Journal of Convex Analysis
Volume 16 (2009), No. 3&4, 687–698

Operator Topologies and Graph Convergence

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Dedicated to Stephen Simons on the occasion of his 70th birthday.

Received: March 12, 2008

Let $\mathbf{B}(X, Y)$ be the continuous linear transformations from a normed linear space X to a normed linear space Y. This article presents two general results - one for the norm topology on Y and one for the weak topology on Y - that explain how convergence of sequences in $\mathbf{B}(X, Y)$ with respect to a topology of uniform convergence on a prescribed family of norm bounded subsets of X is reflected in the bornological convergence of the associated sequence of graphs with respect to a family of subsets of $X \times Y$.

Keywords: Operator topology, polar topology, bornological convergence, Attouch-Wets Convergence, normed linear space, convex set, starshaped set

2000 Mathematics Subject Classification: Primary 47A05; Secondary 46A17, 54B20

1. Introduction

Let $\langle X, ||\cdot||_X\rangle$ and $\langle Y, ||\cdot||_Y\rangle$ be normed linear spaces over a field of scalars $\mathbb{F}$ where either $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. The *operator norm* $||\cdot||_{\text{op}}$ on the space $\mathbf{B}(X, Y)$ of continuous linear transformations from X to Y is defined by the familiar formula

$$||T||_{\text{op}} := \sup \{||Tx||_Y : ||x||_X \leq 1\} \quad (T \in \mathbf{B}(X, Y)).$$

Denoting the unit ball of X by U_X, operator norm convergence of $T_1, T_2, T_3, \ldots$ to T in $\mathbf{B}(X, Y)$ means uniform convergence of $\langle T_n\rangle$ to T on U_X, and thus uniform convergence on each bounded subset B of X, since for $S \in \mathbf{B}(X, Y)$ and $\alpha \in \mathbb{F}$,

$$\sup \{||Sx||_Y : x \in \alpha B\} = |\alpha| \cdot \sup \{||Sx||_Y : x \in B\}.$$

Thus, convergence in the operator norm is convergence with respect to a topology of uniform convergence on a *bornology* [16], that is, a family of nonempty subsets of X that is closed under taking finite unions, that is closed under taking subsets, and that forms a cover of X. In this case it is the bornology $\mathcal{B}(||\cdot||_X)$ of norm bounded subsets of X.

Now the bornology of bounded sets plays a fundamental role in a description of operator norm convergence of sequences in $\mathbf{B}(X, Y)$ in terms of the convergence of the associated sequence of graphs in $X \times Y$. Let us equip the product with the norm $||(x, y)||_{\text{box}} := \max\{||x||_X, ||y||_Y\}$ (any equivalent norm can also be used). The following result was obtained by Penot and Zălinescu [22] for general continuous linear transformations (the

ISSN 0944-6532 / $ 2.50

special case of continuous linear functionals was handled earlier in [3, 7]). For $T \in \mathbf{B}(X,Y)$, let us now put $\mathrm{Gr}(T) := \{(x,y) : Tx = y\}$.

Theorem 1.1. *Let X and Y be normed linear spaces and let $T, T_1, T_2, T_3, \dots$ be a sequence in $\mathbf{B}(X,Y)$. Then $\lim_{n\to\infty} ||T_n - T||_{op} = 0$ if and only if $\forall \varepsilon > 0\ \forall E \in \mathcal{B}(||\cdot||_{box})$ we have eventually both*

$$\mathrm{Gr}(T_n) \cap E \subseteq \mathrm{Gr}(T) + \varepsilon U_{X\times Y},$$
$$\mathrm{Gr}(T) \cap E \subseteq \mathrm{Gr}(T_n) + \varepsilon U_{X\times Y}.$$

Arguably, their result should be well-known among the analysis community whereas exactly the opposite is the case. Convergence of graphs in this sense is a special case of a well-studied convergence notion for closed sets in a normed linear space, called *Attouch-Wets convergence* or *bounded Hausdorff convergence* in the literature. For closed subsets $A, A_1, A_2, A_3, \dots$ of $\langle X, ||\cdot||_X\rangle$, we write $A = AW - \lim A_n$ if for each norm bounded set B and each $\varepsilon > 0$, we have eventually both

$$A_n \cap B \subseteq A + \varepsilon U_X \quad \text{and} \quad A \cap B \subseteq A_n + \varepsilon U_X.$$

This convergence is compatible with a metrizable topology that is completely metrizable when X is a Banach space [1, 5]. It is hard to overstate the importance of this convergence notion in convex and variational analysis [26, 2, 4, 5, 11, 20, 21, 24, 22].

It is the purpose of this note to replace the bornology of bounded subsets of X by an arbitrary subfamily $\mathcal{B}$ and display how uniform convergence in $\mathbf{B}(X,Y)$ on members of $\mathcal{B}$ can be explained in terms of convergence of graphs.

2. Preliminaries

Let $\langle X, ||\cdot||_X\rangle$ be a normed linear space over a field of scalars $\mathbb{F}$. We denote the origin of X by θ and the space $\mathbf{B}(X,\mathbb{F})$ by X^*. If $A \subseteq X$, we denote its *convex hull* by $\mathrm{co}(A)$; by a *polytope*, we mean the convex hull of a finite set. We denote its *absolutely convex hull* by $\mathrm{aco}(A)$; this is the smallest balanced convex set containing A. Elements of $\mathrm{aco}(A)$, consist of all linear combinations of elements of A of the form

$$\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n$$

where $\{a_1, a_2, a_3, \dots, a_n\} \subseteq A$ and $\Sigma_{j=1}^n |\lambda_j| \leq 1$.

A subset A of X is called *starshaped* with respect to $a_0 \in A$ if $\forall a \in A,\ \mathrm{co}(\{a, a_0\}) \subseteq A$ [25]. The smallest set containing $E \subseteq X$ that is starshaped with respect to the origin is star $(E;\theta) := \cup_{e\in E}\ \mathrm{co}(\{e,\theta\}) = \cup_{\lambda\in[0,1]} \lambda E$.

We next introduce two idempotent operators $\downarrow$ and Σ on families of nonempty subsets of X, defined in [19] by

$$\downarrow \mathcal{S} := \{E \neq \emptyset : \exists S \in \mathcal{S} \text{ with } E \subseteq S\};$$
$$\Sigma\mathcal{S} := \{E : E \text{ is a finite union of elements of } \mathcal{S}\}.$$

Given a family of nonempty subsets $\mathcal{S}$ of norm bounded subsets of X that is a cover of X, the *topology $\tau^s_{\mathcal{S}}$ of strong uniform convergence on elements of* $\mathcal{S}$ for $\mathbf{B}(X,Y)$ is a locally convex topology having as a local base at the the zero linear transformation all sets of the form

$$V(S_1, S_2, \ldots S_n; \varepsilon) := \{T \in \mathbf{B}(X,Y) : \forall x \in \cup_{j=1}^n S_j,\ ||Tx||_Y < \varepsilon\},$$

where $\{S_1, S_2, \ldots S_n\} \subseteq \mathcal{S}$ and $\varepsilon > 0$. When $Y = \mathbb{F}$, such topologies are often called for good reason *polar topologies*(see, e.g., [23, p. 46]).

The strongest such topology is obtained when $\mathcal{S} = \{nU_X : n \in \mathbb{N}\}$ and is the operator norm topology. The weakest such topology is the topology of strong pointwise convergence, sometimes called the *strong operator topology* in the literature [15, p. 475], which of course is the *weak-star topology* when the codomain is the field of scalars. As another example, when $Y = \mathbb{F}$, the topology of strong uniform convergence on $\{S : S$ is the range of a norm convergent sequence$\}$ is called the *bounded weak-star topology* (see more generally [15, p. 512].

The requirement that the sets be norm bounded of course is included to guarantee that each neighborhood is absorbing with respect to the vector space $\mathbf{B}(X,Y)$. The requirement that the family be a cover is included so that the convergence implies strong pointwise convergence. More generally, if a second cover $\mathcal{T}$ refines $\mathcal{S}$, that is, $\mathcal{T} \subseteq \downarrow \mathcal{S}$, then $\tau^s_{\mathcal{T}}$ is coarser than $\tau^s_{\mathcal{S}}$. One consequence we shall use in our main results is this:

$$\mathcal{S} \subseteq \mathcal{T} \subseteq \downarrow \mathcal{S} \Rightarrow \tau^s_{\mathcal{T}} = \tau^s_{\mathcal{S}}.$$

It is also easy to see that $\tau^s_{\Sigma\mathcal{S}} = \tau^s_{\mathcal{S}}$, so that $\mathcal{S}$ can be enlarged to $\downarrow \Sigma\mathcal{S} = \Sigma \downarrow \mathcal{S}$, the smallest bornology containing $\mathcal{S}$, without changing the topology. For example, strong pointwise convergence is uniform convergence with respect to the bornology of nonempty finite subsets $\mathcal{F}_X$ of X. Further, it is clear that the $\tau^s_{\mathcal{S}}$ topology for $\mathbf{B}(X,Y)$ is not altered by replacing $\mathcal{S}$ by the larger family $\{\alpha S : \alpha \in \mathbb{F},\ S \in \mathcal{S}\}$.

In the theory of locally convex spaces, topologies of uniform convergence are often presented in terms of families of absolutely convex sets, in view of the historical importance of the Mackey-Arens theorem [23, p. 62]. For example, the topology of uniform convergence on the weakly compact absolutely convex subsets of X for X^* is called the *Mackey topology*; when X is a Banach space, by Krein's Theorem [15, p. 434], the topology of strong uniform convergence on weakly compact absolutely convex subsets agrees with the topology of strong uniform convergence on weakly compact subsets.

The next result, which we view as a folk theorem, attempts to reconcile some of the approaches we have indicated.

Proposition 2.1. *Let $\mathcal{S}$ be a family of norm bounded subsets of X that is a cover of X. Then there a bornology $\mathcal{B}$ containing $\mathcal{S}$ and a subfamily $\mathcal{B}_0$ with $\mathcal{B}_0 \subseteq \mathcal{B} \subseteq \downarrow \mathcal{B}_0$ consisting of norm bounded absolutely convex sets that is closed under homothetic images such that $\tau^s_{\mathcal{S}} = \tau^s_{\mathcal{B}} = \tau^s_{\mathcal{B}_0}$.*

Proof. Let $\mathcal{D}$ be the smallest bornology containing $\mathcal{S}$, and consider this family of subsets of X:

$$\mathcal{B}_0 := \{\alpha \mathrm{aco}(D) : \alpha > 0 \text{ and } D \in \mathcal{D}\}.$$

Note $\mathcal{B}_0$ is a family of norm bounded absolutely convex sets closed under homothetic images. In view of the description of the absolutely convex hull of a set in terms of linear combinations of elements of the set, clearly $\tau^s_{\mathcal{S}} = \tau^s_{\mathcal{B}_0}$. Put $\mathcal{B} :=\downarrow \mathcal{B}_0$. Evidently $\mathcal{B}$ is an hereditary family such that $\tau^s_{\mathcal{S}} = \tau^s_{\mathcal{B}} = \tau^s_{\mathcal{B}_0}$, and so we are done if we can show that $\mathcal{B}$ is closed under finite unions. To this end, suppose $E_1 \subseteq \alpha_1 \text{aco}(D_1)$ and $E_2 \subseteq \alpha_2 \text{aco}(D_2)$ where $\{D_1, D_2\} \subseteq \mathcal{D}$. Then

$$E_1 \cup E_2 \subseteq (\alpha_1 + \alpha_2)(\text{aco}(D_1) \cup \text{aco}(D_2)) \subseteq (\alpha_1 + \alpha_2)\text{aco}(D_1 \cup D_2) \in \mathcal{B}_0$$

as required. □

We will also be interested in the *topology of weak uniform convergence* $\tau^w_{\mathcal{B}}$ with respect to a cover $\mathcal{B}$ of X by norm bounded subsets. A local base at the zero transformation for this locally convex topology consists of all sets of the form

$$W(S, f_1, f_2, \ldots f_m; \varepsilon) := \{T \in \mathbf{B}(X, Y) : \forall x \in S\ \forall j \leq m,\ |(f_j \circ T)x| < \varepsilon\},$$

where $S \in \Sigma\mathcal{B}$, $\varepsilon > 0$, and $\{f_1, f_2, f_3, \ldots f_m\} \subseteq Y^*$. Evidently, $T = \tau^w_{\mathcal{B}}\text{-}\lim T_n$ if and only if $\forall f \in Y^*$ we have $f \circ T = \tau^s_{\mathcal{B}}\text{-}\lim f \circ T_n$.

When $\mathcal{B} = \mathcal{F}_X$ this is the *topology of weak pointwise convergence*, a.k.a. the *weak operator topology* [15], as $T = \tau^w_{\mathcal{F}_X}\text{-}\lim T_n$ if and only if for each $x \in X$, $\langle T_n x\rangle$ converges weakly to Tx. All of our comments for topologies of strong uniform convergence remain valid for these weaker topologies and, in particular, the analog of Proposition 2.1 holds.

Our results that describe convergence in $\mathbf{B}(X, Y)$ with respect to topologies of uniform convergences in terms of the graphs of the transformations involve a generalization of Attouch-Wets convergence studied vigorously over the last five years in the context of metric spaces (see, e.g., [19, 12, 8, 9, 13]). To provide a framework that includes this body of research as well as the present investigation, we give a definition in a general context. Suppose $\langle X, \mathcal{U}\rangle$ is a Hausdorff uniform space where $\mathcal{U}$ is a diagonal uniformity [27, p. 238]. If $A \subseteq X$ and U is an entourage, we write $U(A)$ for $\{x \in X : \exists a \in A \text{ with } (a, x) \in U\}$.

Definition 2.2. Let $\langle X, \mathcal{U}\rangle$ be a Hausdorff uniform space and let $\mathcal{S}$ be a cover of X. A net $\langle A_\lambda\rangle_{\lambda\in\Lambda}$ of closed subsets of X is declared $\mathcal{S}$-*convergent* to a closed subset A if for each $U \in \mathcal{U}$ and $S \in \mathcal{S}$ there exists an index λ_0 in the underlying directed set Λ for the net such that whenever $\lambda \succeq \lambda_0$ we have both

$$A \cap S \subseteq U(A_\lambda) \quad \text{and} \quad A_\lambda \cap S \subseteq U(A).$$

In the sequel, when $\langle A_\lambda\rangle$ is $\mathcal{S}$-*convergent* to A, we will write $A = \mathcal{S}\text{-}\lim A_\lambda$. No finer convergence results if $\mathcal{S}$ is replaced by the bornology generated by $\mathcal{S}$, and thus convergence in this sense is called *bornological convergence*. Again paralleling what we have for topologies of uniform convergence, if $\mathcal{T}$ refines $\mathcal{S}$, then $A = \mathcal{S} - \lim A_\lambda \Rightarrow A = \mathcal{T} - \lim A_\lambda$. As a trivial example, if our family $\mathcal{S}$ consists of all nonempty subsets of X, then for convergence we need only check the definition at $S = X$. Convergence in this sense is just convergence with respect to the *hyperspace uniform topology* [27, p. 250] which generalizes the *Hausdorff metric topology* [18, 5, 20] in the metric setting. We mention that if $\mathcal{S}$ = the nonempty compact subsets of X, then bornological convergence of closed sets is convergence with respect to the *Fell topology* [5, p. 141], a.k.a, the *topology of closed convergence* [18] (see the discussion following Corollary 3.6 *infra*).

In applications the uniformity in question often is a metric uniformity. When the metric comes from a norm, equivalent norms produce the same bornological convergence, as the metrics they induce are uniformly equivalent. Necessary and sufficient conditions for two metric bornological convergences to coincide are presented in [8].

Note also that not all entourages need be checked: a base for the uniformity will do. In particular if X is a Hausdorff locally convex space with a local base $\{V_i : i \in I\}$ at θ, then with respect to the natural uniformity associated with $\{V_i : i \in I\}$, we see that $A = \mathcal{S} - \lim A_\lambda$ if and only if for each $S \in \mathcal{S}$ and $i \in I$, eventually both $A \cap S \subseteq A_\lambda + V_i$ and $A_\lambda \cap S \subseteq A + V_i$.

In our discussion of topologies of uniform convergence on $\mathbf{B}(X, Y)$, we remarked that we could restrict our attention to uniform convergence on certain (absolutely) convex sets. We do not have this luxury in the case of bornological convergence, even for sequences of compact convex sets.

Example 2.3. Consider the plane $\mathbb{R}^2$ equipped with the Euclidean norm (or the box norm) and with the bornology of finite subsets $\mathcal{F}_{\mathbb{R}^2}$. Let $\langle x_n \rangle$ be a sequence of distinct norm one elements, and let $A_n = \text{co}(\{x_n, \theta\})$. While $\{\theta\} = \mathcal{F}_{\mathbb{R}^2} - \lim A_n$, bornological convergence fails with respect to the bornology generated by all polytopes, i.e., convex hulls of members of $\mathcal{F}_{\mathbb{R}^2}$, which of course coincides with the bornology of norm bounded sets in this setting.

3. Results

We first introduce some notation. Let X and Y be normed linear spaces. If $\mathcal{S}$ is a family of nonempty subsets of X and $\mathcal{T}$ is a family of nonempty subsets of Y, we write $\mathcal{S} \otimes \mathcal{T}$ for the associated family of subsets of $X \times Y$ consisting of all products of the form $S \times T$ where $S \in \mathcal{S}$ and $T \in \mathcal{T}$.

Theorem 3.1. *Let $\langle X, ||\cdot||_X \rangle$ and $\langle Y, ||\cdot||_Y \rangle$ be normed linear spaces. Let $\mathcal{B}_0$ be a cover of nonempty norm bounded subsets of X each starshaped with respect to the origin that is closed under multiplication by positive scalars, and suppose $\mathcal{B}_0 \subseteq \mathcal{B} \subseteq\downarrow \mathcal{B}_0$. Suppose $T, T_1, T_2, T_3, \ldots$ is a sequence in $\mathbf{B}(X, Y)$. The following conditions are equivalent:*

(*1*) $\text{Gr}(T) = \mathcal{B} \otimes \{Y\}\text{-}\lim \text{Gr}(T_n)$ *with respect to the uniformity of the box norm for $X \times Y$;*

(*2*) $\text{Gr}(T) = \mathcal{B}_0 \otimes \{\alpha U_Y : \alpha > 0\}\text{-}\lim \text{Gr}(T_n)$ *with respect to the uniformity of the box norm for $X \times Y$;*

(*3*) $T = \tau^s_{\mathcal{B}_0}\text{-}\lim T_n$*;*

(*4*) $T = \tau^s_{\mathcal{B}}\text{-}\lim T_n$.

Proof. Since $\mathcal{B}_0 \otimes \{\alpha U_Y : \alpha > 0\}$ refines $\mathcal{B} \otimes \{Y\}$, (*1*) implies (*2*) and clearly (*3*) and (*4*) are equivalent. We prove (*2*) $\Rightarrow$ (*3*) and (*4*) $\Rightarrow$ (*1*).

Assume (*2*) holds and fix $B \in \mathcal{B}_0$. By scaling we may assume without loss of generality that $B \subseteq \frac{1}{2} U_X$ because by assumption B is norm bounded and uniform convergence on any homothetic image of B implies uniform convergence on B. We first claim that $\cup_{n=1}^\infty T_n(B)$ is a norm bounded subset of Y. If $\cup_{n=1}^\infty T_n(B)$ fails to be bounded, then since linear transformations preserve starshapedness with respect to the origin, there exists for infinitely many $n \in \mathbb{N}$ $x_n \in B$ with

$$||T_n x_n||_Y = 2\left(||T||_{\text{op}} + \frac{1}{2}\right).$$

Choose by graph convergence $k_0 \in \mathbb{N}$ such that $\forall n > k_0$ we have

$$\mathrm{Gr}(T_n) \cap \left(B \times 2\left(||T||_{\mathrm{op}} + \frac{1}{2}\right) U_Y\right) \subseteq \mathrm{Gr}(T) + \frac{1}{2}(U_{X\times Y}).$$

Now pick $n_1 > k_0$ and $x_{n_1} \in B$ with $||T_{n_1} x_{n_1}||_Y = 2(||T||_{\mathrm{op}} + \frac{1}{2})$, and then pick $x \in X$ with

$$(\diamondsuit) \qquad ||(x_{n_1}, T_{n_1} x_{n_1}) - (x, Tx)||_{\mathrm{box}} \leq \frac{1}{2}.$$

Since $x_{n_1} \in B \subseteq \frac{1}{2} U_X$, we conclude $x \in U_X$, and we obtain

$$||T_{n_1} x_{n_1} - Tx||_Y \geq |\ ||T_{n_1} x_{n_1}||_Y - ||Tx||_Y| \geq |2\left(||T||_{\mathrm{op}} + \frac{1}{2}\right) - ||T||_{\mathrm{op}}| > \frac{1}{2}.$$

and this contradicts $(\diamondsuit)$.

The claim established, next choose $\alpha > 0$ where $\cup_{n=1}^{\infty} T_n(B) \subset \alpha U_Y$ and let $\varepsilon > 0$. To show that $\langle T_n \rangle$ converges strongly uniformly to T on B, we consider two cases: (i) $\forall x \in X,\ Tx = 0$; and (ii) $||T||_{\mathrm{op}} > 0$.

In case (i) choose $k \in \mathbb{N}$ such that $\forall n > k$

$$\mathrm{Gr}(T_n) \cap (B \times \alpha U_Y) \subseteq \mathrm{Gr}(T) + \varepsilon U_{X\times Y} = X \times \varepsilon U_Y.$$

By the choice of α for each $x \in B$ and for all n we have $(x, T_n x) \in B \times \alpha U_Y$ and so for $n > k$ we obtain $||T_n x - Tx||_Y \leq \varepsilon$ as required.

In case (ii), put $\delta := \varepsilon(2||T||_{\mathrm{op}} + 2)^{-1}$ and choose $k \in \mathbb{N}$ so large that $\forall n > k$, we have

$$(\heartsuit) \qquad \mathrm{Gr}(T_n) \cap (B \times \alpha U_Y) \subseteq \mathrm{Gr}(T) + \delta U_{X\times Y}.$$

Let $x \in B$ and $n > k$ be arbitrary; by the choice of δ and $(\heartsuit)$, we can find $w \in X$ such that $||w - x||_X < \frac{\varepsilon}{2}||T||_{\mathrm{op}}^{-1}$ and $||Tw - T_n x||_Y < \frac{\varepsilon}{2}$. As a result,

$$||Tw - Tx||_Y \leq ||T||_{\mathrm{op}} \cdot \frac{\varepsilon}{2||T||_{\mathrm{op}}} = \frac{\varepsilon}{2},$$

and it follows that

$$||T_n x - Tx||_Y \leq ||T_n x - Tw||_Y + ||Tw - Tx||_Y < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

In either case we conclude that $\sup\{||Tx - T_n x||_Y : x \in B\} \leq \varepsilon$, and since $B \in \mathcal{B}_0$ was arbitrary, we have $T = \tau^s_{\mathcal{B}_0}\text{-}\lim T_n$. Thus *(3)* holds

The proof of *(4)* $\Rightarrow$ *(1)* is very easy. Suppose $T = \tau^s_{\mathcal{B}}\text{-}\lim T_n$ and $\varepsilon > 0$ and $B \in \mathcal{B}$ are arbitrary. Choose $k \in \mathbb{N}$ so large that $\forall n > k,\ \sup_{x\in B} ||T_n x - Tx||_Y < \varepsilon$. Then for all such n we actually have $\mathrm{Gr}(T) \cap (B \times Y) \subseteq \mathrm{Gr}(T_n) + (\{\theta\} \times \varepsilon U_Y)$ and $\mathrm{Gr}(T_n) \cap (B \times Y) \subseteq \mathrm{Gr}(T) + (\{\theta\} \times \varepsilon U_Y)$, and so both

$$\mathrm{Gr}(T) \cap (B \times Y) \subseteq \mathrm{Gr}(T_n) + \varepsilon U_{X\times Y},$$

and

$$\mathrm{Gr}(T_n) \cap (B \times Y) \subseteq \mathrm{Gr}(T) + \varepsilon U_{X\times Y}.$$

This completes the proof. $\square$

We note that while $\tau^s_{\mathcal{B}}$-convergence implies two-sided bornological graph convergence, the proof of Theorem 3.1 shows that $\tau^s_{\mathcal{B}}$-convergence follows just from *upper* bornological graph convergence. For example, condition (*1*) can be replaced by

$$\forall B \in \mathcal{B}\ \forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n > n_0\ \ \mathrm{Gr}(T_n) \cap (B \times Y) \subseteq \mathrm{Gr}(T) + \varepsilon U_{X\times Y}.$$

From this perspective, the equivalence of conditions (*1*) and (*4*) is a special case of Theorem 6.18 of [10].

In applications of Theorem 3.1, on the bornological side, we can replace $\mathcal{B} \otimes \{Y\}$ and $\mathcal{B}_0 \otimes \{\alpha U_Y : \alpha > 0\}$ by any intermediate family, and thus by families that generate the same bornology as does an intermediate family. Frequently, we will use $\mathcal{B} \otimes \{\alpha U_Y : \alpha > 0\}$. On the uniform convergence side, we can replace $\mathcal{B}$ by various associated families as indicated by but not limited to the constructs of Proposition 2.1. For example, in our first application Proposition 3.2, we replace $\mathcal{B}$ by a family $\mathcal{S}$ from which $\mathcal{B}$ can be recovered by taking convex hulls of finite unions of members of $\mathcal{S}$.

Proposition 3.2. *Let X and Y be a normed linear spaces, and let $\{T, T_1, T_2, \ldots\} \subseteq \mathbf{B}(X, Y)$. The following are equivalent:*

(*1*) $\forall x \in X,\ \lim_{n\to\infty} ||T_n x - Tx||_Y = 0$;

(*2*) *whenever P is a polytope in X and $\alpha > 0$ and $\varepsilon > 0$, then eventually both*

$$\mathrm{Gr}(T_n) \cap (P \times \alpha U_Y) \subseteq \mathrm{Gr}(T) + \varepsilon U_{X\times Y},$$
$$\mathrm{Gr}(T) \cap (P \times \alpha U_Y) \subseteq \mathrm{Gr}(T_n) + \varepsilon U_{X\times Y}.$$

Proof. Strong pointwise convergence implies strong uniform convergence on each finite set $\{a_1, a_2, ..., a_n\}$ and thus on each polytope $\mathrm{co}(\{a_1, a_2, ..., a_n\})$. Apply Theorem 3.1 where $\mathcal{B}$ = the set of all polytopes and $\mathcal{B}_0$ = the set of all polytopes containing the origin. □

We note that in the context of linear transformations between reflexive spaces, Zălinescu [28] has shown that strong pointwise convergence of $\langle T_n \rangle$ to T plus strong pointwise convergence of the induced sequence of adjoints (defined from Y^* to X^*) amounts to the Mosco convergence (see, e.g., [5, 20]) of $\langle \mathrm{Gr}(T_n) \rangle$ to $\mathrm{Gr}(T)$ provided X^* has the Kadec property.

Specializing to linear functionals, Proposition 3.2 becomes

Corollary 3.3. *Let X be a normed linear space, and let $\{f, f_1, f_2, \ldots\} \subseteq X^*$. The following are equivalent:*

(*1*) *$\langle f_n \rangle$ is weak-star convergent to f;*

(*2*) *whenever P is a polytope in $X \times \mathbb{F}$ and $\varepsilon > 0$, then eventually both*

$$\mathrm{Gr}(T_n) \cap P \subseteq \mathrm{Gr}(T) + \varepsilon U_{X\times\mathbb{F}} \quad \textit{and} \quad \mathrm{Gr}(T) \cap P \subseteq \mathrm{Gr}(T_n) + \varepsilon U_{X\times\mathbb{F}}.$$

Proof. Each polytope P in $X \times \mathbb{F}$ is contained in $\pi_X(P) \times \pi_{\mathbb{F}}(P)$ and of course $\pi_{\mathbb{F}}(P)$ is contained in $\alpha U_{\mathbb{F}}$ for some $\alpha > 0$. On the other hand if P' is a polytope in X then $P' \times \alpha U_{\mathbb{F}}$ is contained in some polytope in $X \times \mathbb{F}$, as each closed ball in the field of scalars is contained in a polytope. □

For sequences of functions between metric spaces, uniform convergence on compact subsets amounts to pointwise convergence plus equicontinuity. Specializing to linear transformations, strong uniform convergence of $\langle T_n \rangle$ to T on norm compact subsets amounts to strong pointwise convergence plus uniform boundedness of $\{T_n : n \in \mathbb{N}\}$. We also note for the record that uniform convergence on compacta is equivalent to *sequential strong continuous convergence*: whenever $x_0, x_1, x_2, \ldots$ is a sequence in X with $\lim ||x_n - x_0||_X = 0$, then $\lim ||T_n x_n - Tx||_Y = 0$ [14, p. 268]. In the last two results, one would like to replace the topology of strong pointwise convergence by the topology of strong uniform convergence on norm compact subsets and to compensate on the bornological convergence side, polytopes by norm compact convex sets, but in a general normed linear space neither the closed convex hull nor the closed absolutely convex hull of a norm compact set need be norm compact. It is at this point that starshapedness comes into play.

Lemma 3.4. *Suppose $A \subseteq X$ is norm (resp. weakly) compact and $\alpha > 0$. Then $\alpha star(A;\theta)$ is norm (resp. weakly) compact.*

Proof. The product $[0,\alpha] \times A$ is compact and $\alpha\mathrm{star}(A;\theta)$ is the image of $[0,\alpha] \times A$ under the continuous function $(\alpha, a) \mapsto \alpha a$. □

Applying Theorem 3.1 with $\mathcal{B}$ = the norm compact subsets of X and $\mathcal{B}_0$ = the norm compact subsets of X that are starshaped with respect to the origin, we get

Proposition 3.5. *Let X and Y be normed linear spaces and let $\{T, T_1, T_2, \ldots\} \subseteq \mathbf{B}(X,Y)$. Then $\langle T_n \rangle$ convergences to T uniformly on each norm compact subset of X if and only if for each norm compact subset K of X and $\alpha > 0$ we have eventually both*

$$\mathrm{Gr}(T_n) \cap (K \times \alpha U_Y) \subseteq \mathrm{Gr}(T) + \varepsilon U_{X\times Y},$$
$$\mathrm{Gr}(T) \cap (K \times \alpha U_Y) \subseteq \mathrm{Gr}(T_n) + \varepsilon U_{X\times Y}.$$

Specializing to linear functionals, Proposition 3.5 becomes

Corollary 3.6. *Let X be a normed linear space and let $\{f, f_1, f_2, \ldots\} \subseteq X^*$. Then $\langle f_n \rangle$ is convergent to f uniformly on norm compact subsets if and only if for each $\varepsilon > 0$ and for each norm compact subset K of $X \times \mathbb{F}$ we have eventually both*

$$\mathrm{Gr}(f_n) \cap K \subseteq \mathrm{Gr}(f) + \varepsilon U_{X\times \mathbb{F}},$$
$$\mathrm{Gr}(f) \cap K \subseteq \mathrm{Gr}(f_n) + \varepsilon U_{X\times \mathbb{F}}.$$

It is well-known (see [5, Thms. 5.1.6 and 5.2.10]) that if $A, A_1, A_2, A_3, \ldots$ is a sequence of closed sets in a metric space $\langle X, d \rangle$ and $\mathcal{K}$ is the family of compact subsets of X, then the following three conditions are equivalent:

(1) $A = \mathcal{K}\text{-}\lim A_n$;
(2) $\langle A_n \rangle$ is convergent to A in the *Fell Topology*, that is
  (2a) whenever V is open and $A \cap V \neq \emptyset$, then eventually $A_n \cap V \neq \emptyset$, and
  (2b) whenever K is compact and $A \cap K = \emptyset$, then eventually $A_n \cap K = \emptyset$;
(3) $\langle A_n \rangle$ is *Kuratowski convergent* to A, that is, A is at once the upper closed limit of $\langle A_n \rangle$ and the lower closed limit of $\langle A_n \rangle$.

By our remarks preceding Lemma 3.4 and equivalence of conditions (1) and (3) immediately above, Corollary 3.6 for linear functionals may be restated in this known form [3, Thm. 4.1] : $\langle \mathrm{Gr}(f_n)\rangle$ is Kuratowski convergent to $\mathrm{Gr}(f)$ if and only if $\langle f_n\rangle$ is weak-star convergent to f and $\{||f_n||_{\mathrm{op}} : n \in \mathbb{N}\}$ is a bounded set of reals.

We also note that the statements of Corollaries 3.3 and 3.6 obviously remain valid with minor adjustments for linear transformations with values in $\mathbb{F}^n$, not just for linear functionals.

By Mazur's Theorem [15, p. 416], if X is a Banach space then the closed convex hull of a norm compact set is norm compact and we can give formal results that parallel Proposition 3.2 and Corollary 3.3 where convexity only appears on one side. But by the Uniform Boundedness Principle [15, p. 66], when X is a Banach space, a sequence convergent strongly pointwise automatically convergences strongly uniformly on norm compact subsets, as the sequence is uniformly bounded in the operator norm and hence is equicontinuous. So nothing is really gained by stating such results when X is a Banach space.

We can present an exact analog to Proposition 3.5 for weakly compact sets, but we prefer to state a more aesthetically pleasing result by placing a restriction on the codomain.

Proposition 3.7. *Let X be a normed linear space and let Y be the dual of a Banach space, and let $\{T, T_1, T_2, \ldots\} \subseteq \mathbf{B}(X, Y)$. Then $\langle T_n\rangle$ convergences to T uniformly on each weakly compact subset of X if and only if for each weakly compact subset C of X and each weak-star compact subset K of Y we have eventually both*

$$\mathrm{Gr}(T_n) \cap (C \times K) \subseteq \mathrm{Gr}(T) + \varepsilon U_{X\times Y},$$
$$\mathrm{Gr}(T) \cap (C \times K) \subseteq \mathrm{Gr}(T_n) + \varepsilon U_{X\times Y}.$$

Proof. By the Uniform Boundedness Principle and Alaoglu's Theorem [15, p. 424], the bornology on Y having as a base the weak-star compact sets coincides with $\mathcal{B}(||\cdot||_Y)$. Apply Theorem 3.1. □

Specializing to linear functionals, Proposition 3.7 becomes

Corollary 3.8. *Let X be a normed linear space and let $\{f, f_1, f_2, \ldots\} \subseteq X^*$. Then $\langle f_n\rangle$ is convergent to f strongly uniformly on weakly compact subsets of X if and only if for each $\varepsilon > 0$ for each weakly compact subset K of $X \times \mathbb{F}$ we have eventually both*

$$\mathrm{Gr}(f_n) \cap K \subseteq \mathrm{Gr}(f) + \varepsilon U_{X\times\mathbb{F}},$$
$$\mathrm{Gr}(f) \cap K \subseteq \mathrm{Gr}(f_n) + \varepsilon U_{X\times\mathbb{F}}.$$

By Krein's Theorem, if X is a Banach space, we can convexify our last two results on the bornological convergence side. We leave this as an easy exercise for the reader.

As we must stop somewhere, we choose to finish with an application to Mackey convergence of linear functionals, i.e., uniform convergence on weakly compact absolutely convex subsets of X.

Proposition 3.9. *Let X be a normed linear space, and let $\{f, f_1, f_2, \ldots\} \subseteq X^*$. The following are equivalent:*

(1) $\langle f_n\rangle$ *is Mackey convergent to f;*

(*2*) *whenever* C *is a weakly compact convex subset in* $X \times \mathbb{F}$ *and* $\varepsilon > 0$, *then eventually both*

$$\mathrm{Gr}(f_n) \cap C \subseteq \mathrm{Gr}(f) + \varepsilon U_{X\times\mathbb{F}} \quad \text{and} \quad \mathrm{Gr}(f) \cap C \subseteq \mathrm{Gr}(f_n) + \varepsilon U_{X\times\mathbb{F}}.$$

Proof. Here $\mathcal{B} = \mathcal{B}_0 =$ the weakly compact absolutely convex subsets of X. Let $\mathcal{C}$ denote the convex weakly compact subsets of $X \times \mathbb{F}$. Each element of $\mathcal{B} \otimes \{\alpha U_{\mathbb{F}} : \alpha > 0\}$ belongs to $\mathcal{C}$, and if $C \in \mathcal{C}$, then C is contained in $\mathrm{aco}(\pi_X(C)) \times \alpha U_{\mathbb{F}}$ (note that the absolutely convex hull of a convex weakly compact set is weakly compact without completeness - see the proof of Lemma 3.4 *supra*). Thus, the bornology generated by $\mathcal{C}$ coincides with the bornology generated by $\mathcal{B} \otimes \{\alpha U_{\mathbb{F}} : \alpha > 0\}$. □

Our result for topologies of weak uniform convergence essentially falls out of Theorem 3.1. Given normed linear spaces X and Y let $\mathcal{U}$ be the diagonal uniformity on $X \times Y$ having as a base all sets of the form

$$]f_1, \ldots, f_n; \varepsilon[:= \{((x_1, y_1), (x_2, y_2)) : ||x_1 - x_2||_X \leq \varepsilon \text{ and } \forall i \leq n \ |f_i(y_1 - y_2)| \leq \varepsilon\}$$

where $\varepsilon > 0$ and $\{f_1, f_2, f_3, \ldots, f_n\}$ is a finite subset of Y^*. This of course is the uniformity associated with the usual local basis of absolutely convex neighborhoods of θ for the product of X with the norm topology and Y with the weak topology [23, p. 87]. Observe that for $A \subseteq X \times Y$, $A + (\varepsilon U_X \times \{y : \forall i \leq n \ |f_i(y)| \leq \varepsilon\}) =]f_1, f_2, \ldots, f_n; \varepsilon[(A)$.

Theorem 3.10. *Let* $\langle X, ||\cdot||_X\rangle$ *and* $\langle Y, ||\cdot||_Y\rangle$ *be normed linear spaces. Let* $\mathcal{B}_0$ *be a cover of nonempty norm bounded subsets of* X *each starshaped with respect to the origin that is closed under multiplication by positive scalars, and suppose* $\mathcal{B}_0 \subseteq \mathcal{B} \subseteq\downarrow \mathcal{B}_0$. *Suppose* $T, T_1, T_2, T_3, \ldots$ *is a sequence in* $\mathbf{B}(X, Y)$. *The following conditions are equivalent:*

(*1*) $\mathrm{Gr}(T) = \mathcal{B} \otimes \{Y\}$-$\lim \mathrm{Gr}(T_n)$ *with respect to the uniformity* $\mathcal{U}$ *for* $X \times Y$;

(*2*) $\forall B \in \mathcal{B}_0$, $\forall f \in Y^*$, $\forall \alpha > 0$, $\forall \varepsilon > 0$ *and for each weak neighborhood* V *of the origin in* Y, *we have eventually both*

$$\mathrm{Gr}(T) \cap (B \times \{y : |f(y)| \leq \alpha\}) \subseteq \mathrm{Gr}(T_n) + (\varepsilon U_X \times V), \quad \text{and}$$
$$\mathrm{Gr}(T_n) \cap (B \times \{y : |f(y)| \leq \alpha\}) \subseteq \mathrm{Gr}(T) + (\varepsilon U_X \times V);$$

(*3*) $T = \tau^w_{\mathcal{B}_0}$-$\lim T_n$;

(*4*) $T = \tau^w_{\mathcal{B}}$-$\lim T_n$.

Proof. (*1*) $\Rightarrow$ (*2*). The cover $\{B \times \{y : |f(y)| \leq \alpha\} \ : B \in \mathcal{B}_0, \ f \in Y^* \text{ and } \alpha > 0\}$ of $X \times Y$ refines $\mathcal{B} \otimes \{Y\}$.

(*2*) $\Rightarrow$ (*3*). Assuming (*2*) it suffices to show for each $f \in Y^*$ we have $f \circ T = \tau^s_{\mathcal{B}_0} - \lim f \circ T_n$. Fix $B \in \mathcal{B}_0$, $\varepsilon > 0$ and $\alpha > 0$; with $V = \{y : |f(y)| \leq \varepsilon\}$, choose by (*2*) $k \in \mathbb{N}$ such that whenever $n > k$, both

$$\mathrm{Gr}(T) \cap (B \times \{y : |f(y)| \leq \alpha\}) \subseteq \mathrm{Gr}(T_n) + (\varepsilon U_X \times \{y : |f(y)| \leq \varepsilon\}), \quad \text{and}$$
$$\mathrm{Gr}(T_n) \cap (B \times \{y : |f(y)| \leq \alpha\}) \subseteq \mathrm{Gr}(T) + (\varepsilon U_X \times \{y : |f(y)| \leq \varepsilon\}).$$

Now let $(x, (f \circ T)x) \in B \times \alpha U_{\mathbb{F}}$ be arbitrary. Then $(x, Tx) \in B \times \{y : |f(y)| \leq \alpha\}$ and so $\forall n > k \ \exists (x_n, T_n x_n)$ with $||x - x_n||_X \leq \varepsilon$ and $Tx - T_n x_n \in \{y : |f(y)| \leq \varepsilon\}$, i.e.,

$|(f \circ T_n)x_n - (f \circ T)x| \leq \varepsilon$. This implies $(x, (f \circ T)x)$ belongs to $\mathrm{Gr}(f \circ T_n) + \varepsilon U_{X \times \mathbb{F}}$ and it follows that for all $n > k$

$$\mathrm{Gr}(f \circ T) \cap (B \times \alpha U_{\mathbb{F}}) \subseteq \mathrm{Gr}(f \circ T_n) + \varepsilon U_{X \times \mathbb{F}}.$$

The same argument gives

$$\mathrm{Gr}(f \circ T_n) \cap (B \times \alpha U_{\mathbb{F}}) \subseteq \mathrm{Gr}(f \circ T) + \varepsilon U_{X \times \mathbb{F}},$$

and applying Theorem 3.1 condition *(3)* follows.

(3) $\Rightarrow$ *(4)*. This is obvious.

(4) $\Rightarrow$ *(1)*. Fix $B \in \mathcal{B}$ and let $\{f_1, f_2, \ldots, f_m\}$ in Y^* and $\varepsilon > 0$ be arbitrary. By weak uniform convergence on B, there exists $k \in \mathbb{N}$ such that for all $n > k$ and all $x \in B$, $(T - T_n)x \in \{y : \forall i \leq m \ |f_i(y)| \leq \varepsilon\}$. This immediately implies

$$\mathrm{Gr}(T) \cap (B \times Y) \subseteq \mathrm{Gr}(T_n) + (\varepsilon U_X \times \{y : \forall i \leq m \ |f_i(y)| \leq \varepsilon\}), \quad \text{and}$$
$$\mathrm{Gr}(T_n) \cap (B \times Y) \subseteq \mathrm{Gr}(T) + (\varepsilon U_X \times \{y : \forall i \leq m \ |f_i(y)| \leq \varepsilon\}).$$

□

While the bornology of norm bounded sets seems natural to attach to the norm topology of a normed linear space X to form a so-called *bornological universe* [17, 6], for the weak topology a more appropriate choice may be provided by the smallest bornology containing $\{x : |f(x)| \leq \alpha\}$ where f runs over X^* and α runs over the positive scalars. This *weak bornology* is natural with respect to viewing the weak topology as a topology of pointwise convergence, identifying each $x \in X$ with $\widehat{x} \in X^{**} \subseteq \mathbb{F}^{X^*}$ where of course

$$\widehat{x}(f) = f(x) \quad (f \in X^*).$$

Just as the product topology on $\mathbb{F}^{X^*}$ = the topology of pointwise convergence on the scalar-valued functions defined on X^* is the coarsest topology on the product containing $\pi_f^{-1}(V)$ for each open set V in $\mathbb{F}$ and each $f \in X^*$, the natural product bornology [6] is the coarsest bornology containing $\pi_f^{-1}(B)$ for each bounded set B of scalars and each $f \in X^*$. The trace of this bornology on $\{\widehat{x} : x \in X\}$ is precisely what we have suggested. Theorem 3.10 of course provides additional support for this choice. That said, it is a little disturbing that while the norm and weak topologies agree for a finite dimensional space, the norm and weak bornologies diverge, e.g., in $\mathbb{R}^2$ the union of the coordinate axes belongs to the weak bornology.

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