and } v \text{ running over the group } \mathbb{O}_n \text{ of } n \times n \text{ orthogonal matrices},$$
$$E_0 = \{\operatorname{diag}(s_1, \ldots, s_n) : s_1 \geq \ldots \geq s_n \geq 0\}.$$

Corollary 4.1. *For any of the above subsystems $\mathcal{F}_0$, let K be a linear operator from $\mathbb{M}_n(\mathbb{C})$ to $\mathbb{M}_n(\mathbb{C})$ such that $K\mathbb{M}_n(\mathbb{C}) \subset W_0$.*

(i) *If the restriction $K^*|_{W_0}$ is a simple morphism of $\mathcal{F}_0$ and $\mathcal{E}$, i.e., if* (30) *is satisfied and $K^*E_0 \subset D$, then K is a morphism of $\mathcal{E}$ and $\mathcal{F}_0$, i.e., $KD \subset E_0$ and*

$$s(Kx) \prec_w s(Kx_\downarrow) \quad \textit{for } x \in \mathbb{M}_n(\mathbb{C}), \tag{50}$$

where $x_\downarrow = \operatorname{diag} s(x)$.

(ii) *If the restriction $K^*|_{W_0}$ is the identity on W_0, then inequality* (50) *holds.*

(iii) *If K is symmetric ($K^* = K$) and $D \subset W_0$, and if the restriction $K|_{W_0}$ is the identity, then inequality* (50) *holds in the form*

$$s(Kx) \prec_w s(x) \quad \textit{for } x \in \mathbb{M}_n(\mathbb{C}). \tag{51}$$

Proof. Clearly, (i) ⇒ (ii) ⇒ (iii). To prove (i), use Theorem 2.2. □

In the case of the subsystem described in (b), (51) extends the classical inequality of Fan

$$|d(x)| \prec_w s(x) \quad \text{for } x \in \mathbb{M}_n(\mathbb{C}).$$

In fact, it is sufficient to employ the orthoprojector $Kx = \operatorname{diag} d(x)$ from $\mathbb{M}_n(\mathbb{C})$ onto $\mathbb{D}_n(\mathbb{C})$, where $d(x)$ stands for the diagonal of $x \in \mathbb{M}_n(\mathbb{C})$.

The next result follows from Theorem 3.6.

Corollary 4.2. *Let $K : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_n(\mathbb{C})$ be a partial isometry. Denote $W_0 = KV$ and $E_0 = KD$. Let H_0 be a closed subgroup of $O(W_0)$. Suppose that $KK^*D = E_0$.*

Then the following statements are mutually equivalent:

(i) *$\mathcal{F}_0 = (W_0, H_0, E_0)$ is an E-system.*

(ii) *The following inequality holds*

$$KK^*w \preceq_{H_0} KK^*w_\downarrow \quad \textit{for } w \in \mathbb{M}_n(\mathbb{C}), \tag{52}$$

where $w_\downarrow = \operatorname{diag} s(w)$.

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