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Strongly-Representable Monotone Operators

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Recently in [4] a new class of maximal monotone operators has been introduced. In this note we study domain-range properties as well as connections with other classes and calculus rules for these operators we called strongly-representable. While not every maximal monotone operator is strongly-representable, every maximal monotone NI operator is strongly-representable, and every strongly-representable operator is locally maximal monotone, maximal monotone locally, strongly maximal monotone, and ANA. As a consequence the conjugate of the Fitzpatrick function of a maximal monotone operator is not necessarily a representative function.

1. Introduction

Let X be a non trivial (real) Banach space and X^* its topological dual; set $Z := X \times X^*$ which is a Banach space with respect to the norm $\|(x,x^*)\| := \big(\|x\|^2 + \|x^*\|^2\big)^{1/2}$. We denote by "$s$" the strong topology, by "ω" the weak topology on X, by "ω^*" the weak-star topology on X^*, and by $Z^* := X^* \times X^{**}$ the dual of Z.

For $z := (x, x^*) \in Z$ we set $c(z) := \langle x, x^*\rangle := x^*(x)$. For the sake of simplicity, we use the same notation c for the coupling in Z^*, that is, $c(z^*) := \langle x^*, x^{**}\rangle := x^{**}(x^*)$ for $z^* := (x^*, x^{**}) \in Z^*$, since the contexts of Z or Z^* offer no possibility of confusion for c.

Consider

$$\mathcal{F} := \mathcal{F}(Z) := \{f \in \Lambda(Z) \mid f(z) \geq c(z),\ \forall z \in Z\}, \qquad \mathcal{F}_s := \mathcal{F}_s(Z) := \mathcal{F}(Z) \cap \Gamma_s(Z),$$

where for a locally convex space (E,τ), $\Lambda(E)$ denotes the class of proper convex functions $f : E \to \overline{\mathbb{R}}$ and $\Gamma_\tau(E)$ is the class of those $f \in \Lambda(E)$ which are τ–lower semicontinuous (lsc for short). The elements of $\mathcal{F}(Z)$ are called representative functions in Z.

It is known that whenever $f \in \mathcal{F}(Z)$ the set

$$M_f := [f \leq c] := \{z \in Z \mid f(z) \leq c(z)\} = \{z \in Z \mid f(z) = c(z)\} =: [f = c]$$

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is monotone, that is, $c(z - z') \geq 0$ for all $z, z' \in M_f$; this also follows from Proposition 2.1 below.

For $z_1 := (x_1, x_1^*)$, $z_2 := (x_2, x_2^*) \in Z$ we set

$$\langle z_1, z_2 \rangle := z_1 \cdot z_2 := \langle x_1, x_2^* \rangle + \langle x_2, x_1^* \rangle .$$

Note the following useful relations:

$$c(z \pm z') = c(z) \pm \langle z, z' \rangle + c(z'), \quad c(z) = c(-z) = \tfrac{1}{2} \langle z, z \rangle , \quad \forall z, z' \in Z.$$

For $z = (x, x^*) \in Z$, $\alpha > 0$, and $g : Z \to \overline{\mathbb{R}}$ we denote by g_z and g_α the functions on Z defined by

$$g_z(w) := g(z + w) - c(z + w) + c(w), \quad g_\alpha(w) := \alpha g\left(y, \alpha^{-1} y^*\right), \quad \text{for } w := (y, y^*) \in Z.$$

Hence $g_z(w) = g(z + w) - z \cdot w - c(z)$ for $w \in Z$, and so g_z is convex as the sum of a convex function and an affine function; moreover

$$g_z(w) - c(w) = g(z + w) - c(z + w), \quad \forall z, w \in Z, \tag{1}$$

$$g_\alpha(w) - c(w) = \alpha \left[g(w_\alpha) - c(w_\alpha) \right], \quad \forall \alpha > 0, \ \forall w \in Z, \tag{2}$$

where $w_\alpha := (y, \alpha^{-1} y^*)$ for $w = (y, y^*)$. It follows that

$$f \in \mathcal{F}(Z) \Rightarrow \left[f_\alpha, \ f_z \in \mathcal{F}(Z), \ \forall \alpha > 0, \ \forall z \in Z \right],$$
$$f \in \mathcal{F}_s(Z) \Rightarrow \left[f_\alpha, \ f_z \in \mathcal{F}_s(Z), \ \forall \alpha > 0, \ \forall z \in Z \right],$$

and

$$M_{f_z} = M_f - z, \quad M_{f_\alpha} = \{ (x, \alpha x^*) \mid (x, x^*) \in M_f \} \tag{3}$$

for every $f \in \mathcal{F}(Z)$, $z \in Z$, and $\alpha > 0$.

In the sequel for a proper function $g : Z \to \overline{\mathbb{R}}$, we denote by g^* its usual (convex) conjugate, and by ∂g its usual subdifferential, that is, $g^* : Z^* = X^* \times X^{**} \to \overline{\mathbb{R}}$ and $\partial g : Z \rightrightarrows Z^*$, while the pairing between Z and Z^* is given by

$$\langle (x, x^*), (u^*, u^{**}) \rangle := \langle x, u^* \rangle + \langle x^*, u^{**} \rangle , \quad \forall (x, x^*) \in X \times X^*, \ (u^*, u^{**}) \in X^* \times X^{**}.$$

Let $\widehat{x}$ be the image $J(x)$ of $x \in X$, where $J : X \to X^{**}$ is the canonical injection of X into X^{**}, that is, $J(x)(x^*) := \langle x, x^* \rangle$ for $x^* \in X^*$ and $x \in X$. In the sequel we shall use $\widehat{z}$ for $(x^*, \widehat{x}) \in Z^*$ when $z := (x, x^*) \in Z$. Moreover, for $g : Z \to \overline{\mathbb{R}}$ we consider $g^\square : Z \to \overline{\mathbb{R}}$ defined by $g^\square(z) := g^*(\widehat{z})$; hence $g^\square$ is convex and $s \times \omega^*$–lsc.

For $M \subset X \times X^*$, its *Fitzpatrick function* φ_M is defined as

$$\varphi_M(z) = \sup\{ \langle z, w \rangle - c_M(w) \mid w \in Z \} = \sup\{ \langle z, w \rangle - c(w) \mid w \in M \},$$

where $c_M(z) := c(z)$, for $z \in M$ and $c_M(z) := \infty$, for $z \in Z \setminus M$; in simpler words $\varphi_M(x, x^*) = c_M^*(x^*, \widehat{x}) = c_M^\square(x, x^*)$ or $\varphi_M(z) = c_M^*(\widehat{z}) = c_M^\square(z)$, for $z = (x, x^*) \in Z$.

The *Penot function* of a non-empty monotone set $M \subset X \times X^*$ is defined by $\psi_M(z) = \varphi_M^*(\widehat{z}) = \varphi_M^\square(z)$ for $z \in Z$, and is the greatest $\omega \times \omega^*$–lsc proper convex function majorized by c_M in Z.

Let $g : X \times X^* \to \overline{\mathbb{R}}$ be a proper function and $z := (x, x^*) \in Z$. Then

$$\begin{aligned}&(g_{(x,x^*)})^*(u^*, u^{**}) \\ &= g^*\left(u^* + x^*, u^{**} + \widehat{x}\right) - \langle x, u^*\rangle - \langle x^*, u^{**}\rangle - \langle x, x^*\rangle\,, \quad \forall (u^*, u^{**}) \in X^* \times X^{**},\end{aligned}$$

that is,

$$(g_z)^*(w^*) = g^*\left(w^* + \widehat{z}\right) - c(w^* + \widehat{z}) + c(w^*), \quad \forall w^* \in Z^*,$$

or equivalently

$$(g_z)^* = (g^*)_{\widehat{z}},$$

and

$$\partial g_z(w) = \{w^* \in Z^* \mid w^* + \widehat{z} \in \partial g(w + z)\} = \partial g(w + z) - \widehat{z} \;\; \forall w, z \in Z.$$

In particular, $\operatorname{Im}\partial g_z = \operatorname{Im}\partial g - \widehat{z}$.

For $\alpha > 0$, $x \in X$, $x^*, u^* \in X^*$, $u^{**} \in X^{**}$, we have

$$\begin{aligned}(g_\alpha)^*(u^*, u^{**}) &= \alpha g^*(\alpha^{-1}u^*, u^{**}), \\ (u^*, u^{**}) \in \partial g_\alpha(x, x^*) &\Leftrightarrow (\alpha^{-1}u^*, u^{**}) \in \partial g(x, \alpha^{-1}x^*).\end{aligned}$$

Let us consider the more restrictive classes

$$\mathcal{G} := \mathcal{G}(Z) := \{f \in \mathcal{F}(Z) \mid f^*(z^*) \ge c(z^*), \ \forall z^* \in Z^*\}\,, \qquad \mathcal{G}_s := \mathcal{G}_s(Z) := \mathcal{G}(Z) \cap \Gamma_s(Z).$$

The classes $\mathcal{F}_\tau(Z)$, $\mathcal{G}_\tau(Z)$ are defined similarly, for any other topology τ on Z.

Using the formulas above for $(g_z)^*$ and $(g_\alpha)^*$ we get

$$f \in \mathcal{G}_s(Z) \Rightarrow [f_\alpha, \ f_z \in \mathcal{G}_s(Z), \ \forall \alpha > 0, \ \forall z \in Z]\,. \tag{4}$$

A set $M \subset Z$ is called *strongly-representable* in Z whenever there is $f \in \mathcal{G}_s(Z)$ such that $M = M_f$. In this case f is called a *strong-representative* of M.

It has been proven in [4, Th. 4.2] that every strongly-representable operator is maximal monotone. In this paper we show that not every maximal monotone operator is strongly-representable by providing the property of convexity for the closure of the range; property that distinguishes between these two classes.

Consider

$$h : X \times X^* \to \mathbb{R}, \quad h(x, x^*) = \tfrac{1}{2}\left\|(x, x^*)\right\|^2 = \tfrac{1}{2}\left\|x\right\|^2 + \tfrac{1}{2}\left\|x^*\right\|^2, \quad (x, x^*) \in X \times X^*.$$

Since the dual norm on $X^* \times X^{**}$ is given by $\|(u^*, u^{**})\| = \left(\|u^*\|^2 + \|u^{**}\|^2\right)^{1/2}$ we know that $h^*(u^*, u^{**}) = \frac{1}{2}\left\|(u^*, u^{**})\right\|^2$. Notice that

$$h \ge \pm c, \qquad h^* \ge \pm c. \tag{5}$$

Moreover,

$$\partial h(x, x^*) = F_X(x) \times F_{X^*}(x^*) \;\; \forall (x, x^*) \in X \times X^*,$$

where $F_X : X \rightrightarrows X^*$ is the duality mapping of X, that is,

$$F_X(x) := \partial\left(\tfrac{1}{2}\left\|\cdot\right\|^2\right)(x) = \left\{x^* \in X^* \mid \|x\|^2 = \|x^*\|^2 = \langle x, x^*\rangle\right\}, \quad x \in X,$$

and similarly for F_{X^*}. Note that

$$|\langle z, z'\rangle| \leq \|z\| \cdot \|z'\|, \quad |c(z) - c(z')| \leq \tfrac{1}{2}\|z - z'\|^2 + \|z'\| \cdot \|z - z'\|, \quad \forall z, z' \in Z. \tag{6}$$

Taking $z' = z$ in the first inequality or $z' = 0$ in the second we get $|c(z)| \leq \frac{1}{2}\|z\|^2$ for $z \in Z$.

When there is no risk of confusion a multifunction $S : E \rightrightarrows F$ is identified with its graph $\operatorname{gph} S := \{(x, y) \mid y \in S(x)\}$; moreover, $\operatorname{dom} S := \operatorname{Pr}_E(\operatorname{gph} S)$, $\operatorname{Im} S := \operatorname{Pr}_F(\operatorname{gph} S)$, and $S^{-1} : F \rightrightarrows E$ has $\operatorname{gph} S^{-1} := \{(y, x) \mid (x, y) \in \operatorname{gph} S\}$.

When E, F are (real) linear spaces, $A, B \subset E$, and $\alpha \in \mathbb{R}$, we set $A + B := \{a + b \mid a \in A,\ b \in B\}$ and $\alpha A := \{\alpha a \mid a \in A\}$ with $A + \emptyset := \emptyset$ and $\alpha\emptyset := \emptyset$ by convention. For $S, T : E \rightrightarrows F$ and $\alpha \in \mathbb{R}$, the multifunctions $S + T : E \rightrightarrows F$ and $\alpha S : E \rightrightarrows F$ have the graphs $\operatorname{gph}(S+T) := \{(x, y + v) \mid (x, y) \in \operatorname{gph} S,\ (x, v) \in \operatorname{gph} T\}$, that is, $(S + T)(x) = S(x) + T(x)$, and $\operatorname{gph}(\alpha S) := \{(x, \alpha y) \mid (x, y) \in \operatorname{gph} S\}$, that is, $(\alpha S)(x) = \alpha S(x)$. Hence $\operatorname{dom}(S + T) = \operatorname{dom} S \cap \operatorname{dom} T$, $\operatorname{Im}(S + T) \subset \operatorname{Im} S + \operatorname{Im} T$, $\operatorname{dom}(\alpha S) = \operatorname{dom} S$, and $\operatorname{Im}(\alpha S) = \alpha \operatorname{Im} S$.

Generally $\operatorname{gph}(S + T)$ is different from $\operatorname{gph} S + \operatorname{gph} T$ and $\operatorname{gph}(\alpha S)$ is different from $\alpha \operatorname{gph} S$.

As usual, for a subset A of a normed vector space X and $x \in X$, we set $d(x, A) := \inf\{\|x - u\| \mid u \in A\}$ with the convention that $\inf \emptyset := +\infty$.

2. Domain-range properties

Proposition 2.1. *Suppose that $f \in \mathcal{F}(Z)$, $z_1, z_2 \in Z$, and $\varepsilon_1, \varepsilon_2 \geq 0$ are such that $f(z_1) \leq c(z_1) + \varepsilon_1$ and $f(z_2) \leq c(z_2) + \varepsilon_2$. Then*

$$c(z_1 - z_2) \geq -2(\varepsilon_1 + \varepsilon_2).$$

Proof. Indeed,

$$c\left(\tfrac{1}{2}z_1 + \tfrac{1}{2}z_2\right) \leq f\left(\tfrac{1}{2}z_1 + \tfrac{1}{2}z_2\right) \leq \tfrac{1}{2}f(z_1) + \tfrac{1}{2}f(z_2) \leq \tfrac{1}{2}\left(c(z_1) + \varepsilon_1\right) + \tfrac{1}{2}\left(c(z_2) + \varepsilon_2\right),$$

whence $-\frac{1}{2}(\varepsilon_1 + \varepsilon_2) \leq \frac{1}{4}c(z_1 - z_2)$. The conclusion follows. □

Proposition 2.2. *Let $f \in \mathcal{G}(Z)$. Then:*

(*i*) *For every $z \in Z$ one has*

$$\inf_{w \in Z}\left(f_z(w) + h(w)\right) = -\min_{w^* \in Z^*}\left[\left(f^*(\widehat{z} + w^*) - c(\widehat{z} + w^*)\right) + \left(h^*(w^*) + c(w^*)\right)\right] = 0.$$

(*ii*) *For every $z \in Z$ there is $z^* \in M_{f^*}$ such that $\widehat{z} - z^* \in \operatorname{gph}(-F_{X^*})$ and $\|\widehat{z} - z^*\|^2 \leq 2(f^*(\widehat{z}) - c(\widehat{z}))$. Moreover*

$$\left(\sqrt{2} - 1\right)\|\widehat{z} - z^*\| \leq d(\widehat{z}, M_{f^*}) \leq \sqrt{2\left(f^*(\widehat{z}) - c(\widehat{z})\right)} = \sqrt{2\left(f^{\square}(z) - c(z)\right)}. \tag{7}$$

(*iii*) *For every $\alpha > 0$, $\operatorname{Im}\left((M_{f^*})^{-1} + \alpha(F_{X^*})^{-1}\right) = X^*$.*

Proof. (i) Taking into account the formulas related to f_z we may (and do) assume that $z=0$. Because $f \geq c$ and $f^* \geq c$, we obtain from (5) that $f+h \geq 0$ and $f^*+h^* \geq 0$. Since f is convex and h is finite, convex, and continuous on Z, using the Fenchel duality theorem (see e.g. [16, Cor. 2.8.5]) we obtain

$$\begin{aligned} 0 \leq \inf_{z\in Z}[f(z)+h(z)] &= -\min_{z^*\in Z^*}[f^*(z^*)+h^*(-z^*)] = -\min_{z^*\in Z^*}[f^*(z^*)+h^*(z^*)] \\ &= -\inf_{z^*\in Z^*}[f^*(z^*)+h^*(z^*)] \leq 0. \end{aligned}$$

The conclusion of (i) follows because $f_z \in \mathcal{G}(Z)$ whenever $f \in \mathcal{G}(Z)$.

(ii) Fix $z \in Z$. We get from (i) an element $z^* \in Z^*$ such that

$$[f^*(z^*) - c(z^*)] + [h^*(z^* - \widehat{z}) + c(z^* - \widehat{z})] = 0.$$

Because the terms in square brackets are non-negative, we see that $f^*(z^*) - c(z^*) = 0$, that is, $z^* \in M_{f^*}$ and $h^*(z^*-\widehat{z}) + c(z^*-\widehat{z}) = 0$, whence $\widehat{z} - z^* \in \operatorname{gph}(-F_{X^*})$. Since $f^*(z^*) = c(z^*)$ we have $f^*(\widehat{z}) \geq \varphi_{M_{f^*}}(\widehat{z}) \geq \langle \widehat{z}, z^*\rangle - c(z^*)$ (for more details see [13, Remark 3.6]). Therefore

$$f^*(\widehat{z}) - c(\widehat{z}) \geq \langle \widehat{z}, z^*\rangle - c(z^*) - c(\widehat{z}) = -c(\widehat{z} - z^*) = h^*(z^* - \widehat{z}) = \tfrac{1}{2}\|\widehat{z} - z^*\|^2.$$

This yields the second inequality in relation (7) because $\delta := d(\widehat{z}, M_{f^*}) \leq \|\widehat{z} - z^*\|$.

Since M_{f^*} is monotone, for every $w^* \in M_{f^*}$ we have that

$$\begin{aligned} 0 \leq c\left(z^* - w^*\right) &= c(z^* - \widehat{z}) + \langle z^* - \widehat{z}, \widehat{z} - w^*\rangle + c(\widehat{z} - w^*) \\ &\leq -\tfrac{1}{2}\|\widehat{z} - z^*\|^2 + \|z^* - \widehat{z}\| \cdot \|\widehat{z} - w^*\| + \tfrac{1}{2}\left\|\widehat{z} - w^*\right\|^2. \end{aligned}$$

It follows that $0 \leq -\|\widehat{z} - z^*\|^2 + 2\delta\left\|z^* - \widehat{z}\right\| + \delta^2$, whence $\|\widehat{z} - z^*\| \leq (1+\sqrt{2})\delta$, i.e., the first inequality in (7) holds.

(iii) If necessary, replacing f by f_α, we may assume that $\alpha = 1$. Let $u^* \in X^*$. Applying (ii) for $z = (0, u^*)$ we get $z^* = (x^*, x^{**}) \in M_{f^*}$ such that $u^* - x^* \in (F_{X^*})^{-1}(x^{**})$. The conclusion follows. □

Remark 2.3. From assertion (i) of the preceding proposition we have that $f \in \mathcal{G}_s(Z)$ implies $f \in \mathcal{F}_s(Z)$ and $\inf(f_z + h) = 0$ for every $z \in Z$.

Remark 2.4. The first part of assertion (ii) of the previous proposition can be interpreted as

$$\widehat{Z} := X^* \times J(X) \subset \operatorname{gph} M_{f^*} + \operatorname{gph}(-F_{X^*}), \tag{8}$$

and is a generalization to non-reflexive spaces for the "$-J$" criterion for the maximality of operators in reflexive spaces (see [8]); moreover, (8) can be obtained from [9, Lem. 35.5] by taking $g := h$. In reflexive spaces, an operator is maximal monotone iff it is strongly-representable, a situation that is no longer valid in the non-reflexive context in the sense that there exist maximal monotone operators that are not strongly-representable as we will see in the sequel. The second part of assertion (ii) extends [6, Lem. 2.3] to the non-reflexive case.

A partial converse of Proposition 2.2 follows.

Proposition 2.5. *If $f : Z \to \overline{\mathbb{R}}$ is such that $\inf_{w\in Z}(f_z(w) + h(w)) = 0$ for every $z \in Z$ then $f \geq c$; moreover, if f is convex then $f \in \mathcal{F}(Z)$ and $f^*(z^*) \geq c(z^*)$ for every $z^* \in \widehat{Z} + \operatorname{gph}(-F_{X^*})$.*

Proof. The condition $\inf(f_z + h) = 0$, for every $z \in Z$ implies

$$f_z(w) + h(w) = f(z+w) - c(z+w) + h(w) + c(w) \geq 0, \quad \forall z, w \in Z.$$

Taking $w = 0$ we get $f \geq c$ in Z.

Assume now that f is convex. Then $f \in \Lambda(Z)$ and so $f \in \mathcal{F}(Z)$. Again, the fundamental duality formula yields

$$\inf_{w\in Z}(f_z(w) + h(w)) = -\min_{z^*\in Z^*}\left[(f^*(\widehat{z} + z^*) - c(\widehat{z} + z^*)) + (h^*(z^*) + c(z^*))\right] = 0,$$

which implies $f^*(z^*) \geq c(z^*)$, for every $z^* \in \widehat{Z} + \operatorname{gph}(-F_{X^*})$, since $[h^* + c = 0] = \operatorname{gph}(-F_{X^*})$. □

Theorem 2.6. *Let $f \in \Gamma_s(Z)$ be such that $\inf_{w\in Z}(f_z(w) + h(w)) = 0$, for every $z \in Z$. Then M_f is nonempty, monotone and*

$$d((x, x^*), M_f) \leq 2\sqrt{f(x, x^*) - \langle x, x^*\rangle}, \quad \forall (x, x^*) \in X \times X^*. \tag{9}$$

Proof. From Proposition 2.5 we know that $f \in \mathcal{F}(Z)$ and so M_f is monotone. Fix $z := (x, x^*) \in X \times X^*$. If $f(z) = \infty$ or $f(z) = c(z)$ there is nothing to prove. Let $\varepsilon := f(z) - c(z) \in (0, \infty)$ and set $\varepsilon_0 := \varepsilon$, $z_0 := z$. Fix $\beta \in (1, \infty)$, $\gamma \in (2, \infty)$ and consider a sequence $(\varepsilon_n)_{n\geq 0} \subset (0, \infty)$ satisfying

$$4\varepsilon_n + 6\varepsilon_{n+1} \leq \gamma^2\varepsilon_n, \quad \forall n \geq 0 \quad \text{and} \quad \sum_{n\geq 0}\sqrt{\varepsilon_n} < \beta\sqrt{\varepsilon} \tag{10}$$

(for example $\varepsilon_n = \varepsilon_0 r^{2n}$, $n \geq 0$, where $r = \min\{((\gamma^2 - 4)/6)^{1/2}, (1 - \beta^{-1})/2\}$).

Because $\inf(f_{z_0} + h) = 0$, there exists $z_1 \in Z$ such that

$$f_{z_0}(z_1 - z_0) + h(z_1 - z_0) \leq \varepsilon_1.$$

Using the definition of f_{z_0} given in (1) and since $f, f_{z_0} \geq c$ and $h \geq -c$ we get

$$\begin{aligned} 0 \leq f(z_1) - c(z_1) = f_{z_0}(z_1 - z_0) - c(z_1 - z_0) \leq \varepsilon_1, \\ 0 \leq \tfrac{1}{2}\|z_1 - z_0\|^2 + c(z_1 - z_0) \leq \varepsilon_1. \end{aligned} \tag{11}$$

Using Proposition 2.1 we obtain that $c(z_1 - z_0) \geq -2(\varepsilon_0 + \varepsilon_1)$, and so, by (11),

$$\|z_1 - z_0\|^2 \leq 2\varepsilon_1 + 4(\varepsilon_0 + \varepsilon_1) = 4\varepsilon_0 + 6\varepsilon_1 \leq \gamma^2\varepsilon_0,$$

whence

$$\|z_1 - z_0\| \leq \gamma\sqrt{\varepsilon_0}.$$

Continuing this procedure we obtain a sequence $(z_n)_{n\geq 0} \subset Z$ such that

$$f(z_n) \leq c(z_n) + \varepsilon_n, \quad \|z_{n+1} - z_n\| \leq \gamma\sqrt{\varepsilon_n}, \quad \forall n \geq 0.$$

We obtain from (10) that

$$\sum_{n\geq 0}\|z_{n+1}-z_n\| \leq \gamma\sum_{n\geq 0}\sqrt{\varepsilon_n} < \gamma\beta\sqrt{\varepsilon}.$$

Since Z is complete, it follows that the sequence $(z_n)_{n\geq 0}$ is strongly convergent to some $z_\varepsilon \in Z$ and $\|z-z_\varepsilon\| \leq \gamma\beta\sqrt{\varepsilon}$. Since f is s–lsc and $\varepsilon_n \to 0$, from the inequality $f(z_n) \leq c(z_n)+\varepsilon_n$ we get

$$c(z_\varepsilon) \leq f(z_\varepsilon) \leq \liminf_{n\to\infty} f(z_n) \leq \lim_{n\to\infty}(c(z_n)+\varepsilon_n) = c(z_\varepsilon).$$

Therefore, $f(z_\varepsilon) = c(z_\varepsilon)$, that is, $z_\varepsilon \in M_f \neq \emptyset$. Moreover, $d(z, M_f) \leq \gamma\beta\sqrt{\varepsilon}$. Since $\beta > 1$ and $\gamma > 2$ are arbitrarily chosen, we find that $d(z, M_f) \leq 2\sqrt{\varepsilon}$, that is, (9) holds. □

As a consequence of the previous theorem, every strongly-representable operator has the following Brøndsted–Rockafellar property. For other results of this type see [4].

Corollary 2.7. *Let $f \in \Gamma_s(Z)$ be such that $\inf_{w\in Z}(f_z(w)+h(w)) = 0$, for every $z \in Z$. For every $\varepsilon > 0$ and every $(x, x^*) \in X \times X^*$ with $f(x, x^*) < \langle x, x^*\rangle + \varepsilon$ there exists $(x_\varepsilon, x_\varepsilon^*) \in M_f$ such that $\|x - x_\varepsilon\|^2 + \|x^* - x_\varepsilon^*\|^2 < 4\varepsilon$.*

The next result corresponds to [7, Prop. 2] (established in reflexive Banach spaces).

Corollary 2.8. *Let $f \in \mathcal{G}_s(Z)$ and $\gamma > 4$. For every $(x, x^*) \in X \times X^*$ and every $\alpha > 0$ there exists $(x_\alpha, x_\alpha^*) \in M_f$ such that*

$$\|x_\alpha - x\|^2 + \alpha^2\|x_\alpha^* - x^*\|^2 \leq \gamma\alpha\left(f(x, x^*) - \langle x, x^*\rangle\right). \tag{12}$$

Proof. If $(x, x^*) \notin \operatorname{dom} f$ we can take an arbitrary $(x_\alpha, x_\alpha^*) \in M_f$, while if $f(x, x^*) = \langle x, x^*\rangle$ we take $(x_\alpha, x_\alpha^*) = (x, x^*)$, for every $\alpha > 0$.

Let (x, x^*) be such that $f(x, x^*) - \langle x, x^*\rangle \in (0, \infty)$ and fix $\alpha > 0$. By (4) we know that $f_\alpha \in \mathcal{G}_s(Z)$ and from (2) we have

$$f_\alpha(x, \alpha x^*) - \langle x, \alpha x^*\rangle = \alpha\left(f(x, x^*) - \langle x, x^*\rangle\right) \in (0, \infty).$$

Applying Theorem 2.6 for f_α and $(x, \alpha x^*)$, taking into account that $\gamma > 4$, we get $(x_\alpha, x_\alpha^*) \in M_f$ (that is, $(x_\alpha, \alpha x_\alpha^*) \in M_{f_\alpha}$) such that (12) holds. □

In the sequel we also interpret M_f as a multifunction from X to X^*, and so $\operatorname{dom} M_f$ is $\Pr_X(M_f)$ and $\operatorname{Im} M_f$ is $\Pr_{X^*}(M_f)$.

Corollary 2.9. *Let $f \in \mathcal{G}_s(Z)$. Then*

$$\operatorname{cl}(\operatorname{dom} M_f) = \operatorname{cl}\left(\Pr_X(\operatorname{dom} f)\right), \qquad \operatorname{cl}(\operatorname{Im} M_f) = \operatorname{cl}\left(\Pr_{X^*}(\operatorname{dom} f)\right).$$

In particular $\operatorname{cl}(\operatorname{dom} M_f)$ and $\operatorname{cl}(\operatorname{Im} M_f)$ are convex sets. Here "cl" stands for the closure with respect to the strong topology.

Proof. The inclusions $\operatorname{dom} M_f \subset \operatorname{Pr}_X(\operatorname{dom} f)$ and $\operatorname{Im} M_f \subset \operatorname{Pr}_{X^*}(\operatorname{dom} f)$ are obvious. It suffices to prove the converse inclusions.

Let $x^* \in \operatorname{Pr}_{X^*}(\operatorname{dom} f)$, that is, $(x, x^*) \in \operatorname{dom} f$ for some $x \in X$. Applying Corollary 2.8 (with some fixed $\gamma > 4$), we get that for every $\alpha > 0$, there is $(x_\alpha, x_\alpha^*) \in M_f$ satisfying (12). Therefore, $x_\alpha^* \in \operatorname{Im} M_f$ and $\alpha \|x_\alpha^* - x^*\|^2 \leq \gamma (f(x, x^*) - \langle x, x^* \rangle)$. This shows that $s\text{–}\lim_{\alpha\to\infty} x_\alpha^* = x^*$, from which $x^* \in \operatorname{cl}(\operatorname{Im} M_f)$.

Similarly, if $x \in \operatorname{Pr}_X(\operatorname{dom} f)$, i.e., $(x, x^*) \in \operatorname{dom} f$ for some $x^* \in X^*$, then, according to Corollary 2.8, taking for every $\alpha > 0$ an $(x_\alpha, x_\alpha^*) \in M_f$ satisfying (12), we have that $x_\alpha \in \operatorname{dom} M_f$ and $\|x_\alpha - x\|^2 \leq \gamma\alpha (f(x, x^*) - \langle x, x^* \rangle)$. Hence $s\text{–}\lim_{\alpha\to 0} x_\alpha = x$, which proves that $x \in \operatorname{cl}(\operatorname{dom} M_f)$. □

Remark 2.10. The previous result shows that the strong closures of a strongly-representable operator domain and range are convex. Since, in general, the closure of the range of a maximal monotone operator is not necessarily convex (see e.g. [3]), this shows that not every maximal monotone operator is strongly-representable.

Remark 2.11. Let M be a maximal monotone operator that is not strongly-representable. Then $M = [\varphi_M = c]$, $\varphi_M \in \mathcal{F}_s(Z)$ and if we assume that $\varphi_M^* \geq c$ in Z^* then $\varphi_M \in \mathcal{G}_s(Z)$ and M would be strongly-representable; a contradiction. Hence the inequality $\varphi_M^* \geq c$ fails in Z^*, that is, the conjugate of the Fitzpatrick function of a maximal monotone operator is not necessarily a representative function.

The next result has been proved in [4, Th. 4.2] for $f \in \mathcal{G}_s(Z)$. For convenience we provide the reader with a short proof.

Theorem 2.12. *Let $f \in \Gamma_s(Z)$ be such that $\inf_{w\in Z} (f_z(w) + h(w)) = 0$, for every $z \in Z$. Then M_f is maximal monotone in Z. In particular every strongly-representable operator is maximal monotone.*

Proof. Let z_0 be monotonically related to M_f. Replacing f by f_{z_0} if necessary, we may assume without loss of generality that $z_0 = 0$, that is

$$c(z) \geq 0, \quad \forall z \in M_f. \tag{13}$$

Since $\inf(f + h) = 0$, there is $z_n \in Z$ such that $f(z_n) + h(z_n) < 1/n^2$, for every $n \geq 1$. The function $f + h$ is coercive. Indeed, fixing some $\overline{z}^* \in \operatorname{dom} f^*$ we have that

$$f(z) + h(z) \geq \tfrac{1}{2} \|z\|^2 + \langle z, \overline{z}^* \rangle - f^*(\overline{z}^*) \geq \tfrac{1}{2} \|z\|^2 - \|z\| \|\overline{z}^*\| - f^*(\overline{z}^*) \quad \forall z \in Z.$$

Therefore, the sequence $(z_n)_{n\geq 1}$ is bounded. Since $f \geq c$ and $h \geq -c$ we see that $f(z_n) < c(z_n) + 1/n^2$ and $h(z_n) + c(z_n) \leq 1/n^2$. Applying Corollary 2.7 for z_n, f, and $\varepsilon = 1/n^2$ we get $w_n \in M_f$ such that $\|w_n - z_n\| < 2/n$, for $n \geq 1$.

According to (13) and (6) we get

$$\begin{aligned} \|z_n\|^2 = 2h(z_n) &\leq -2c(z_n) + 2n^{-2} \leq -2c(w_n) + 2\,|c(w_n) - c(z_n)| + 2n^{-2} \\ &\leq \|w_n - z_n\|^2 + 2\|z_n\| \cdot \|w_n - z_n\| + 2n^{-2} \leq 6n^{-2} + 4n^{-1} \|z_n\|, \end{aligned}$$

for $n \geq 1$. Since (z_n) is bounded we have that $\|z_n\| \to 0$. Letting $n \to \infty$ in the inequality $f(z_n) < c(z_n) + 1/n^2$ and taking into account that $f \in \Gamma_s(Z)$ we get $z_0 = 0 \in M_f$. □

Remark 2.13. When X is a Banach space, the subdifferential $\partial\varphi$ of a the function $\varphi \in \Gamma_s(X)$ is strongly-representable thus maximal monotone. A strong-representative for $\partial\varphi$ is given by $f(x, x^*) = \varphi(x) + \varphi^*(x^*)$ for $x \in X$, $x^* \in X^*$.

Corollary 2.14. *Let $f \in \Gamma_s(Z)$ be such that $\inf_{w\in Z}(f_z(w) + h(w)) = 0$, for every $z \in Z$. Then*

$$f \geq \mathrm{cl}_{s\times\omega^*} f = \mathrm{cl}_{\omega\times\omega^*} f \geq \varphi_{M_f} \geq c \quad \textit{in } Z, \tag{14}$$

$M_f = M_{\mathrm{cl}_{\omega\times\omega^*} f} \subset [f^\square = c]$, and $\inf_{w\in Z}((\mathrm{cl}_{\omega\times\omega^*} f)_z(w) + h(w)) = 0$, for every $z \in Z$. Here "$\mathrm{cl}_{s(\omega)\times\omega^*} f$" stands for the greatest convex $s(\omega) \times \omega^*-$lsc function majorized by f in Z.

Proof. According to Theorem 2.12, M_f is maximal monotone. By [1, Th. 2.4], if $z \in M_f$ then $z \in \partial f(z)$. This implies $f^\square(z) = c(z)$, for every $z \in M_f$, that is, $M_f \subset [f^\square = c]$ and so $f^\square \leq c_{M_f}$. Hence $f \geq \mathrm{cl}_{\omega\times\omega^*} f = f^{\square\square} \geq \varphi_{M_f} \geq c$ in Z. Therefore $0 \leq (\mathrm{cl}_{\omega\times\omega^*} f)_z + h \leq f_z + h$; whence $\inf_{w\in Z}((\mathrm{cl}_{\omega\times\omega^*} f)_z(w) + h(w)) = 0$, for every $z \in Z$.

From $f \geq \mathrm{cl}_{\omega\times\omega^*} f \geq c$ we get $M_f \subset M_{\mathrm{cl}_{\omega\times\omega^*} f}$. Because M_f is maximal and $M_{\mathrm{cl}_{\omega\times\omega^*} f}$ is monotone the equality ensues. □

As a direct consequence of the previous corollary and Proposition 2.2 (see Remark 2.3), the next result shows that the representative of a strongly-representable operator can be picked to be lower semicontinuous with respect to the topology $\omega \times \omega^*$ on Z. Recall that $\mathcal{G}_{\omega\times\omega^*}(Z) := \mathcal{G}(Z) \cap \Gamma_{\omega\times\omega^*}(Z) = \mathcal{G}(Z) \cap \Gamma_{s\times\omega^*}(Z) =: \mathcal{G}_{s\times\omega^*}(Z)$.

Corollary 2.15. *For every $f \in \mathcal{G}_s(Z)$ one has $\mathrm{cl}_{\omega\times\omega^*} f \in \mathcal{G}_{\omega\times\omega^*}(Z)$ and $M_f = M_{\mathrm{cl}_{\omega\times\omega^*} f} = M_{f^\square}$. In particular $\{M_f \mid f \in \mathcal{G}_s(Z)\} = \{M_f \mid f \in \mathcal{G}_{\omega\times\omega^*}(Z)\}$. Moreover, if f is a strong representative of $M \subset Z$ then so are $\mathrm{cl}_{\omega\times\omega^*} f$ and φ_M.*

Proof. As previously seen in Corollary 2.14, $M_f = M_{\mathrm{cl}_{\omega\times\omega^*} f} \subset [f^\square = c]$ and from $f^* \geq c$ we know that $f^\square \geq c$ and $M_{f^\square} = [f^\square = c]$ is monotone. Since M_f is maximal monotone the equality holds. Moreover, from (14) we get $\varphi^*_{M_f} \geq (\mathrm{cl}_{\omega\times\omega^*} f)^* \geq f^* \geq c$ which proves that $\mathrm{cl}_{\omega\times\omega^*} f$ and φ_{M_f} are strong representatives of M_f. □

Corollary 2.16. *For every $f \in \mathcal{G}(Z)$, $\overline{f} := \mathrm{cl}_s f \in \mathcal{G}_s(Z)$ and $M_{\overline{f}} = M_{f^\square}$ is a maximal monotone extension of M_f.*

Proof. Since $f \geq c$ and c is continuous on Z for the strong topology we have that $f \geq \overline{f} \geq c$, $M_f \subset M_{\overline{f}} = M_{\overline{f}^\square} = M_{f^\square}$, and $\overline{f} \in \mathcal{G}_s(X \times X^*)$ because $f^* = \overline{f}^* \geq c$ and $\overline{f}^\square = f^\square$. □

An immediate consequence of the preceding results is the following characterization of strongly-representable operators. Recall that $\mathcal{F}_{\omega\times\omega^*}(Z) := \mathcal{F}(Z) \cap \Gamma_{\omega\times\omega^*}(Z)$.

Theorem 2.17. *Let $N \subset X \times X^*$ be monotone. The following are equivalent:*

(i) *N is strongly representable,*

(ii) *$\varphi_N \in \mathcal{G}(X \times X^*)$ and N is representable, that is, there is $f \in \mathcal{F}_{\omega\times\omega^*}(X \times X^*)$ such that $N = M_f$,*

(iii) *N is maximal monotone and $\varphi^*_N \geq c$.*

Proof. The implication $(i) \Rightarrow (ii)$ follows from Corollary 2.15 with $f = \varphi_N$ and Theorem 2.12.

For $(ii) \Rightarrow (iii)$ it suffices to prove that N is maximal monotone. According to [13, Th. 3.4], the condition $N = M_f$ for some $f \in \mathcal{F}_{\omega\times\omega^*}(X \times X^*)$ together with $\varphi_N \geq c$ imply that N is maximal monotone.

If (iii) holds then $N = M_{\varphi_N}$, $\varphi_N \geq c$, and $\varphi_N^* \geq c$. Therefore φ_N is a strong-representative of N. $\square$

Recall that an operator $M : X \rightrightarrows X^*$ is called *locally bounded at* $x \in \operatorname{cl}_s(\operatorname{dom} M)$ if there exists an s–open neighborhood V of x and $K > 0$ such that

$$\|x^*\| \leq K, \quad \forall x \in V,\ \forall x^* \in M(x),$$

and it is known that every monotone operator M is locally bounded at x for every $x \in (\operatorname{co}\operatorname{dom} M)^i$, where for a $A \subset X$ we denoted by "A^i, $\operatorname{int} A$" the algebraic respectively the strong-topological interior of A (see e.g. [16, Th. 3.11.14]).

Taking Corollary 2.9, Theorem 2.12, and [16, Th. 3.11.15] into account, we realize that a strongly-representable operator is locally bounded only inside the interior of its domain.

Corollary 2.18. *Let* $f \in \Gamma_s(Z)$ *be such that* $\inf_{w\in Z}(f_z(w) + h(w)) = 0$*, for every* $z \in Z$*. If* M_f *is locally bounded at* $x \in \operatorname{cl}_s(\operatorname{dom} M_f)$ *then* $x \in \operatorname{int}(\operatorname{dom} M_f)$*.*

3. Calculus rules for strongly-representable operators

We base our argument on the construction used in [5]. For X, Y locally convex spaces and $F : X \times Y \rightrightarrows X^* \times Y^*$ we define the multifunction $G := G(F) : X \rightrightarrows X^*$ by

$$\operatorname{gph} G := \{(x, x^*) \in X \times X^* \mid \exists y^* \in Y^* :\ (x, 0, x^*, y^*) \in \operatorname{gph} F\}.$$

Note that $\operatorname{gph} G$ is non-empty iff $0 \in \operatorname{Pr}_Y(\operatorname{gph} F)$ and as noticed in [5], G is monotone whenever F is monotone.

In general, for a locally convex space E, we denote by $\mathcal{M}(E)$ the class of monotone subsets of $E \times E^*$ and by $\mathfrak{M}(E)$ the class of maximal monotone subsets of $E \times E^*$. Also, we denote by $\operatorname{aff} A$ and $\overline{\operatorname{aff}} A$ the affine hull and the closed affine hull of $A \subset E$, respectively.

First consider the following slight generalization of [5, Lem. 3.1].

Lemma 3.1. *Let* X, Y *be separated locally convex spaces.*

(i) *If* $F \in \mathcal{M}(X \times Y)$ *and* $Y_0 \subset Y$ *is a closed linear subspace such that*

$$F(x, y) = F(x, y) + \{0\} \times Y_0^{\perp}, \quad \forall (x, y) \in X \times Y, \tag{15}$$

then $\operatorname{Pr}_Y(\operatorname{dom}\varphi_F) \subset y + Y_0$*, for every* $y \in \operatorname{Pr}_Y(\operatorname{gph} F)$*.*

(ii) *If* $F \in \mathfrak{M}(X \times Y)$*, then* $\operatorname{Pr}_Y(\operatorname{dom}\varphi_F) \subset \overline{\operatorname{aff}}(\operatorname{Pr}_Y(\operatorname{gph} F))$*.*

Proof. (i) Fix $y \in \operatorname{Pr}_Y(\operatorname{gph} F)$, that is, $(x, y, x^*, y^*) \in \operatorname{gph} F$ for some $(x, x^*, y^*) \in X \times X^* \times Y^*$. By (15), we have $(x, y, x^*, y^* + v^*) \in \operatorname{gph} F$, for every $v^* \in Y_0^{\perp}$.

For every $\overline{y} \in \mathrm{Pr}_Y(\operatorname{dom}\varphi_F)$ there exist $(\overline{x}, \overline{x}^*, \overline{y}^*) \in X \times X^* \times Y^*$, $\gamma \in \mathbb{R}$ such that $\varphi_F(\overline{x}, \overline{y}, \overline{x}^*, \overline{y}^*) \leq \gamma$. From the definition of φ_F we have

$$\begin{aligned}\gamma \geq\ & \langle(\overline{x}, \overline{y}), (x^*, y^* + v^*)\rangle + \langle(x, y), (\overline{x}^*, \overline{y}^*)\rangle - \langle(x, y), (x^*, y^* + v^*)\rangle \\ =\ & \langle\overline{x} - x, x^*\rangle + \langle\overline{y}, y^*\rangle + \langle x, \overline{x}^*\rangle + \langle y, \overline{y}^* - y^*\rangle + \langle\overline{y} - y, v^*\rangle\,,\end{aligned}$$

which provides us with

$$\langle y - \overline{y}, v^*\rangle \geq 0, \quad \forall v^* \in Y_0^{\perp}.$$

This implies that $\overline{y} - y \in (Y_0^{\perp})^{\perp} = Y_0$. Hence $\mathrm{Pr}_Y(\operatorname{dom}\varphi_F) \subset y + Y_0$.

(*ii*) Take $Y_0 := \overline{\operatorname{aff}}(\mathrm{Pr}_Y(\operatorname{gph} F)) - y$ for $y \in \mathrm{Pr}_Y(\operatorname{gph} F)$ fixed. The operator $F + \Phi : X \times Y \rightrightarrows X^* \times Y^*$, where $\operatorname{gph}\Phi := X \times (y + Y_0) \times \{0\} \times Y_0^{\perp}$, is monotone and its graph contains the graph of the maximal monotone operator F, so it coincides with F, from which (15) follows. We get from (*i*) the conclusion. $\square$

As in [5], we use the notation "$\operatorname{ri} A$" for the topological interior of A with respect to $\overline{\operatorname{aff}}A$ and "${}^{ic}A$" for the relative algebraic interior of A with respect to $\overline{\operatorname{aff}}A$; thus $\operatorname{ri} A$ and ${}^{ic}A$ are empty if $\operatorname{aff} A$ is not closed and one always has $\operatorname{ri} A \subset {}^{ic}A$. In the sequel, we use the facts that for C convex with ${}^{ic}C$ nonempty, we have $\operatorname{aff} C = \operatorname{aff}({}^{ic}C)$ and

$${}^{ic}C \subset A \subset C \Longrightarrow [\operatorname{aff} C = \operatorname{aff} A \text{ and } {}^{ic}C = {}^{ic}A]. \tag{16}$$

Theorem 3.2. *Let X, Y be Banach spaces and $f \in \mathcal{G}_s(X \times Y \times X^* \times Y^*)$.*

(*i*) *If $0 \in {}^{ic}(\mathrm{Pr}_Y(\operatorname{dom} f))$ and $g : X \times X^* \to \overline{\mathbb{R}}$ is given by*

$$g(x, x^*) := \inf\{f(x, 0, x^*, y^*) \mid y^* \in Y^*\}, \quad (x, x^*) \in X \times X^*, \tag{17}$$

then $g \in \mathcal{G}(X \times X^*)$,

$$g^*(u^*, u^{**}) = \min\{f^*(u^*, v^*, u^{**}, 0) \mid v^* \in Y^*\}, \quad \forall\,(u^*, u^{**}) \in X^* \times X^{**}, \tag{18}$$

$\overline{g} = \operatorname{cl}_s g \in \mathcal{G}_s(X \times X^*)$ and

$$G(M_f) = M_g = M_{\overline{g}} = M_{g^\square}. \tag{19}$$

Moreover, $G(M_f)$ is strongly representable and $\overline{g}$ is a strong representative of $G(M_f)$; in particular $G(M_f)$ is maximal monotone.

(*ii*) *One has*

$$\begin{aligned}&{}^{ic}(\mathrm{Pr}_Y(\operatorname{dom} f)) = {}^{ic}(\operatorname{conv}(\mathrm{Pr}_Y(M_f))) \\ &= {}^{ic}(\mathrm{Pr}_Y(M_f)) = \operatorname{ri}(\mathrm{Pr}_Y(M_f)) = {}^{ic}(\mathrm{Pr}_Y(\operatorname{dom}\varphi_{M_f})).\end{aligned} \tag{20}$$

Therefore, if $0 \in {}^{ic}(\mathrm{Pr}_Y(M_f))$ then $G(M_f)$ is maximal monotone.

Proof. (*i*) First observe, from their definitions, that $g \geq c$ and $G(M_f) \subset M_g$. To get (18) we follow the proof of [5, Lem. 3.2]; just observe that this time the graph of $\mathcal{C} : X \times X^* \rightrightarrows X \times Y \times X^* \times Y^*$ given by

$$\mathcal{C}(x, x^*) := \{x\} \times \{0\} \times \{x^*\} \times Y^*, \quad (x, x^*) \in X \times X^*,$$

is a closed linear subspace and $\mathcal{C}^*(x^*, y^*, x^{**}, y^{**}) = \{(x^*, x^{**})\}$, if $y^{**} = 0$; $\mathcal{C}^*(x^*, y^*, x^{**}, y^{**}) = \emptyset$, otherwise.

Notice that $g(x, x^*) = \inf\{f(u, v, u^*, v^*) \mid (u, v, u^*, v^*) \in \mathcal{C}(x, x^*)\}$, for $(x, x^*) \in X \times X^*$ and

$$\operatorname{dom} f - \operatorname{Im}\mathcal{C} = X \times \operatorname{Pr}_Y(\operatorname{dom} f) \times X^* \times Y^*,$$

from which $0 \in {}^{ic}(\operatorname{dom} f - \operatorname{Im}\mathcal{C})$.

By the fundamental duality formula (more precisely, see [16, Th. 2.8.6 (v)]) we get (18). Since $f^* \geq c$, from (18), we see that $g^* \geq c$, and so $g \in \mathcal{G}(X \times X^*)$.

Since $g \in \mathcal{G}(X \times X^*)$, we know, by Corollary 2.16, that $M_g \subset M_{\overline{g}} = M_{g^\square}$ and $\overline{g} \in \mathcal{G}_s(X \times X^*)$. Therefore, according to Corollary 2.15 and again from (18)

$$M_{g^\square} = G(M_{f^\square}) = G(M_f) \subset M_g \subset M_{\overline{g}} = M_{g^\square}.$$

Hence (19) holds.

(ii) Set $F := M_f$. We first claim that

$${}^{ic}(\operatorname{Pr}_Y(\operatorname{dom} f)) \subset \operatorname{Pr}_Y(F) \subset \operatorname{Pr}_Y(\operatorname{dom} f). \tag{21}$$

Indeed, let $y \in {}^{ic}(\operatorname{Pr}_Y(\operatorname{dom} f))$. Then $0 \in {}^{ic}(\operatorname{Pr}_Y(\operatorname{dom} f'))$ with $f' := f_{(0,y,0,0)}$ because $\operatorname{dom} f' = \operatorname{dom} f - (0, y, 0, 0)$. Since $f' \in \mathcal{G}_s$, by (i) we get $G(M_{f'}) = \{(x, x^*) \mid \exists y^* : (x, y, x^*, y^*) \in M_f\}$ is maximal monotone; in particular $G(M_{f'})$ is nonempty, and so $y \in \operatorname{Pr}_Y(F)$. Hence the first inclusion of (21) holds while the second one is obvious.

Because $f \in \mathcal{G}_s$, from (14) (see also Remark 2.3), we have that $\varphi_F \leq f \leq \operatorname{conv} c_F$. Here "$\operatorname{conv} c_F$" stands for the greatest convex function majorized by c_F in $X \times Y \times X^* \times Y^*$.

It follows that

$$F \subset \operatorname{conv} F \subset \operatorname{dom}(\operatorname{conv} c_F) \subset \operatorname{dom} f \subset \operatorname{dom} \varphi_F,$$

whence

$$\begin{aligned}\operatorname{Pr}_Y(F) \subset \operatorname{Pr}_Y(\operatorname{conv} F) = \operatorname{conv}(\operatorname{Pr}_Y(F)) \subset \operatorname{Pr}_Y(\operatorname{dom}(\operatorname{conv} c_F)) \\ \subset \operatorname{Pr}_Y(\operatorname{dom} f) \subset \operatorname{Pr}_Y(\operatorname{dom} \varphi_F).\end{aligned} \tag{22}$$

This together with Lemma 3.1 (ii) yield

$$\begin{aligned}\operatorname{aff}(\operatorname{Pr}_Y(F)) = {} & \operatorname{aff}(\operatorname{Pr}_Y(\operatorname{conv} F)) \subset \operatorname{aff}(\operatorname{Pr}_Y(\operatorname{dom}(\operatorname{conv} c_F)) \\ \subset {} & \operatorname{aff}(\operatorname{Pr}_Y(\operatorname{dom} f)) \subset \operatorname{aff}(\operatorname{Pr}_Y(\operatorname{dom} \varphi_F)) \subset \overline{\operatorname{aff}}\,(\operatorname{Pr}_Y(F)).\end{aligned} \tag{23}$$

If $\operatorname{aff}(\operatorname{Pr}_Y(\operatorname{conv} F))$ $(= \operatorname{aff}(\operatorname{Pr}_Y(F)))$ is closed, all inclusions in (23) become equalities; hence in this case

$${}^{ic}(\operatorname{Pr}_Y(F)) \subset {}^{ic}(\operatorname{Pr}_Y(\operatorname{conv} F)) \subset {}^{ic}(\operatorname{Pr}_Y(\operatorname{dom} f)) \subset {}^{ic}(\operatorname{Pr}_Y(\operatorname{dom} \varphi_F)). \tag{24}$$

Assume that ${}^{ic}(\operatorname{Pr}_Y(\operatorname{dom} f)) \neq \emptyset$. Taking (16) and (21) into account, we know that $\operatorname{aff}(\operatorname{Pr}_Y(\operatorname{dom} f)) = \operatorname{aff}(\operatorname{Pr}_Y(F))$ is closed and ${}^{ic}(\operatorname{Pr}_Y(F)) = {}^{ic}(\operatorname{Pr}_Y(\operatorname{dom} f))$. Relation (24) provides

$${}^{ic}(\operatorname{Pr}_Y(F)) = {}^{ic}(\operatorname{Pr}_Y(\operatorname{conv} F)) = {}^{ic}(\operatorname{Pr}_Y(\operatorname{dom} f)). \tag{25}$$

If ${}^{ic}(\mathrm{Pr}_Y(F)) \neq \emptyset$ then $\mathrm{aff}(\mathrm{Pr}_Y(F))$ is closed and (24) is true. Therefore ${}^{ic}(\mathrm{Pr}_Y(\mathrm{dom}\, f)) \neq \emptyset$, whence (25) holds again.

We proved that (25) is true regardless of whether or not ${}^{ic}(\mathrm{Pr}_Y(F))$ is non-empty. (Indeed, if ${}^{ic}(\mathrm{Pr}_Y(F)) \neq \emptyset$, we have seen above that (25) holds. Assume that ${}^{ic}(\mathrm{Pr}_Y(F)) = \emptyset$ $[= {}^{ic}(\mathrm{Pr}_Y(\mathrm{dom}\, f))]$. If $\mathrm{aff}(\mathrm{Pr}_Y(F))$ is closed then (25) follows from (24); if $\mathrm{aff}(\mathrm{Pr}_Y(F))$ $[= \mathrm{aff}(\mathrm{Pr}_Y(\mathrm{conv}\, F))]$ is not closed then ${}^{ic}(\mathrm{Pr}_Y(\mathrm{conv}\, F)) = \emptyset$, and so (25) holds again.)

Since $X \times Y \times X^* \times Y^*$ is a Banach space and $f \in \Gamma_s(X \times Y \times X^* \times Y^*)$, by [16, Prop. 2.7.2], we have ${}^{ic}(\mathrm{Pr}_Y(\mathrm{dom}\, f)) = \mathrm{ri}\,(\mathrm{Pr}_Y(\mathrm{dom}\, f))$. (Indeed, because $\mathrm{epi}\, f$ is closed and the involved spaces are Banach spaces, condition $H(x, x^*, y^*)$ holds. By the last part of [16, Prop. 2.7.2] we get ${}^{ib}D = \mathrm{rint}\, D$, where $D := \mathrm{Pr}_Y(\mathrm{dom}\, f)$. If $\mathrm{aff}\, D$ is not closed then ${}^{ic}D = \emptyset = \mathrm{ri}\, D$; if $\mathrm{aff}\, D$ is closed then ${}^{ic}D = {}^{ib}D = {}^{i}D$ and $\mathrm{ri}\, D = \mathrm{rint}\, D$.) This gives

$$ {}^{ic}(\mathrm{Pr}_Y(F)) = {}^{ic}(\mathrm{Pr}_Y(\mathrm{conv}\, F)) = {}^{ic}(\mathrm{Pr}_Y(\mathrm{dom}\, f)) = \mathrm{ri}\,(\mathrm{Pr}_Y(\mathrm{dom}\, f)) . \tag{26} $$

From Corollary 2.15 we know that φ_F is a strong representative of $F = M_f$. Relation (26) applied for φ_F states that ${}^{ic}(\mathrm{Pr}_Y(\mathrm{dom}\, \varphi_F)) = {}^{ic}(\mathrm{Pr}_Y(F))$, thereby completing the proof of (20). □

For $F : X \times Y \rightrightarrows X^* \times Y^*$ and $A : X \to Y$ a continuous linear operator, we consider $F_A : X \times Y \rightrightarrows X^* \times Y^*$ defined by

$$ \mathrm{gph}\, F_A := \{(x, y, x^*, y^*) \in X \times Y \times X^* \times Y^* \mid (x^* - A^\top y^*, y^*) \in F(x, Ax + y)\}, $$

where $A^\top : Y^* \to X^*$ is the adjoint of A. Equivalently, $F_A(x, y) = B^\top F B(x, y)$ with $B(x, y) := (x, y + Ax)$, for $(x, y) \in X \times Y$.

Since $B : X \times Y \to X \times Y$ is an isomorphism of normed vector spaces (with $B^\top(x^*, y^*) = (x^* + A^\top y^*, y^*)$), if F is strongly–representable, (maximal) monotone then F_A is strongly–representable, (maximal) monotone. Moreover, if f is a (strong) representative of F then $f_A := f \circ L$ is a (strong) representative of F_A, where $L := B \times (B^{-1})^\top$. In an extended form

$$ f_A(x, y, x^*, y^*) = f(x, y + Ax, x^* - A^\top y^*, y^*), \quad (x, y, x^*, y^*) \in X \times Y \times X^* \times Y^*. $$

Note that $y \in \mathrm{Pr}_Y(\mathrm{dom}\, f_A)$ iff $y = y' - Ax'$ for some $(x', y') \in \mathrm{Pr}_{X\times Y}(\mathrm{dom}\, f)$, $(x, y, x^*, y^*) \in M_{f_A}$ iff $(x, y + Ax, x^* - A^\top y^*, y^*) \in M_f$, and $(M_f)_A = M_{f_A}$, for every $f \in \mathcal{F}$.

Using the previous result for F_A we get the next two consequences.

Corollary 3.3. *Assume that X, Y are Banach spaces, $f \in \mathcal{G}_s(X \times Y \times X^* \times Y^*)$ and $A \in L(X, Y)$. Then*

$$ \begin{aligned} {}^{ic}\{y - Ax \mid (x, y) \in \mathrm{dom}\, M_f\} &= {}^{ic}\{y - Ax \mid (x, y) \in \mathrm{conv}(\mathrm{dom}\, M_f)\} \\ &= {}^{ic}\{y - Ax \mid (x, y) \in \mathrm{Pr}_{X\times Y}(\mathrm{dom}\, f)\} \\ &= \mathrm{ri}(\{y - Ax \mid (x, y) \in \mathrm{dom}\, M_f\}). \end{aligned} $$

Assume that $0 \in {}^{ic}\{y - Ax \mid (x, y) \in \mathrm{Pr}_{X\times Y}(\mathrm{dom}\, f)\}$ (or equivalently $0 \in {}^{ic}\{y - Ax \mid (x, y) \in \mathrm{dom}\, M_f\}$). Then the multifunction $G(F_A)$ whose graph is

$$ \{(x, x^*) \in X \times X^* \mid \exists y^* \in Y^* : (x^* - A^\top y^*, y^*) \in M_f(x, Ax)\} $$

is strongly-representable, a strong representative is given by $\overline{g}$, *where* $g : X \times X^* \to \overline{\mathbb{R}}$ *is defined by*

$$g(x, x^*) = \inf\{f(x, Ax, x^* - A^\top y^*, y^*) \mid y^* \in Y^*\}, \quad \forall (x, x^*) \in X \times X^*;$$

in particular $G(F_A)$ *is maximal monotone. More precisely,* $G(F_A) = M_g = M_{\overline{g}} = M_{g^\square}$ *and*

$$g^\square(x, x^*) = \min\{f^\square(x, Ax, x^* - A^\top y^*, y^*) \mid y^* \in Y^*\}, \quad \forall (x, x^*) \in X \times X^*.$$

Theorem 3.4. *Assume that* X, Y *are Banach spaces,* $f \in \mathcal{G}_s(X \times X^*)$, $g \in \mathcal{G}_s(Y \times Y^*)$ *and* $A \in L(X, Y)$. *Then*

$$\begin{aligned} {}^{ic}(\operatorname{dom} M_g - A(\operatorname{dom} M_f)) &= {}^{ic}(\operatorname{conv}(\operatorname{dom} M_g - A(\operatorname{dom} M_f))) \\ &= {}^{ic}(\Pr_Y(\operatorname{dom} g) - A(\Pr_X(\operatorname{dom} f))) \\ &= \operatorname{ri}(\operatorname{dom} M_g - A(\operatorname{dom} M_f)). \end{aligned}$$

If, in addition, $0 \in {}^{ic}(\operatorname{dom} g - A(\operatorname{dom} f))$ *(or equivalently* $0 \in {}^{ic}(\operatorname{dom} M_g - A(\operatorname{dom} M_f))$*) then* $M_f + A^\top M_g A$ *is strongly representable (and maximal monotone) having as strong representative the function* $\overline{k}$, *where*

$$k : X \times X^* \to \overline{\mathbb{R}}, \qquad k(x, x^*) := \inf\{f(x, x^* - A^\top y^*) + g(Ax, y^*) \mid y^* \in Y^*\}. \tag{27}$$

Moreover, $M_f + A^\top M_g A = M_k = M_{\overline{k}} = M_{k^\square}$ *and*

$$k^\square(x, x^*) := \min\{f^\square(x, x^* - A^\top y^*) + g^\square(Ax, y^*) \mid y^* \in Y^*\} \quad \forall (x, x^*) \in X \times X^*.$$

Proof. Consider $\phi : X \times Y \times X^* \times Y^*$ defined by $\phi(x, y, x^*, y^*) := f(x, x^*) + g(y, y^*)$. Then $\phi^*(x^*, y^*, x^{**}, y^{**}) = f^*(x^*, x^{**}) + g^*(y^*, y^{**})$, and so $\phi \in \mathcal{G}_s(X \times Y \times X^* \times Y^*)$. Moreover, for $F := M_\phi$ we have $G(F_A) = M_f + A^\top M_g A$. The conclusion follows using the preceding corollary. □

Taking $X = Y$ and $A = \mathrm{Id}_X$ in the previous theorem, the next result shows that the Rockafellar Conjecture on the sum of maximal monotone operators is true in the strongly-representable case.

Corollary 3.5. *Let* X *be a Banach space and let* $M, N : X \rightrightarrows X^*$ *be strongly representable. Then* ${}^{ic}(\operatorname{dom} M - \operatorname{dom} N) = {}^{ic}(\operatorname{conv}(\operatorname{dom} M) - \operatorname{conv}(\operatorname{dom} N))$ *(is a convex set). If* $0 \in {}^{ic}(\operatorname{dom} M - \operatorname{dom} N)$ *then* $M + N$ *is strongly representable; in particular* $M + N$ *is maximal monotone. Moreover,* $\operatorname{cl}(\operatorname{dom}(M + N))$ *and* $\operatorname{cl}(\operatorname{Im}(M + N))$ *are convex sets.*

Remark 3.6. Since every subdifferential is strongly-representable, the previous corollary together with [8, Th. 26.1] or [9, Th. 44.1] show that every strongly-representable operator is maximal monotone locally or of type FPV (see [8, Def. 25.4], [9, Def. 36.7]).

Theorem 3.7. *If* X *is a Banach space,* $M : X \rightrightarrows X^*$ *is strongly representable, and* $N : X \rightrightarrows X^*$ *is maximal monotone with* $\operatorname{dom} N = X$, *then* $M + N$ *is maximal monotone.*

Proof. In order to prove that $M + N$ is maximal monotone we wish to apply [13, Th. 3.4], that is, to show that $M + N$ is representable and $\varphi_{M+N} \geq c$. Since $M + N$ is representable by [13, Cor. 5.6], we have only to prove that $\varphi_{M+N} \geq c$. To this end it suffices to prove that $\overline{x} \in \operatorname{dom}(M+N)$ whenever $\overline{z} := (\overline{x}, \overline{x}^*) \in [\varphi_{M+N} \leq c]$, i.e., $\overline{z}$ is monotonically related to $M+N$ (because for a monotone operator $S : X \rightrightarrows X^*$ one always has $(\operatorname{dom} S) \times X^* \subset [\varphi_S \geq c]$; see e.g. [12, Prop. 2 (i)] or [13, Prop. 2.1 (d)]).

According to Corollary 2.15, we may choose f to be a strong representative for M such that $f \in \mathcal{G}_{s\times\omega^*}(X \times X^*)$. Let $\overline{z} = (\overline{x}, \overline{x}^*)$ be monotonically related to $M+N$. Taking $M_0 := M - \overline{z}$ and $N_0 = N - (\overline{x}, 0)$, then $\operatorname{gph}(M+N) - \overline{z} = \operatorname{gph}(M_0 + N_0)$ and $(0,0)$ is monotonically related to $M_0 + N_0$; moreover, $f_{\overline{z}} \in \mathcal{G}_{s\times\omega^*}(X \times X^*)$, $f_{\overline{z}}$ is a strong representative of M_0 and $\operatorname{dom} N_0 = X$. If we prove that $0 \in \operatorname{dom}(M_0 + N_0)$ then $\overline{x} \in \operatorname{dom}(M+N)$. Hence, without loss of generality, we assume that $\overline{z} = 0 \in [\varphi_{M+N} \leq c]$, that is

$$c(u, u^* + v^*) \geq 0 \quad \text{for all } u \in X,\ u^*, v^* \in X^* \text{ with } (u, u^*) \in M,\ (u, v^*) \in N. \tag{28}$$

Fix $(x_0, x_0^*) \in \operatorname{dom} f$ and let $[0, x_0] := \{t x_0 \mid 0 \leq t \leq 1\}$ and $C_\varepsilon := [0, x_0] + \varepsilon U$, for $\varepsilon > 0$, where $U := \{x \in X \mid \|x\| \leq 1\}$. Since N is locally bounded and $[0, x_0]$ is compact there is $\varepsilon_0 > 0$ such that N is bounded on C_{ε_0}, that is, there is $K > 0$ such that

$$\|v^*\| \leq K, \quad \forall v \in C_{\varepsilon_0},\ \forall v^* \in N(v). \tag{29}$$

Take $C := C_{\varepsilon_0/2}$ and for $n \geq 1$ consider

$$\phi_n(x) := \iota_C(x) + \frac{n}{2}\|x\|^2, \quad \Phi_n(x, x^*) = \phi_n(x) + \phi_n^*(-x^*), \quad x \in X,\ x^* \in X^*.$$

It is clear that $\Phi_n \geq -c$. Moreover,

$$\phi_n^*(x^*) = \min\left\{\sigma_C(u^*) + \tfrac{1}{2n}\left\|x^* - u^*\right\|^2 \mid u^* \in X^*\right\} \geq 0, \quad \forall x^* \in X^*, \tag{30}$$

and ϕ_n^* is finite and continuous on X^*, where for $A \subset X$ and $x^* \in X^*$, $\sigma_A(x^*) := \iota_A^*(x^*) = \sup_{x\in A}\langle x, x^*\rangle$. Since Φ_n is continuous at (x_0, x_0^*), $f \geq c$, $\Phi_n \geq -c$, $f^* \geq c$, and $\Phi_n^* \geq -c$, applying the fundamental duality formula (as in the proof of Proposition 2.2), we get

$$\inf_{w\in X\times X^*}(f(w) + \Phi_n(w)) = 0, \quad \forall n \geq 1.$$

Therefore, for every $n \geq 1$ there is $z_n := (x_n, x_n^*)$ such that $f(z_n) + \Phi_n(z_n) < n^{-2}$. Since $x_n \in C$, we know that $\|x_n\| \leq \|x_0\| + \varepsilon_0/2$, for every $n \geq 1$.

As previously seen, $f \geq c$ and $\Phi_n \geq -c$ imply that

$$\Phi_n(z_n) + c(z_n) \leq n^{-2}, \quad f(z_n) < c(z_n) + n^{-2}, \quad \forall n \geq 1. \tag{31}$$

From (31), Corollary 2.7 provides $w_n := (y_n, y_n^*) \in M$ such that $\|w_n - z_n\| < 2/n$, for $n \geq 1$.

Pick $v_n^* \in N(y_n)$. For every $n \geq 4/\varepsilon_0$ we have that $y_n \in C_{\varepsilon_0}$, and so $\|v_n^*\| \leq K$ by (29).

Using (6) and (28), this yields

$$\begin{aligned}
&\frac{n}{2}\|x_n\|^2 + \phi_n^*(-x_n^*) \\
&= \Phi_n(z_n) \le -c(z_n) + n^{-2} \le -c(w_n) + |c(w_n) - c(z_n)| + n^{-2} \\
&\le -c(w_n) + \tfrac{1}{2}\|w_n - z_n\|^2 + \|z_n\|\|z_n - w_n\| + n^{-2} \\
&\le -c(w_n) + 2n^{-1}\|z_n\| + 3n^{-2} \le -c(w_n) + 2n^{-1}\|x_n\| + 2n^{-1}\|x_n^*\| + 3n^{-2} \\
&= -c(y_n, y_n^* + v_n^*) + c(y_n, v_n^*) + 2n^{-1}\|x_n\| + 2n^{-1}\|x_n^*\| + 3n^{-2} \\
&\le K(\|x_n\| + 2n^{-1}) + 2n^{-1}\|x_n\| + 2n^{-1}\|x_n^*\| + 3n^{-2} \\
&\le K\|x_n\| + 2n^{-1}\|x_n^*\| + Ln^{-1},
\end{aligned}$$

for $n \ge 4/\varepsilon_0$, where $L := 2K + 2\|x_0\| + \varepsilon_0 + 3$. Hence, for $n \ge 4/\varepsilon_0$ we have

$$\frac{n}{2}\|x_n\|^2 - K\|x_n\| + [\phi_n^*(-x_n^*) - 2n^{-1}\|x_n^*\| - Ln^{-1}] \le 0,$$

or equivalently

$$\tfrac{1}{2}(\|nx_n\| - K)^2 + [n\phi_n^*(-x_n^*) - 2\|x_n^*\|] \le \tfrac{1}{2}K^2 + L. \tag{32}$$

We claim that

$$n\phi_n^*(x^*) \ge 3\|x^*\| - 18, \quad \forall x^* \in X^*, \ \forall n \ge 6/\varepsilon_0. \tag{33}$$

Notice that $n\phi_n^*(x^*) = \min\left\{\sigma_{nC}(u^*) + \frac{1}{2}\|x^* - u^*\|^2 \mid u^* \in X^*\right\}$, $x^* \in X^*$, $n \ge 1$.

The condition $n \ge 6/\varepsilon_0$ implies $nC \supset 3U$; whence $n\sigma_C(u^*) = \sigma_{nC}(u^*) \ge \sigma_{3U}(u^*) = 3\|u^*\|$, for every $u^* \in X^*$.

For fixed $x^* \in X^*$ we consider two cases: a) $\|x^* - u^*\| < 6$ and b) $\|x^* - u^*\| \ge 6$.

If a) holds then $\|u^*\| \ge \|x^*\| - \|x^* - u^*\| > \|x^*\| - 6$ and so

$$\sigma_{nC}(u^*) + \tfrac{1}{2}\|x^* - u^*\|^2 \ge \sigma_{nC}(u^*) \ge 3\|u^*\| \ge 3\|x^*\| - 18.$$

If b) holds then $\frac{1}{2}\|x^* - u^*\|^2 \ge 3\|x^* - u^*\|$ and so

$$\sigma_{nC}(u^*) + \tfrac{1}{2}\|x^* - u^*\|^2 \ge 3\|u^*\| + 3\|x^* - u^*\| \ge 3\|x^*\|.$$

In both cases we obtain that our claim is true. Using (33), from (32) we get

$$\tfrac{1}{2}(\|nx_n\| - K)^2 + \|x_n^*\| \le \tfrac{1}{2}K^2 + L + 18, \quad \forall n \ge 6/\varepsilon_0.$$

Hence necessarily $\|x_n\| \to 0$ and $(x_n^*)_n$ is bounded. On a subnet, denoted for simplicity by the same index, $x_n^* \to x^*$ weakly-star in X^*. Passing to limit in (31) we get $(0, x^*) \in [f = c] = M$ and so $\overline{x} = 0 \in \operatorname{dom} M = \operatorname{dom}(M + N)$. The proof is complete. $\square$

The previous theorem allows us to recover the result in [8, Th. 42.2] and its extension [11, Cor. 2.9(a)] (see also [9, Th. 53.1]).

Corollary 3.8. *If X is a Banach space, $\varphi \in \Gamma_s(X)$ and $L : X \to X^*$ is linear and positive then $\partial\varphi + L$ is maximal monotone.*

Corollary 3.9. *If X is a Banach space, $\varphi \in \Gamma_s(X)$ and $N : X \rightrightarrows X^*$ is maximal monotone with $\operatorname{dom} N = X$ then $\partial\varphi + N$ is maximal monotone.*

4. Comparison with other classes of operators

Recall that $M : X \rightrightarrows X^*$ is called *locally maximal monotone* or of type *FP* (see [8, Def. 25.2], [9, Def. 36.5] and [2]) if for every open convex set $U \subset X^*$ such that $U \cap \operatorname{Im} M \neq \emptyset$, if $z \in X \times U$ is such that $c(z - w) \geq 0$ for all $w \in \operatorname{gph} M \cap (X \times U)$ then $z \in \operatorname{gph} M$.

Theorem 4.1. *Every strongly-representable operator is locally maximal monotone.*

Proof. Let M be a strongly-representable operator with a strong-representative $f \in \mathcal{G}_{\omega \times \omega^*}(X \times X^*)$. Consider $U \subset X^*$ an open convex set such that $U \cap \operatorname{Im} M \neq \emptyset$ and $z := (x, x^*) \in X \times U$ such that $c(z - w) \geq 0$ for all $w \in \operatorname{gph} M \cap (X \times U)$. Doing a translation (in fact replacing f by f_z which implies that U is replaced by $U - x^*$) we may (and do) assume that $z = 0 \in X \times U$. Hence

$$c(w) \geq 0, \quad \forall w \in \operatorname{gph} M \cap (X \times U), \tag{34}$$

and we have to show that $(z =)\ 0 \in \operatorname{gph} M$. Fix $u^* \in U \cap \operatorname{Im} M$, $u \in X$ such that $(u, u^*) \in M$, and set $C_r := [0, u^*] + rU_{X^*}$ for $r > 0$, where $U_{X^*} = \{x^* \in X^* \mid \|x^*\| \leq 1\}$. For $\alpha \in \mathbb{R}$, let $\alpha_+ := \max(\alpha, 0)$ and $\alpha_- := (-\alpha)_+$. For $r > 0$, consider

$$\phi_r(x) = r\|x\| + \langle x, u^* \rangle_+ , \quad \Phi_r(x, x^*) := \phi_r(-x) + \phi_r^*(x^*), \quad x \in X, \ x^* \in X^*.$$

Then $\phi_r^* = \iota_{C_r}$, $\Phi_r \geq -c$, and $\Phi_r^* \geq -c$. We know that Φ_r is continuous at $(u, u^*) \in M \subset \operatorname{dom} f$, and from the fundamental duality formula combined with $\Phi_r \geq -c$, $\Phi_r^* \geq -c$, we get, as in the proof of Proposition 2.2, that $\inf(f + \Phi_r) = 0$, for every $r > 0$.

Because $[0, u^*]$ is a compact subset of the open set U, there exists $r_0 \in (0, 1]$ such that $C_{r_0} \subset U$. Consider a sequence $(r_n)_{n \geq 1} \subset (0, r_0/3]$ with $r_n \to 0$. Since $\inf(f + \Phi_{r_n}) = 0$, for every $n \geq 1$ there exists $z_n := (x_n, x_n^*)$ such that $f(z_n) + \Phi_{r_n}(z_n) < r_n^4$.

Again, $f \geq c$ and $\Phi_{r_n} \geq -c$ imply that $\Phi_{r_n}(z_n) + c(z_n) < r_n^4$ and $f(z_n) < c(z_n) + r_n^4$ for $n \geq 1$. Corollary 2.7 provides $w_n := (y_n, y_n^*) \in M$ such that $\|w_n - z_n\| < 2r_n^2$, for $n \geq 1$. Since $x_n^* \in C_{r_n}$ and $\|x_n^* - y_n^*\| < 2r_n^2 \leq 2r_n$, we find that $y_n^* \in U$ and $w_n \in \operatorname{gph} M \cap (X \times U)$.

Hence, according to (34), $c(w_n) \geq 0$, for every $n \geq 1$. Taking (6) into account, we get

$$\begin{aligned} r_n\|x_n\| &\leq r_n\|x_n\| + \langle x_n, u^* \rangle_- = \Phi_r(z_n) \leq -c(z_n) + r_n^4 \leq -c(w_n) + |c(z_n) - c(w_n)| + r_n^4 \\ &\leq \tfrac{1}{2}\|w_n - z_n\|^2 + \|z_n\| \cdot \|w_n - z_n\| + r_n^4 \leq 2r_n^2\|x_n\| + 2r_n^2(r_n + \|u^*\|) + 3r_n^4, \end{aligned}$$

whence

$$(1 - 2r_n)\|x_n\| \leq 2r_n(r_n + \|u^*\|) + 3r_n^3, \quad \forall n \geq 1.$$

Hence $x_n \to 0$, strongly in X, as $n \to \infty$. Since $x_n^* \in C_{r_n}$, we have that $x_n^* = t_n u^* + r_n u_n^*$ with $t_n \in [0, 1]$ and $u_n^* \in U_{X^*}$. Taking a subsequence if necessary, we have that $t_n \to \bar{t} \in [0, 1]$, and this implies $x_n^* \to \overline{x}^* := \bar{t}u^* \in U$, strongly in X^*. Let $n \to \infty$ in $f(z_n) < c(z_n) + r_n^4$ to find $(0, \overline{x}^*) \in [f = c] = M$.

In particular, we proved that whenever $z := (x, x^*) \in X \times U$ is monotonically related to $M_U := \operatorname{gph} M \cap (X \times U)$ then there is $\overline{x}^* \in U$ such that $(x, \overline{x}^*) \in M_U$. In other words we showed that $[\varphi_{M_U} \leq c] \subset (\operatorname{dom} M_U) \times X^*$. Since $(\operatorname{dom} M_U) \times X^* \subset [\varphi_{M_U} \geq c]$ (see e.g. [12, Prop. 2 (i)]), this implies

$$\varphi_{M_U}(z) = c(z), \qquad \varphi_{M_U} \geq c \ \text{ in } X \times U;$$

hence z is a local minimum point for $\varphi_{M_U} - c$ and so $\widehat{z} \in \partial\varphi_{M_U}(z)$. Therefore $\varphi_{M_U}(z) + \varphi_{M_U}^{\square}(z) = z \cdot z = 2c(z)$ which in turn gives $\psi_{M_U}(z) = \varphi_{M_U}^{\square}(z) = c(z)$ because $\varphi_{M_U}(z) = c(z)$. From $f \leq c_M \leq c_{M_U}$ and $f \in \Gamma_{\omega\times\omega^*}(X \times X^*)$ we know that $c \leq f \leq \psi_{M_U}$; whence $f(z) = c(z)$, that is $z \in M$. The proof is complete. $\square$

Using a different argument the previous result allows us to recover the convexity of the closure for the range of a strongly-representable operator (see [2, Th. 3.5]).

Recall that $M \subset Z := X \times X^*$ is called *strongly maximal monotone* (see [8, Def. 25.8], [9, Def. 36.9]) if M is monotone and whenever the non-empty convex weakly-compact set $C \subset X$ and $x^* \in X^*$ are such that

$$\forall (y, y^*) \in M,\ \exists x \in C\ :\ \langle x - y, x^* - y^* \rangle \geq 0 \tag{35}$$

then $C \cap M^{-1}(x^*) \neq \emptyset$, and, further, whenever the non-empty convex weakly-star compact set $C \subset X^*$ and $x \in X^*$ are such that

$$\forall (y, y^*) \in M,\ \exists x^* \in C\ :\ \langle x - y, x^* - y^* \rangle \geq 0 \tag{36}$$

then $C \cap M(x) \neq \emptyset$.

Theorem 4.2. *Every strongly-representable operator is strongly maximal monotone.*

Proof. Let M be a strongly-representable operator with a strong-representative $f \in \mathcal{G}_{\omega\times\omega^*}(X \times X^*)$ (according to Theorem 2.17).

Consider first the non-empty convex weakly-compact set $C \subset X$ and $x^* \in X^*$ such that (35) holds. Of course, $R := \sup\{\|x\| \mid x \in C\} < \infty$. Moreover,

$$d(x, C) = \min\{\|x - u\| \mid u \in C\} \geq \inf\{\|x - u\| \mid \|u\| \leq R\} = (\|x\| - R)_+. \tag{37}$$

Replacing if necessary f by $f_{(0,x^*)}$, we may assume that $x^* = 0$. Therefore (35) reduces to

$$\forall (y, y^*) \in M,\ \exists x \in C\ :\ \langle y, y^* \rangle \geq \langle x, y^* \rangle\,. \tag{38}$$

For $r > 0$, consider $\phi_r(x) = rd^2(x, C) = r\min\{\|x - u\|^2 \mid u \in C\}$ for $x \in X$.

Then $\phi_r^*(x^*) = \sigma_C(x^*) + \frac{1}{4r}\|x^*\|^2$, $x^* \in X^*$, and ϕ_r, ϕ_r^* are continuous, for every $r > 0$. As usual $\sigma_C(x^*) = \sup_{u\in C}\langle u, x^*\rangle$ for $x^* \in X^*$.

Let $\Phi_r(x, x^*) := \phi_r(x) + \phi_r^*(-x^*)$ for $x \in X$, $x^* \in X^*$. Then $\Phi_r \geq -c$, $\Phi_r^* \geq -c$, and Φ_r is continuous; hence allowing us to use the fundamental duality formula to get $\inf(f + \Phi_r) = 0$, for every $r > 0$.

Consider $(r_n)_{n\geq 1} \subset (0, 1]$ with $r_n \to 0$. Since $\inf(f + \Phi_{r_n}) = 0$, for every $n \geq 1$ there exists $z_n := (x_n, x_n^*)$ such that $f(z_n) + \Phi_{r_n}(z_n) < r_n^4$. Because $f \geq c$ and $\Phi_{r_n} \geq -c$, we get $\Phi_{r_n}(z_n) + c(z_n) < r_n^4$ and $f(z_n) < c(z_n) + r_n^4$, for every $n \geq 1$.

Corollary 2.7 provides $w_n := (y_n, y_n^*) \in M$ such that $\|w_n - z_n\| < 2r_n^2$ for $n \geq 1$. Using (38) we know that there is $\tilde{x}_n \in C$ such that $\langle y_n, y_n^*\rangle \geq \langle \tilde{x}_n, y_n^*\rangle$, and so

$$-c(w_n) = -\langle y_n, y_n^*\rangle \leq -\langle \tilde{x}_n, y_n^*\rangle = -\langle \tilde{x}_n, y_n^* - x_n^*\rangle + \langle \tilde{x}_n, -x_n^*\rangle \leq 2Rr_n^2 + \sigma_C(-x_n^*),$$

for every $n \geq 1$. Together with (6), this yields

$$\begin{aligned} & r_n d^2(x_n, C) + \sigma_C(-x_n^*) + \tfrac{1}{4r_n}\|x_n^*\|^2 = \Phi_{r_n}(z_n) \\ \leq & \; -c(z_n) + r_n^4 \leq -c(w_n) + |c(z_n) - c(w_n)| + r_n^4 \\ \leq & \; 2Rr_n^2 + \sigma_C(-x_n^*) + \tfrac{1}{2}\|w_n - z_n\|^2 + \|z_n\| \cdot \|w_n - z_n\| + r_n^4 \\ \leq & \; 2Rr_n^2 + \sigma_C(-x_n^*) + 2r_n^2(\|x_n\| + \|x_n^*\|) + 3r_n^4, \end{aligned}$$

for every $n \geq 1$. We get

$$r_n d^2(x_n, C) - 2r_n^2\|x_n\| + \tfrac{1}{4r_n}\|x_n^*\|^2 - 2r_n^2\|x_n^*\| \leq 2Rr_n^2 + 3r_n^4,$$

and after dividing by r_n we find

$$d^2(x_n, C) - 2r_n\|x_n\| + \left(\frac{1}{2r_n}\|x_n^*\| - 2r_n^2\right)^2 \leq 4r_n^4 + 2Rr_n + 3r_n^3, \quad \forall n \geq 1. \tag{39}$$

Therefore the sequence $\big(d(x_n, C) - 2r_n\|x_n\|\big)_n$ is bounded above and, using (37), we see that $(x_n)_n$ is bounded. Hence $\lim_{n\to\infty} r_n\|x_n\| = 0$ and from (39) we find subsequently that $(\frac{1}{2r_n}\|x_n^*\|)_n$ is bounded, $x_n^* \to 0$, strongly in X^*, as $n \to \infty$, and $\lim_{n\to\infty} d(x_n, C) = 0$. Since C is weakly-compact, this shows that, at least on a subnet, denoted for simplicity by the same index, $x_n \to x \in C$, weakly in X. As usual, we find that $(x, 0) \in [f = c] = M$, that is $x \in C \cap M^{-1}(0)$, if we let $n \to \infty$ in $f(z_n) < c(z_n) + r_n^4$.

Now, consider the nonempty convex weakly-star compact set $C \subset X^*$ and $x \in X$ such that (36) holds. By a translation (f replaced by $f_{(x,0)}$) we may assume that $x = 0$. In this way, relation (36) spells

$$\forall (y, y^*) \in M, \ \exists x^* \in C \ : \ \langle y, y^* \rangle \geq \langle y, x^* \rangle . \tag{40}$$

For $r > 0$, take $\phi_r(x) = \sigma_C(x) + \frac{1}{4r}\|x\|^2$, where $\sigma_C(x) = \max_{u^* \in C} \langle x, u^* \rangle$, $x \in X$. Then $\phi_r^*(x^*) = rd^2(x^*, C)$ for $x^* \in X^*$, and ϕ_r, ϕ_r^* are continuous.

Let $\Phi_r(x, x^*) = \phi_r(-x) + \phi_r(x^*)$, for $(x, x^*) \in X \times X^*$ and $r > 0$. Then $\Phi_r \geq -c$, $\Phi_r^* \geq -c$, and Φ_r is continuous. This allows us to apply the fundamental duality formula to get $\inf(f + \Phi_r) = 0$, for every $r > 0$.

Consider $(r_n)_{n\geq 1} \subset (0, 1]$ with $r_n \to 0$. Since $\inf(f + \Phi_{r_n}) = 0$, for every $n \geq 1$ there exists $z_n := (x_n, x_n^*)$ such that $f(z_n) + \Phi_{r_n}(z_n) < r_n^4$. Because $f \geq c$ and $\Phi_{r_n} \geq -c$, we get $\Phi_{r_n}(z_n) + c(z_n) < r_n^4$ and $f(z_n) < c(z_n) + r_n^4$, for every $n \geq 1$.

Corollary 2.7 provides $w_n := (y_n, y_n^*) \in M$ such that $\|w_n - z_n\| < 2r_n^2$, for $n \geq 1$. Using (40) we know that for w_n there is $\tilde{x}_n^* \in C$ such that $\langle y_n, y_n^* \rangle \geq \langle y_n, \tilde{x}_n^* \rangle$, and so

$$-c(w_n) = -\langle y_n, y_n^* \rangle \leq -\langle y_n, \tilde{x}_n^* \rangle = -\langle y_n - x_n, \tilde{x}_n^* \rangle + \langle -x_n, \tilde{x}_n^* \rangle \leq 2Rr_n^2 + \sigma_C(-x_n)$$

for every $n \geq 1$, where $R := \sup\{\|x^*\| \mid x^* \in C\} < \infty$. Together with (6) this implies that, for every $n \geq 1$, we have

$$\begin{aligned} & \sigma_C(-x_n) + \tfrac{1}{4r_n}\|x_n\|^2 + r_n d^2(x_n^*, C) = \Phi_{r_n}(z_n) \\ \leq & \; -c(z_n) + r_n^4 \leq -c(w_n) + |c(z_n) - c(w_n)| + r_n^4 \\ \leq & \; 2Rr_n^2 + \sigma_C(-x_n) + \tfrac{1}{2}\|w_n - z_n\|^2 + \|z_n\| \cdot \|w_n - z_n\| + r_n^4 \\ \leq & \; 2Rr_n^2 + \sigma_C(-x_n) + 2r_n^2(\|x_n\| + \|x_n^*\|) + 3r_n^4. \end{aligned}$$

Therefore

$$\frac{1}{4r_n}\|x_n\|^2 - 2r_n^2\|x_n\| + r_n d^2(x_n^*, C) - 2r_n^2\|x_n^*\| \le 2Rr_n^2 + 3r_n^4,$$

or equivalently,

$$\left(\frac{1}{2r_n}\|x_n\| - 2r_n^2\right)^2 + d^2(x_n^*, C) - 2r_n\|x_n^*\| \le 4r_n^4 + 2Rr_n + 3r_n^3.$$

As in the first part, this implies that, at least on a subnet, $z_n \to (0, x^*)$ strongly $\times$ weakly-star in $X \times X^*$, for some $x^* \in C$. Again, by passing to limit in $f(z_n) < c(z_n) + r_n^4$ we find $x^* \in C \cap M(0)$. The proof is complete. □

Remark 4.3. From the above considerations we see that every strongly-representable operator is X–regular in the sense introduced in [10]. This can be deduced from Theorem 4.2 and [10, Prop. 1] or from Corollary 3.5 and [10, Th. 1]. As seen in [10, Th. 2], the X–regularity provides a different argument for the convexity of the closure of a strongly-representable operator domain.

Corollary 4.4. *Let $f \in \mathcal{G}_s(Z)$. For every $(x, x^*) \in X \times X^*$ and every $\varepsilon > 0$ there exists $(x_\varepsilon, x_\varepsilon^*) \in M_f$ such that $\{(x_\varepsilon, x_\varepsilon^*) \mid \varepsilon > 0\}$ is bounded and*

$$\|x - x_\varepsilon\|^2 + 2\langle x - x_\varepsilon, x^* - x_\varepsilon^*\rangle + \|x^* - x_\varepsilon^*\|^2 \le \varepsilon.$$

Proof. Replacing f by $f_{(x,x^*)}$ if necessary, we may assume that $(x, x^*) = (0, 0)$. As seen in the proof of Theorem 2.12, $f + h$ is (strongly) coercive. Hence there exists $r > 0$ such that $\{z \in Z \mid f(z) + h(z) \le 1\} \subset rU_Z$, where $U_Z = \{z \in Z \mid \|z\| \le 1\}$.

For $\varepsilon \in (0, 1]$ take $\varepsilon' \in (0, \varepsilon)$ such that $10\varepsilon' + 8r\sqrt{\varepsilon'} = \varepsilon$. Since $\inf(f + h) = 0$, there exists $w_\varepsilon \in Z$, such that $f(w_\varepsilon) + h(w_\varepsilon) < \varepsilon'$ and $\|w_\varepsilon\| \le r$. From $f \ge c$ and $h \ge -c$ it follows that

$$f(w_\varepsilon) < c(w_\varepsilon) + \varepsilon', \qquad \tfrac{1}{2}\|w_\varepsilon\|^2 + c(w_\varepsilon) \le \varepsilon'.$$

Corollary 2.7 applied for $\varepsilon' > 0$ and w_ε provides $z_\varepsilon \in M_f$ such that $\|w_\varepsilon - z_\varepsilon\| < \delta := 2\sqrt{\varepsilon'}$.

Using (6) we get

$$\|z_\varepsilon\|^2 \le (\|w_\varepsilon\| + \|z_\varepsilon - w_\varepsilon\|)^2 \le \|w_\varepsilon\|^2 + 2r\|z_\varepsilon - w_\varepsilon\| + \|z_\varepsilon - w_\varepsilon\|^2,$$
$$c(z_\varepsilon) \le c(w_\varepsilon) + |c(w_\varepsilon) - c(z_\varepsilon)| \le c(w_\varepsilon) + r\|z_\varepsilon - w_\varepsilon\| + \tfrac{1}{2}\|z_\varepsilon - w_\varepsilon\|^2.$$

Therefore,

$$\|z_\varepsilon\|^2 + 2c(z_\varepsilon) \le \|w_\varepsilon\|^2 + 2c(w_\varepsilon) + 4r\delta + 2\delta^2 \le 2\varepsilon' + 4r\delta + 2\delta^2 = 10\varepsilon' + 8r\sqrt{\varepsilon'} = \varepsilon.$$

For $\varepsilon \ge 1$ we take $z_\varepsilon := z_1$. The proof is complete. □

The next result shows that every strongly-representable operator is of type ANA (for this notion see [8, Def. 25.10], [9, Def. 36.11]).

Corollary 4.5. *Let $f \in \mathcal{G}_s(Z)$. Then for every $(x, x^*) \in X \times X^* \setminus M_f$ there exists a bounded sequence $((x_n, x_n^*))_{n\geq 1} \subset M_f$ such that $x_n \neq x$, $x_n^* \neq x^*$ for every $n \geq 1$, and*

$$\lim_{n\to\infty} \frac{\langle x_n - x, x_n^* - x^* \rangle}{\|x_n - x\| \cdot \|x_n^* - x^*\|} = -1.$$

Proof. Let $(x, x^*) \in X \times X^* \setminus M_f$. Fix $(\varepsilon_n)_{n\geq 1} \subset (0, \infty)$ with $\varepsilon_n \to 0$. Using Corollary 4.4 we get a bounded sequence $((x_n, x_n^*))_{n\geq 1} \subset M_f$ such that

$$\|x - x_n\|^2 + 2\langle x - x_n, x^* - x_n^* \rangle + \|x^* - x_n^*\|^2 \leq \varepsilon_n^2, \quad \forall n \geq 1. \tag{41}$$

Hence

$$\big|\, \|x - x_n\| - \|x^* - x_n^*\| \,\big| \leq \varepsilon_n, \quad \forall n \geq 1. \tag{42}$$

There exist $\gamma > 0$ and $n_0 \geq 1$ such that $\|x - x_n\| \geq 2\gamma$ for all $n \geq n_0$, since otherwise, on a subsequence, $x_{n_k} \to x$, strongly in X, and because of (42), $x_{n_k}^* \to x^*$, strongly in X^*. This yields the contradiction

$$\langle x, x^* \rangle < f(x, x^*) \leq \liminf_{k\to\infty} f(x_{n_k}, x_{n_k}^*) = \lim_{k\to\infty} \left\langle x_{n_k}, x_{n_k}^* \right\rangle = \langle x, x^* \rangle.$$

From (42) we obtain

$$\left| \frac{\|x^* - x_n^*\|}{\|x - x_n\|} - 1 \right| \leq \frac{\varepsilon_n}{2\gamma}, \quad \forall n \geq n_0,$$

whence $\lim_{n\to\infty} \|x^* - x_n^*\| / \|x - x_n\| = 1$.

Hence $\|x^* - x_n^*\| \geq \gamma$, for every $n \geq n_1$, for some $n_1 \geq n_0$ and $\lim_{n\to\infty} \|x - x_n\| / \|x^* - x_n^*\| = 1$.

From (41) we get

$$-2 \leq \frac{2\langle x_n - x, x_n^* - x^* \rangle}{\|x_n - x\| \cdot \|x_n^* - x^*\|} \leq \frac{\varepsilon_n^2}{2\gamma^2} - \frac{\|x - x_n\|}{\|x^* - x_n^*\|} - \frac{\|x^* - x_n^*\|}{\|x - x_n\|}, \quad \forall n \geq n_1,$$

whence the conclusion follows. □

Recall that $N \subset Z := X \times X^*$ is called *NI*, or of *negative-infimum type* (see [8, Def. 25.5], [9, Def. 36.2]), if

$$\inf_{(u,u^*)\in N} \langle u^* - x^*, \widehat{u} - x^{**} \rangle \leq 0, \quad \forall (x^*, x^{**}) \in Z^*,$$

or equivalently $c_N^*(x^*, x^{**}) \geq \langle x^*, x^{**} \rangle$, for every $(x^*, x^{**}) \in X^* \times X^{**}$.

Proposition 4.6. *Let $N \subset X \times X^*$ be maximal monotone and NI. Then N is strongly-representable.*

Proof. Since N is maximal monotone we have that $c_N \geq \varphi_N \geq c$ and $N = M_{\varphi_N} = [\varphi_N = c]$. It follows that $\varphi_N^* \geq c_N^*$. Since N is NI we have $c_N^* \geq c$, and so $\varphi_N \in \mathcal{G}_s(Z)$. To conclude we see that φ_N is a strong representative of N or we use Theorem 2.17. □

Corollary 2.9 allows us to recover with simple proofs the results in [15, Cors. 3.5, 3.6], which at their turn solve [8, Problem 27.7] (see also [9, Problem 43.3]).

Corollary 4.7. *Let $M : X \rightrightarrows X^*$ be a maximal monotone operator of type NI. Then* $\operatorname{cl}(\operatorname{dom} M)$ *and* $\operatorname{cl}(\operatorname{Im} M)$ *are convex sets.*

Remark 4.8. Implicitly, from the above results, we have proved that every maximal monotone operator of type NI is automatically maximal monotone locally, locally maximal monotone, strongly maximal monotone, and ANA and the closures of the domain and range of a maximal monotone operator of type NI are convex. This answers partially the open problem stated in [8, Problems 25.9, 25.11] and [8, Problems 36.10, 36.12].

For skew bounded operators the converse of Proposition 4.6 holds.

Corollary 4.9. *Let $S : X \rightrightarrows X^*$ be skew, that is,* $\operatorname{gph} S$ *is a linear subspace and* $\langle x, x^* \rangle = 0$*, for all* $(x, x^*) \in \operatorname{gph} S$*. Consider the conditions:*

(i) *S is maximal monotone and NI,*
(ii) *S is $s \times \omega^*$–closed in $X \times X^*$ and S^* is monotone in $X^{**} \times X^*$,*
(iii) *S is strongly-representable.*

Then $(i) \Leftrightarrow (ii) \Rightarrow (iii)$. If in addition $S : X \to X^*$ has $\operatorname{dom} S = X$*, then $(iii) \Rightarrow (i)$. Here $(x^{**}, x^*) \in \operatorname{gph} S^*$ iff $\langle u, x^* \rangle = \langle u^*, x^{**} \rangle$, for every $(u, u^*) \in \operatorname{gph} S$.*

Proof. First note that $z^* \in (-S^*)^{-1} \subset Z^*$ iff $\langle z, z^* \rangle = 0$, for every $z \in S$. Moreover, from its definition, $(-S^*)^{-1}$ is weakly-star closed in Z^*. This implies that for $L \subset Z$ linear we have L is skew iff $L \subset (-L^*)^{-1}$ iff $\operatorname{cl}_{s \times w^*} L$ is skew.

For a skew S we have

$$\psi_S = \iota_{\operatorname{cl}_{s \times w^*} S}, \qquad \varphi_S = \iota_{(-S^*) \cap Z}, \qquad c_S^* = \iota_S^* = \iota_{(-S^*)^{-1}}.$$

Indeed, $c_S = \iota_S$ is convex, whence $\psi_S = \operatorname{cl}_{s \times w^*} \iota_S = \iota_{\operatorname{cl}_{s \times w^*} S}$ and

$$c_S^*(z^*) = \iota_S^*(z^*) = \sup\{\langle z, z^* \rangle - c(z) \mid z \in S\} = \sup\{\langle z, z^* \rangle \mid z \in S\} = \iota_{(-S^*)^{-1}}(z^*)$$

for every $z^* \in Z^*$. Similarly $\varphi_S = \iota_{(-S^*) \cap Z}$.

Notice also that S^* is monotone in $X^{**} \times X^*$ iff $\iota_S^* = c_S^* \geq c$ in Z^* iff S is NI.

$(i) \Rightarrow (ii)$ Because S is maximal monotone, $S \subset \operatorname{cl}_{s \times w^*} S$ and $\operatorname{cl}_{s \times w^*} S$ is monotone (being skew), we have that S $(= \operatorname{cl}_{s \times w^*} S)$ is $s \times w^*$–closed in $X \times X^*$. Since S is NI, as seen above, S^* is monotone in $X^{**} \times X^*$.

For $(ii) \Rightarrow (iii)$ and $(ii) \Rightarrow (i)$ notice that $c_S = \iota_S$ is a strong-representative of S.

$(iii) \Rightarrow (i)$ Assume that $\operatorname{dom} S = X$ and let $f \in \mathcal{G}_s(Z)$ be a strong-representative of S. Then S is maximal monotone by Theorem 2.12 and $f \geq \varphi_S = \iota_{(-S^*) \cap Z}$ by Corollary 2.14. Since $\operatorname{dom} S = X$ and S is skew, S is single-valued and for all $x, y \in X$, $0 = \langle x + y, S(x + y) \rangle = \langle y, Sx \rangle + \langle x, Sy \rangle$, that is, $\langle Sx, y \rangle = -\langle Sy, x \rangle$; whence $(-S^*) \cap Z = S$. This implies $f \geq \varphi_S = \iota_S = c_S$ and so $c_S^* = \iota_S^* = \iota_{(-S^*)^{-1}} \geq f^* \geq c$, i.e., S is NI. $\square$

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References

[1] S. Fitzpatrick: Representing monotone operators by convex functions, in: Functional Analysis and Optimization, Workshop / Miniconference (Canberra, 1988), Proc. Cent. Math. Anal. Aust. Natl. Univ. 20, Australian National University, Canberra (1988) 59–65.

[2] S. P. Fitzpatrick, R. R. Phelps: Bounded approximants to monotone operators on Banach spaces, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 9 (1992) 573–595.

[3] S. P. Fitzpatrick, R. R. Phelps: Some properties of maximal monotone operators on nonreflexive Banach spaces, Set-Valued Anal. 3 (1995) 51–69.

[4] M. Marques Alves, B. F. Svaiter: Brønsted-Rockafellar property and maximality of monotone operators representable by convex functions in non-reflexive Banach spaces, J. Convex Analysis 15(3) (2008) 693–706.

[5] J.-P. Penot, C. Zălinescu: Some problems about the representation of monotone operators by convex functions, ANZIAM J. 47 (2005) 1–20.

[6] J.-P. Penot, C. Zălinescu: On the convergence of maximal monotone operators, Proc. Amer. Math. Soc. 134 (2006) 1937–1946.

[7] J.-P. Penot, C. Zălinescu: Convex analysis can be helpful for the asymptotic analysis of monotone operators, Math. Program. B 116 (2009) 481–498.

[8] S. Simons: Minimax and Monotonicity, Springer, Berlin (1998).

[9] S. Simons: From Hahn-Banach to Monotonicity, Springer, Berlin (2008).

[10] A. Verona, M. E. Verona: Regular maximal monotone operators, Set-Valued Anal. 6 (1998) 303–312.

[11] A. Verona, M. E. Verona: Regular maximal monotone operators and the sum theorem, J. Convex Analysis 7 (2000) 115–128.

[12] M. D. Voisei: Calculus rules for maximal monotone operators in general Banach spaces, J. Convex Analysis 15 (2008) 73–85.

[13] M. D. Voisei: The sum and chain rules for maximal monotone operators, Set-Valued Anal. 16 (2008) 461–476.

[14] M. D. Voisei: A maximality theorem for the sum of maximal monotone operators in nonreflexive Banach spaces, Math. Sci. Res. J. 10 (2006) 36–41.

[15] D. Zagrodny: The convexity of the closure of the domain and the range of a maximal monotone multifunction of type NI, Set-Valued Anal. 16 (2008) 759–783.

[16] C. Zălinescu: Convex Analysis in General Vector Spaces, World Scientific, Singapore (2002).