Journal of Convex Analysis
Volume 16 (2009), No. 3&4, 973–986

Positive Sets, Conservative Sets and Dissipative Sets

Jean-Paul Penot
*Laboratory of Applied Mathematics, Faculty of Sciences,
University of Pau, UMR CNRS 5142, BP 1155, 64013 Pau Cedex, France*
jean-paul.penot@univ-pau.fr

Dedicated to Stephen Simons on the occasion of his 70th birthday.

Received: April 24, 2008
Revised manuscript received: June 19, 2009

We look for a simple general framework which would encompass the notion of symmetric self-dual spaces introduced by S. Simons and the notion of self-paired product space proposed by the author in [28]. Such a framework is appropriate for the study of a notion generalizing the concept of monotone operator. The representation of such operators by functions is the main purpose of the study.

Keywords: Balanced space, conservative set, dissipative set, Fitzpatrick function, monotone operator, positive set, Simons space

2000 Mathematics Subject Classification: 47H05, 47H06, 47B44, 47A07, 11E05

1. Introduction

Mathematics and strip tease have several commun features: both are exciting and both lead to maximal bareness. For what concerns mathematics, the sole subject of interest here, the aim is to reach a generality and simplicity as great as possible. The simple examples of the concepts of group, normed vector space, topological space, differential manifold, fiber bundle show that useful, general notions emerged with difficulties. Still, the advantages of simple models are twofold: first, proofs are likely to be more direct when they do not use specific features or heavily structured frameworks and second, such models may be detected in unexpected cases.

After several attempts of representations of maximal monotone operators by convex functions made by several authors [2]–[6], [8], [10], [13]–[19], [24], [25], [29], [34]–[35], S. Simons introduced the notion of symmetric self-dual space, in brief SSD-space [34]. We call it here *S-space* or *Simons space*. It consists in a pair (Z, b) in which Z is a nonzero real vector space and $b : Z \times Z \to \mathbb{R}$ is a symmetric bilinear form. We denote by q the associated quadratic form defined by

$$q(z) := (1/2)b(z, z), \quad z \in Z.$$

A subset A of Z satisfying $q(a - a') \geq 0$ for all a, a' in A is called a *positive subset*. Among the most important examples of Simons spaces are Hilbert spaces, Lorentz spaces

ISSN 0944-6532 / $ 2.50

which are products $W \times \mathbb{R}$, where W is a Hilbert space, on which one takes the bilinear form given by $b((w,r),(w',r')) = (w \mid w') - rr'$ for (w,r), $(w',r') \in Z := W \times \mathbb{R}$. A generalization of these examples consists of products of pairs $(V \times W, b)$ where V, W are Hilbert spaces and b is given by $b((v,w),(v',w')) := (v \mid v') - (w \mid w')$; then the positive subsets are the graphs of nonexpansive maps from a subset of V to W. Another class of examples is described below.

On the other hand, the author introduced in [28] a notion of *self-paired product space*, SPP-space. It consists in a triple (X, Y, c) formed by two sets X, Y paired by a coupling function $c : X \times Y \to \mathbb{R}_{-\infty} := \mathbb{R} \cup \{-\infty\}$. Then, the product set $Z := X \times Y$ is endowed with the map $b : Z \times Z \to \mathbb{R}_{-\infty}$ given by

$$b(z, z') := c(x, y') + c(x', y) \tag{1}$$

for $z := (x, y)$, $z' := (x', y')$, which is obviously symmetric and such that $c(z) = q(z) := (1/2)b(z, z)$ for all $z \in Z$. Thus Z has no linear structure (so that the notion of positive subset has no meaning) but Z is a product space. This product structure enables one to study the so-called *c-monotone operators* which are the multimaps (or operators) $M : X \rightrightarrows Y$ from X into Y which satisfy the relation

$$c(x, y') + c(x', y) \leq c(x, y) + c(x', y')$$

for all $x, x' \in X$, $y \in M(x)$, $y' \in M(x')$. When X is a normed vector space and Y is the dual space X^* of X, c being the usual pairing $\langle \cdot, \cdot \rangle$, such c-monotone multimaps are just usual monotone operators. When X and Y are real linear spaces and c is bilinear, the space $Z := X \times Y$ endowed with b given by (1) becomes a Simons space and the graphs of c-monotone maps are just the positive subsets of (Z, b). However, several examples of self-paired product spaces are displayed in [28] which are not of that type; in particular, accretive operators on a reflexive Banach space can be studied by taking for c the derivative of half the square of a smooth norm. Another example is of importance in mechanics (see [5] and its references): it is the case in which X and Y are normed vector spaces in duality and $Z := X \times Y$ is provided with a bipotential, i.e. a function $p : X \times Y \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ which is convex and lower semicontinuous in each argument, which satisfies $p \geq \langle \cdot, \cdot \rangle$ and is such that for every $(x, y) \in X \times Y$ one has

$$y \in \partial p(\cdot, y)(x) \Leftrightarrow p(x, y) = \langle x, y \rangle \Leftrightarrow x \in \partial p(x, \cdot)(y).$$

Then the set M_p represented by p, i.e. such that $M_p := \{(x, y) \in Z : p(x, y) = \langle x, y \rangle\}$ is the subject of the study of [5]. We shall see below that when p is jointly convex, the set M_p is monotone.

The purpose of the present paper is to devise a framework which encompasses both the notion of Simons space and the notion of self-paired product space. Our interest remains the study of a special class of subsets of such spaces which generalizes the class of (graphs of) monotone operators.

2. Balanced spaces

The notion we introduce now fulfills the objective mentioned in the introduction. It is so general it encompasses the notion of metric space.

Definition 2.1. A balanced space is a pair (Z, b), where Z is a set and $b : Z \times Z \to \mathbb{R}_{-\infty}$ is a symmetric map.

Thus, when (Z, b) is a Simons space, i.e. when Z is a vector space and b is bilinear and separates points, one has a balanced space. On the other hand, when (X, Y, c) is a self-paired product space, the set $Z := X \times Y$ is a balanced space for the map b given by (1).

For every balanced space (Z, b), one can introduce the map $q : Z \to \mathbb{R}_{-\infty}$ given by

$$q(z) := (1/2)b(z, z). \tag{2}$$

When (Z, b) is a Simons space, q coincides with the quadratic form q introduced above. When (X, Y, c) is a self-paired product space and b is given by (1), $q(\cdot) = (1/2)b(\cdot, \cdot)$ coincides with the given coupling function c. When (Z, d) is a metric space and $c : Z \to \mathbb{R}_{-\infty}$ is a function on Z, setting

$$b(w, z) := c(w) + c(z) + d(w, z), \tag{3}$$

one gets $q = c$ and one also can recover b from q. In general, b cannot be recovered from q, as it is the case with S-spaces (by the familiar polarization identity) and SPP-spaces (for which $c = q$). Nonetheless, one can introduce a notion of conservative subset of a balanced space (Z, b) which encompasses the notion of positive set and the notion of c-monotone subset.

Definition 2.2. A subset S of a balanced space (Z, b) is said to be conservative if for every $z, z' \in S$ one has

$$q(z) + q(z') \geq b(z, z'). \tag{4}$$

It is said to be maximal conservative if there is no conservative subset of Z which contains it properly.

If (Z, b) is a Simons space, then a subset S of Z is positive if, and only if, it is conservative. This assertion follows from the fact that for any $z, z' \in Z$ one has

$$b(z - z', z - z') \geq 0 \Longleftrightarrow b(z, z) + b(z', z') \geq 2b(z, z').$$

On the other hand, when (X, Y, c) is a self-paired product space, a multimap $M : X \rightrightarrows Y$ is c-monotone if, and only if, its graph $S := \{(x, y) \in X \times Y : y \in M(x)\}$ is a conservative subset of the balanced space (Z, b), where $Z := X \times Y$ and b is given by (1). Such an assertion is an immediate consequence of the definitions and of relation (1).

Example. Let (X, Y, c) be a self-paired product space, and let $f : X \to \mathbb{R}_{\infty} := \mathbb{R} \cup \{+\infty\}$ be an arbitrary function. The subdifferential of f (with respect to c) is the multimap $M : X \rightrightarrows Y$ whose graph S is given by

$$S := \{(x, y) : x \in f^{-1}(\mathbb{R}), \forall u \in X \ f(u) + c(x, y) \geq f(x) + c(u, y)\}.$$

Then S is a conservative subset of (Z, b), with b given by $b((u, v), (x, y)) := c(x, v) + c(u, y)$ as in (1). □

One can extend the concept of monotone polarity introduced in [18] to the framework of balanced spaces. Given a subset S of a balanced space (Z, b), we introduce its *conservative polar* S^γ by

$$S^\gamma := \{z \in Z : \forall w \in S \ q(w) + q(z) \geq b(w, z)\}.$$

Clearly, $S \mapsto S^\gamma$ is a polarity, i.e. satisfies the relation

$$\left(\bigcup_{i\in I} S_i\right)^\gamma = \bigcap_{i\in I} S_i^\gamma$$

for any family $(S_i)_{i\in I}$ of subsets of Z. In the case (Z, b) is the balanced space associated with a self-paired product space (X, Y, c), the polarity $S \mapsto S^\gamma$ coincides with the polarity $S \mapsto S^\mu$ defined in [28] which extends the polarity introduced to us by Martínez-Legaz (see [18]).

As in the classical case, the conservative polarity is devised in order to get the equivalence

$$S \text{ is conservative} \Longleftrightarrow S \subset S^\gamma$$

and one has the properties

$$\begin{aligned} S \subset T &\Longrightarrow T^\gamma \subset S^\gamma, \\ S &\subset S^{\gamma\gamma} := (S^\gamma)^\gamma, \\ S^{\gamma\gamma\gamma} &= S^\gamma. \end{aligned}$$

Proposition 2.3. *A subset S of Z is maximal conservative if, and only if, one has $S = S^\gamma$.*

Proof. Suppose S is maximal conservative, i.e. is conservative and not properly contained in a conservative subset. Then S is contained in S^γ and, since for all $z \in Z$ one has $2q(z) = b(z, z)$, for any $z \in S^\gamma$ the set $T := S \cup \{z\}$ is conservative, hence $T = S$ and $z \in S$, so that $S^\gamma \subset S$ and equality holds. Conversely, when $S = S^\gamma$, the set S is conservative and even maximal conservative since for any $z \in Z \backslash S$ there exists some $z' \in S$ such that $q(z') + q(z) < b(z, z')$, so that $S \cup \{z\}$ is not conservative. □

3. Correspondences between conservative sets and functions

A special means to get conservative subsets of a balanced space occurs when Z is provided with a map $m : Z \times Z \to Z$ such that for all $(z, z') \in Z \times Z$ one has

$$q(m(z, z')) \geq \frac{1}{4}q(z) + \frac{1}{4}q(z') + \frac{1}{4}b(z, z'). \tag{5}$$

Such a relation is satisfied when (Z, b) is a Simons space and $m(z, z') = (1/2)z + (1/2)z'$, as easily seen. Slightly modifying a calculation made in [17], [23] and [28, Lemma 1], given an *m-convex* function, i.e. a function g satisfying $g(m(z, z')) \leq (1/2)g(z) + (1/2)g(z')$ for all $z, z' \in Z$, we get a simple construction.

Proposition 3.1. *Suppose* (Z, b) *is a balanced space endowed with a map* m *satisfying* (5)*,* q *being finitely valued. Then if* $g : Z \to \overline{\mathbb{R}}$ *is an* m*-convex function satisfying* $g \geq q$*, the set* M_g *given by* $M_g := \{z \in Z : g(z) = q(z)\}$ *is conservative.*

Proof. Let $z, z' \in M_g$. Our assumptions yield

$$\begin{aligned}\frac{1}{2}q(z) + \frac{1}{2}q(z') = \frac{1}{2}g(z) + \frac{1}{2}g(z') &\geq g(m(z,z')) \geq q(m(z,z')) \\ &\geq \frac{1}{4}q(z) + \frac{1}{4}q(z') + \frac{1}{4}b(z,z').\end{aligned}$$

Simplifying, we get $q(z) + q(z') \geq b(z, z')$. □

Remark. The conclusion also holds when q is arbitrary but g takes its values in $\mathbb{R}_\infty$, since then q is finite on M_g. □

Let us describe a more general means to associate a conservative subset of a balanced space (Z, b) to a function $f : Z \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ on Z. For such a purpose, we define the Fenchel transform $f^b : Z \to \overline{\mathbb{R}}$ of f by

$$f^b(z) := -\inf\{f(w) - b(w, z) \,:\, w \in Z\}, \quad z \in Z, \tag{6}$$

with the convention $r - s = r + (-s)$ for $r, s \in \mathbb{R}$, and $(-\infty) - (-\infty) = +\infty$. Such a transform keeps many of the properties of the classical Fenchel transform. In particular, it is a *conjugacy*, i.e, it is such that $(f + r)^b = f^b - r$ for all $r \in \mathbb{R}$ and it satisfies the following relation for any family $(f_i)_{i\in I}$ of functions on Z :

$$(\inf_{i\in I} f_i)^b = \sup_{i\in I} f_i^b.$$

Moreover, one has $q^b \geq q$, as noted in [28, Lemma 4] in the special case of self-paired product spaces. The proof in the case of a balanced space (Z, b) is similar: given $z \in Z$, plugging $w := z$ in relation (6) with $f := q$, we get $q^b(z) \geq -(q(z) - b(z, z)) = q(z)$ since for all $r \in \mathbb{R}_{-\infty}$ one has $r - 2r = -r$. When (Z, b) is a Simons space, one can say more: if q is a nonnegative quadratic form (a not interesting case since any subset is positive) one has $q^b = q$; otherwise $q^b = +\infty$ since picking some $z_0 \in Z$ such that $q(z_0) < 0$, for all $z \in Z$ one has

$$q^b(z) = \sup_{w\in Z}(b(w, z) - \frac{1}{2}b(w, w)) \geq \sup_{t\in\mathbb{R}}(tb(z_0, z) - \frac{1}{2}t^2 b(z_0, z_0)) = +\infty.$$

Proposition 3.2. *For any function* $f : Z \to \overline{\mathbb{R}}$ *the set* $M := \{z \in Z : f(z) \leq q(z),\ f^b(z) \leq q(z)\}$ *is conservative.*

Proof. Given $z, z' \in M$ one has

$$q(z) \geq f^b(z) \geq -(f(z') - b(z, z')) \geq -(q(z') - b(z, z')).$$

In view of the implication

$$s \geq -(r - t) \Longrightarrow r + s \geq t$$

for r, s, t in $\mathbb{R}_{-\infty}$ (see [28, Lemma 1]), we get relation (4). □

Corollary 3.3. *Let $f : Z \to \overline{\mathbb{R}}$ be such that $f \geq f^b$. Then $M_f := \{z \in Z : f(z) = q(z)\}$ and $M_f^{\leq} := \{z \in Z : f(z) \leq q(z)\}$ are conservative subsets of Z.*

Proof. For all $z \in M_f$ or even $z \in M_f^{\leq}$, one has $z \in M$ as defined in the preceding proposition. Thus M_f and $M_f^{\leq}$ are conservative. □

Corollary 3.4. *Let $f : Z \to \overline{\mathbb{R}}$ be such that $f^b \geq f$. Then $M_{f^b} := \{z \in Z : f^b(z) = q(z)\}$ and $M_f^{\leq} := \{z \in Z : f^b(z) \leq q(z)\}$ are conservative subsets of Z.*

Proof. For all $z \in M_{f^b}$ or even $z \in M_{f^b}^{\leq}$, one has $z \in M$ as defined in the preceding proposition. Thus M_{f^b} and $M_{f^b}^{\leq}$ are conservative. □

In order to introduce a generalization of the representations introduced in [8], [10] and [25], let us recall that the *indicator function* ι_S of a subset S of Z is the function given by $\iota_S(z) := 0$ if $z \in S$, $\iota_S(z) = +\infty$ if $z \in Z \backslash S$.

Definition 3.5. For any subset S of the balanced space (Z, b) the function $f_S := q_S^b$ where $q_S := q + \iota_S$ is called the *Fitzpatrick function* of S.

The *predominant function* of S is the function $g_S := q_S^{bb} := f_S^b$.

In the classical case, the terminology (but not the notation) for the Fitzpatrick function is widely accepted. The terminology for the predominant function g_S stems from the domination property detected in [25]. Such a property is still valid in the framework of balanced spaces: for any subset S of the balanced space (Z, b), g_S is greater than any function f on Z such that $f^{bb} = f$ and $f \leq q$ on S (since $f^{bb} \leq q_S^{bb}$ when $f \leq q_S$). When the closure operation $\varphi \longmapsto \varphi^{bb}$ is simple, the passage from q_S to g_S is rather intuitive while the passage from q_S to f_S requires solving the maximization problem

$$f_S(z) = \sup_{w \in S} -(q(w) - b(w, z)).$$

In the classical case in which $Z := X \times X^*$, where X is a n.v.s. with dual X^* and b is given by

$$b((x, y), (u, v)) := \langle x, v \rangle + \langle u, y \rangle,$$

the Fitzpatrick function of $S \subset Z$ is given by the optimization problem

$$f_S(x, y) = \sup_{(u,v) \in S} (\langle x, v \rangle + \langle u, y \rangle - \langle u, v \rangle),$$

while g_S is simply obtained through convexification and closure of the epigraph of q_S when $\mathrm{cl}(\mathrm{co}(q_S))$ is proper.

Proposition 3.6. *For every subset S of Z, its Fitzpatrick function $f_S := q_S^b$ satisfies $f_S \geq q$ on S.*

Proof. Given $z \in S$, taking $w := z$ in relation (6) with $f := q_S$, we get $f_S(z) := q_S^b(z) \geq -(q(z) - b(z, z)) = q(z)$ as above. □

Let us note that S is conservative if, and only if, $f_S(z) \leq q(z)$ for every $z \in S$, or, equivalently, in view of the preceding proposition, if, and only if, $f_S(z) = q(z)$ for every $z \in S$. Other characterizations generalizing [25, Prop. 4] and using the function $g_S := q_S^{bb}$ introduced there in the special case just mentioned are presented in the next proposition.

Proposition 3.7. *For any subset S of a balanced space (Z, b) the following assertions are equivalent:*

- (a) *S is conservative;*
- (b) $f_S \leq q$ *on* S;
- (b') $f_S = q$ *on* S;
- (c) $f_S \leq q_S$;
- (d) $f_S \leq g_S$;
- (e) $g_S(w) + g_S(z) \geq b(w, z)$ *for every* $w, z \in Z$.

Proof. The implication $(a) \Rightarrow (b)$ stems from the implication $r + s \geq t \Rightarrow s \geq -(r - t)$ for every $r, s, t \in \mathbb{R}_{-\infty}$, so that for all $w, z \in S$, one deduces from the relation $q(z) + q(w) \geq b(w, z)$ the inequality $q(z) \geq -\inf\{q(w) - b(w, z) : w \in S\} := q_S^b(z) = f_S(z)$. The equivalence $(b) \Leftrightarrow (b')$ is a consequence of the preceding proposition, while the equivalence $(b) \Leftrightarrow (c)$ is obvious. The implication $(c) \Rightarrow (d)$ stems from the fact that the conjugacy $f \mapsto f^b$ is antitone. For $(d) \Rightarrow (e)$ we observe that, since $f_S = q_S^b = q_S^{bbb} = g_S^b$, the relation $g_S \geq f_S$ can be written $g_S(z) \geq -(g_S(w) - b(w, z))$ for every $w, z \in Z$. Since for $r, s, t \in \mathbb{R}_{-\infty}$ the implication $s \geq -(r - t) \Rightarrow r + s \geq t$ holds, the preceding inequality can be written as in (e). To prove the implication $(e) \Rightarrow (a)$ we note that since $q_S \geq q_S^{bb} = g_S$, for $w, z \in S$, we have

$$q(w) + q(z) = q_S(w) + q_S(z) \geq g_S(w) + g_S(z) \geq b(w, z),$$

so that S is conservative. □

Using (d) and taking $w = z$ in (e) one gets $2g_S(z) \geq b(z, z) = 2q(z)$ and the first assertion of the next corollary, mentioned in [28, Corollary 1] (with a misprint) in the case Z is a self-paired product space. The second assertion follows from the relation $q_S \geq q_S^{bb} = g_S$.

Corollary 3.8. *For any conservative subset S of a balanced space (Z, b) one has $g_S \geq f$ and $g_S \geq q$. Moreover f_S and g_S coincide with q on S.*

When S is maximal conservative, one has another property, similar to the one detected in [10] and [25, Thm. 5].

Theorem 3.9. *If S is a maximal conservative subset of a balanced space (Z, b) for which b is finite, one has $g_S \geq f_S \geq q$. Moreover*

$$S = \{z \in Z : f_S(z) = q(z)\} = \{z \in Z : g_S(z) = q(z)\}.$$

Proof. We already know that $f_S = q$ on S. Now a conservative subset S is maximal conservative if, and only if, for all $z \in Z \backslash S$ there exists some $w \in S$ such that $q(w) + q(z) < b(w, z)$. When that occurs, since $b(w, z)$ is finite, one has $q(w) - b(w, z) < -q(z)$ hence $f_S(z) > q(z)$. Thus $\{z \in Z : f_S(z) = q(z)\} \subset S$ and equality holds. Since $g_S \geq f_S \geq q$, one has

$$\{z \in Z : g_S(z) = q(z)\} \subset \{z \in Z : f_S(z) = q(z)\} \subset S$$

and since g_S coincides with q on S by Corollary 3.8, the second equality follows. □

Let us give a converse to the preceding theorem.

Theorem 3.10. *Let S be a subset of a balanced space (Z, b) such that $f_S \geq q$ and $S = \{z \in Z : f_S(z) = q(z)\}$. Then S is maximal conservative.*

Proof. Since f_S and q coincide on S, S is conservative by Proposition 3.7. Let $w \in Z$ be such that $T := S \cup \{w\}$ is conservative. Then for all $z \in S$ one has $q(w) + q(z) \geq b(w, z)$, and since the implication $r + s \geq t \Rightarrow s \geq -(r - t)$ holds for every $r, s, t \in \mathbb{R}_{-\infty}$, one gets $q(w) \geq -(q(z) - b(w, z))$ for all $z \in S$, hence $q(w) \geq f_S(w)$. Since $f_S \geq q$ one obtains $w \in \{z \in Z : f_S(z) = q(z)\} = S$. Thus S is maximal conservative. □

Although a balanced space has no product structure and no linear structure, one can introduce a notion of dissipative set. This notion has an interest when b is finitely valued, or, more generally, when one is given a function b on Z which takes its values in $\mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$; then we say that (Z, b) is a *twisted* balanced space: $(Z, -b)$ is a balanced space. Again, one defines q by $q(z) := (1/2)b(z, z)$.

Definition 3.11. A subset S of a twisted balanced space (Z, b) is said to be *dissipative* if for every $w, z \in S$ one has

$$q(w) + q(z) \leq b(w, z).$$

Such a definition incites to introduce an adapted conjugacy. Thus, for a function f on Z, we denote by f^{-b} the function given by

$$f^{-b}(z) := -\inf_{w \in Z} (f(w) + b(w, z)).$$

Such a conjugacy satisfies the usual rules of conjugacies. We associate to a subset S of Z the function $\widetilde{q}_S := (-q)_S = -q + \iota_S$ and the twisted Fitzpatrick function and the twisted predominant function on Z respectively by

$$\widetilde{f}_S = \widetilde{q}_S{}^{-b}, \qquad \widetilde{g}_S := \widetilde{q}_S{}^{-b-b} := (\widetilde{q}_S{}^{-b})^{-b} = \widetilde{f}_S^{-b}.$$

Since relation (2) is still valid when b is changed into $-b$ and q is changed into $-q$, one can derive analogues of the preceding results.

Proposition 3.12. *For any subset S of a twisted balanced space (Z, b) the following assertions are equivalent:*

(*a*) *S is dissipative;*
(*b*) *$\widetilde{f}_S \leq \widetilde{q} := -q$ on S;*
(*b′*) *$\widetilde{f}_S = -q$ on S;*
(*c*) *$\widetilde{f}_S \leq \widetilde{q}_S$;*
(*d*) *$\widetilde{f}_S \leq \widetilde{g}_S$;*
(*e*) *$\widetilde{g}_S(w) + \widetilde{g}_S(z) \geq -b(w, z)$ for every $w, z \in Z$.*

Using (*d*) and taking $w = z$ in (*e*) one gets $2\widetilde{g}_S(z) \geq -b(z, z) = -2q(z)$ and the next corollary.

Corollary 3.13. *For any dissipative subset S of a twisted balanced space (Z, b) one has $\widetilde{g}_S \geq \widetilde{f}_S$, $\widetilde{g}_S \geq -q$. Moreover $\widetilde{f}_S$ and $\widetilde{g}_S$ coincide with $-q$ on S.*

When S is maximal dissipative, one has another property, similar to the one detected in [10] and [25, Thm. 5].

Theorem 3.14. *If S is a maximal dissipative subset of a twisted balanced space (Z, b) one has $\widetilde{g}_S \geq \widetilde{f}_S \geq -q$. Moreover $S = \{z \in Z : \widetilde{f}_S(z) = -q(z)\} = \{z \in Z : \widetilde{g}_S(z) = -q(z)\}$.*

Conversely, let S be a dissipative subset of a twisted balanced space (Z, b) such that $\widetilde{f}_S \geq -q$ and $S = \{z \in Z : \widetilde{f}_S(z) = -q(z)\}$. Then S is maximal dissipative.

4. Topological balanced spaces

We consider now the benefit one can draw from an adapted topology on a balanced space.

Definition 4.1. A topological balanced space is a balanced space (Z, b) endowed with a topology $\mathcal{T}$ for which the mapping b is finitely valued, separately continuous and q is continuous. The weakest topology on Z satisfying these properties is called the natural topology.

In [27], given a self-paired product space formed by the product of a n.v.s. X with its dual space Y and the canonical coupling $\langle \cdot, \cdot \rangle$, we defined the natural topology on $Z := X \times Y$ as the weakest topology for which the functions $(x, y) \mapsto \langle x, v \rangle$ $(v \in Y)$, $(x, y) \mapsto \langle u, y \rangle$ $(u \in X)$ and $(x, y) \mapsto \langle x, y \rangle$ are continuous. Then, for every $w := (u, v) \in Z$, the partial mappings

$$z := (x, y) \mapsto b(w, z) := \langle x, v \rangle + \langle u, y \rangle,$$
$$z := (x, y) \mapsto b(z, w) := \langle u, y \rangle + \langle x, v \rangle$$

and the map q are continuous. Conversely, if these maps are continuous, then, for all $w := (u, v) \in Z$, the functions $(x, y) \mapsto \langle x, v \rangle$, $(x, y) \mapsto \langle u, y \rangle$ and $(x, y) \mapsto \langle x, y \rangle$ are continuous since $\langle x, v \rangle = b((0, v), (x, y))$, $\langle u, y \rangle = b((x, y), (u, 0))$ and $\langle x, y \rangle = q(x, y)$. Therefore, the definition we give here coincides with the one given in [27] in the case of a self-paired product space built as above. Albeit the natural topology does not give the structure of a topological vector space to $Z := X \times Y$, the translations and the dilations of Z are continuous and the dual of Z (i.e. the set of linear forms which are continuous for the natural topology) coincides with Z.

The framework of topological balanced spaces suffices to get an extension of [27, Prop. 3].

Proposition 4.2. *The closure $\overline{S}$ of a conservative subset S of a topological balanced space $(Z, b, \mathcal{T})$ is conservative. In particular, the natural closure of S (i.e. the closure of S for the natural topology) is conservative.*

Proof. Let $w, z \in \overline{S}$ and let $(w_i)_{i \in I}$, $(z_j)_{j \in J}$ be nets in S converging to w and z respectively. Passing to the limit on I and then to the limit on J we get

$$b(w, z_j) = \lim_{i \in I} b(w_i, z_j) \leq \lim_{i \in I} q(w_i) + q(z_j) = q(w) + q(z_j),$$
$$b(w, z) = \lim_{j \in J} b(w, z_j) \leq q(w) + \lim_{j \in J} q(w_j) = q(w) + q(z),$$

so that $\overline{S}$ is conservative. □

Corollary 4.3. *If S is a maximal conservative subset of a topological balanced space $(Z, b, \mathcal{T})$, then S is closed. In particular, S is closed in the natural topology.*

5. A decomposition property

In the sequel, we are given a topological balanced space $(Z, b, \mathcal{T})$. For the sake of simplicity, we assume b takes only finite values. We denote by $A_b(Z)$ the set of b-affine functions on Z, i.e. the set of functions of the form $z \mapsto b(w, z) + r$ for some $w \in Z$, $r \in \mathbb{R}$, and we denote by $\Gamma_b(Z)$ the set of suprema of families of elements of $A_b(Z)$ which are proper, i.e. which do not take the value $-\infty$ and take at least one finite value. We observe that the elements of $\Gamma_b(Z)$ are lower semicontinuous (l.s.c.) for the natural topology on Z. Let us denote by $\beta : Z \to A_b(Z) \subset \mathbb{R}^Z$ the map $z \mapsto b(z, \cdot)$ and write $L_b(Z)$ for the linear space generated by the image of β.

Let us introduce the following sandwich property. Here, for a function $h : Z \to \overline{\mathbb{R}}$, we set $\operatorname{dom} h := \{z \in Z : h(z) < +\infty\}$. We say that the topological balanced space $(Z, b, \mathcal{T})$ satisfies the *sandwich property* if the following condition holds:

(S) for every $f \in \Gamma_b(Z)$, $g \in \Gamma_{-b}(Z)$ such that $\mathbb{R}_+(\beta(\operatorname{dom} f) - \beta(\operatorname{dom} g)) = L_b(Z)$ and $f \geq -g$ there exist some $z \in Z$, $r \in \mathbb{R}$ such that $f \geq b(\cdot, z) + r \geq -g$.

Such a condition is inspired by the Attouch-Brézis condition of convex analysis since it is satisfied whenever X is a Fréchet space and $Z := X \times X^*$ with the natural coupling. Note that we use $-b$ because $b(\cdot, -z)$ has no meaning in a balanced space. In a Simons space one has $\Gamma_{-b}(Z) = \Gamma_b(Z)$ and condition (S) takes a more familiar form. Condition (S) yields a rough form of the decomposition result of [34, Thm. 19.5]. It has links with general duality properties (see [11], [12] and their references). As above, for a function h on Z, we set

$$M_h := \{z \in Z : h(z) = q(z)\}.$$

Proposition 5.1. *Suppose $(Z, b, \mathcal{T})$ satisfies the sandwich property. Let $f \in \Gamma_b(Z)$, $g \in \Gamma_{-b}(Z)$ be such that $\mathbb{R}_+(\beta(\operatorname{dom} f) - \beta(\operatorname{dom} g)) = L_b(Z)$ and $f^b \geq f \geq q$, $g^{-b} \geq g \geq -q$. Then one has $M_f \cap M_{-g} \neq \varnothing$.*

Proof. In view of our assumption, we have $f \geq q \geq -g$ and since the Attouch-Brézis type qualification condition holds, we get some $z \in Z$ and $r \in \mathbb{R}$ such that $f \geq b(\cdot, z) + r \geq -g$. Then we have $-r \geq f^b(z)$ and $r \geq g^{-b}(z)$. Therefore $-r \geq f(z) \geq q(z)$ and $q(z) \geq -g(z) \geq -g^{-b}(z) \geq -r$ and these inequalities are equalities. Thus $z \in M_f \cap M_{-g}$. □

When (Z, b) is a Simons space, the preceding result is a crucial step towards a decomposition of Z into $M_f - M_{-g}$ ([34, Thm. 19.16]). In turn, such a result entails an important criterion for maximality of a conservative (or positive) subset. For reaching such aims, a richer structure on a balanced space is required. Here, instead of a vector space structure of Z and the linearity of b as in a Simons space, we suppose that Z operates on itself. That means that we suppose that a map $t : Z \times Z \to Z$ is given; we set $t_w(z) := t(w, z)$ but we do not suppose $(t_w)_{w \in Z}$ is a group of transformations, although t can be imagined as a generalization of translations in the case Z is vector space or a group. Instead, we assume a compatibility between b and these transformations. Namely, we suppose that for every $a, w, z \in Z$ one has

$$b(t_a(w), t_a(z)) \geq b(w, z) + b(w, a) + b(a, z) + b(a, a), \tag{7}$$

$$q(t_a(z)) \leq q(z) + q(a) + b(a, z). \tag{8}$$

Note that taking $w = z$ in (7) yields the opposite inequality in (8), so that (8) does express a certain compatibility between b and the "translations" t_a.

Lemma 5.2. *Given $a \in Z$ such that $t_a(Z) = Z$ and a function g on Z such that $g^{-b} \geq g$, let h be the function given by $h(z) := g(t_a(z)) + b(z, a) + q(a)$. If (7) holds, then one has $h^{-b} \geq h$. If moreover one assumes that $g \geq -q$ and (8) holds, then one has $h \geq -q$ and $t_a(M_{-h}) \subset M_{-g}$.*

Proof. Since $t_a(Z) = Z$, for all $v \in Z$ one can find $w \in Z$ such that $t_a(w) = v$ so that equality holds between the second and the third lines below:

$$\begin{aligned} -h^{-b}(z) &= \inf_{w \in Z} \{b(w, z) + g(t_a(w)) + b(w, a) + q(a)\} \\ &\leq \inf_{w \in Z} \{b(t_a(w), t_a(z)) - b(z, a) - b(a, a) + g(t_a(w)) + q(a)\} \\ &= \inf_{v \in Z} \{b(v, t_a(z)) + g(v) - b(z, a) - q(a)\} \\ &= -g^{-b}(t_a(z)) - b(z, a) - q(a) \\ &\leq -(g(t_a(z)) + b(z, a) + q(a)) = -h(z). \end{aligned}$$

The relation $h \geq -q$ is a direct consequence of the definition of h, of the relation $g \geq -q$ and of (8). Now, given $z \in M_{-h}$ one has

$$g(t_a(z)) = h(z) - b(z, a) - q(a) = -q(z) - b(z, a) - q(a) \leq -q(t_a(z))$$

hence $t_a(z) \in M_{-g}$ since $g \geq -q$. $\square$

Now suppose that for all $z \in Z$ the transformation t_z is invertible and that the following relation holds for all $w, z \in Z$:

$$w = t_a(z) \Longleftrightarrow a = t_z^{-1}(w). \tag{9}$$

Corollary 5.3. *Suppose the transformations $(t_z)_{z \in Z}$ are invertible and satisfy relations (7) and (8) and $(Z, b, \mathcal{T})$ satisfies the sandwich property. Let $f \in \Gamma_b(Z)$, $g \in \Gamma_{-b}(Z)$ be such that* $\operatorname{dom} g = Z$ *and $f^b \geq f \geq q$, $g^{-b} \geq g \geq -q$. Then for all $a \in Z$, one has $t_a(M_f) \cap M_{-g} \neq \varnothing$.*

If moreover (9) is satisfied, then for all $a \in Z$ there exist $w \in M_{-g}$ and $z \in M_f$ such that $a = t_z^{-1}(w)$.

Proof. Given $a \in Z$, let us apply Proposition 5.1 with g replaced by h as defined in the preceding lemma. We can do so because $\operatorname{dom} h = t_a^{-1}(\operatorname{dom} g) = Z$. Taking $z \in M_f \cap M_{-h}$, we know that $w := t_a(z) \in M_{-g}$. When (9) holds, we get $a = t_z^{-1}(w)$. □

Theorem 5.4. *Let (Z, b) be a Simons space endowed with a topology $\mathcal{T}$ for which b is continuous and the sandwich property holds. Let F (resp. G) be a maximal conservative (resp. dissipative) subset of Z. Suppose the twisted Fitzpatrick function g of G is finite. Then*

$$Z = F - G.$$

Proof. For $w, z \in Z$ let us set $t_z(w) = w + z$. Then t_z is invertible and $t_z^{-1}(v) = v - z$; moreover (9) holds. The bilinearity of b ensures that relations (7) and (8) are satisfied. Let f be the Fitzpatrick function of F and let g be the twisted Fitzpatrick function of G. Since $\operatorname{dom} g = Z$ and $f^b \geq f \geq q$, $g^{-b} \geq g \geq -q$, the conclusion of the preceding corollary holds. It reads: for every $a \in Z$ there exist $w \in G$, $z \in F$ such that $a = w - z$. □

Note that the sandwich property holds when $\beta : Z \to Z^*$ is surjective and $\mathcal{T}$ is locally convex. In particular, we get the following well known property.

Corollary 5.5. *Let X be a reflexive Banach space with dual space Y. Let G be the graph of $-J$, where J is the duality mapping of X and let F be the graph of a maximal monotone operator. Then*

$$X \times Y = F + G.$$

Proof. Let $j := (1/2) \|\cdot\|^2$, so that $J = \partial j$. By [1, Prop. 2.1] the Fitzpatrick function of J satisfies

$$f_J(x, y) \leq j(x) + j^*(y).$$

Since $j^* = (1/2) \|\cdot\|^2$, the Fitzpatrick function f_J of J and the twisted Fitzpatrick function $g := \widetilde{f}_G$ of G are everywhere finite, as $\widetilde{f}_G(x, y) = f_J(x, -y)$. Moreover, $J(-x) = -J(x)$ and $G = -G$, as easily checked. □

Although we have used strong assumptions to get this Minty decomposition, the bareness of the concept of balanced space may enable one to detect crucial properties of conservative subsets and to consider new situations in which classes of functions sharing with convex functions some properties play some role, as it is the case with submodular functions. Detecting more links with duality in general spaces may also be of some interest (see [11], [12], [20], [21], [22], [26] for instance).

Acknowledgements. The author is grateful to two anonymous referees for their remarks.

References

[1] H. H. Bauschke, D. A. MacLaren, H. S. Sendov: Fitzpatrick functions: inequalities, examples and remarks on a problem of Fitzpatrick, J. Convex Analysis 13 (2006) 499–523.

[2] J. Borwein: Maximal monotonicity via convex analysis, J. Convex Analysis 13 (2006) 561–586.

[3] J. Borwein: Maximality of sums of two maximal monotone operators in general Banach space, Proc. Amer. Math. Soc. 135 (2007) 3917–3934.

[4] R. M. Bot, S. I. Grad, G. Wanka: Maximal monotonicity for the precomposition with a linear operator, SIAM J. Optim. 17 (2007) 1239–1252.

[5] M. Buliga, G. de Saxcé, C. Vallée: Existence and construction of bipotentials for graphs of multivalued laws, J. Convex Analysis 15(1) (2008) 87–104.

[6] R. S. Burachik, S. Fitzpatrick: On a family of convex functions attached to a subdifferential operator, J. Nonlinear Convex Anal. 6 (2005) 165–172.

[7] R. S. Burachik, A. M. Rubinov: On abstract convexity and set-valued analysis, J. Nonlinear Convex Anal. 9(1) (2008) 105–123.

[8] R. S. Burachik, B. F. Svaiter: Maximal monotone operators, convex functions and a special family of enlargements, Set-Valued Anal. 10(4) (2002) 297–316.

[9] R. S. Burachik, B. F. Svaiter: Maximal monotonicity, conjugation and the duality product, Proc. Amer. Math. Soc. 131(8) (2003) 2379–2383.

[10] S. Fitzpatrick: Representing monotone operators by convex functions, in: Functional Analysis and Optimization, Workshop / Miniconference (Canberra, 1988), Proc. Cent. Math. Anal. Aust. Natl. Univ. 20, Australian National University, Canberra (1988) 59–65.

[11] F. Flores-Bazán: On a notion of subdifferentiability for non-convex functions, Optimization 33 (1995) 1–8.

[12] F. Flores-Bazán, J. E. Martínez-Legaz: Simplified global optimality conditions in generalized conjugation theory, in: Generalized Convexity, Generalized Monotonicity: Recent Results, J.-P. Crouzeix, J. E. Martínez-Legaz, M. Volle (eds.), Kluwer, Dordrecht (1998) 303–329.

[13] E. Krauss: A representation of maximal monotone operators by saddle functions, Rev. Roum. Math. Pures Appl. 30 (1985) 823–836.

[14] E. Krauss: A representation of arbitrary maximal monotone operators via subgradients of skew-symmetric saddle functions, Nonlinear Anal., Theory Methods Appl. 9 (1985) 1381–1399.

[15] E. Krauss: Maximal monotone operators and saddle functions. I., Z. Anal. Anwend. 5 (1986) 333–346.

[16] J.-E. Martínez-Legaz: Fenchel duality and related properties in generalized conjugation theory, Southeast Asian Bull. Math. 19 (1995) 99–106.

[17] J.-E. Martínez-Legaz, I. Singer: Dualities between complete lattices, Optimization 21 (1990) 481–508.

[18] J.-E. Martínez-Legaz, B. F. Svaiter: Monotone operators representable by l.s.c. convex functions, Set-Valued Anal. 13(1) (2005) 21–46.

[19] J. E. Martinez-Legaz, M. Théra: A convex representation of maximal monotone operators, J. Nonlinear Convex Anal. 2 (2001) 243–247.

[20] J.-J. Moreau: Inf-convolution, sous-additivité, convexité des fonctions numériques, J. Math. Pures Appl. 49 (1970) 109–154.

[21] D. Pallaschke, S. Rolewicz: Foundations of Mathematical Optimization: Convex Analysis Without Linearity, Kluwer, Dordrecht (1998).

[22] J.-P. Penot: Rotundity, smoothness and duality, Control Cybern. 32(4) (2003) 711–733.

[23] J.-P. Penot: Autoconjugate functions and representations of monotone operators, Bull. Aust. Math. Soc. 67 (2003) 277–284.

[24] J.-P. Penot: A representation of maximal monotone operators by closed convex functions and its impact on calculus rules, C. R., Math., Acad. Sci. Paris 338(11) (2004) 853–858.

[25] J.-P. Penot: The relevance of convex analysis for the study of monotonicity, Nonlinear Anal., Theory Methods Appl. 58A(7–8) (2004) 855–871.

[26] J.-P. Penot: Unilateral analysis and duality, in: GERAD, Essays and Surveys in Global Optimization, G. Savard et al. (eds.), Springer, New York (2005) 1–37.

[27] J.-P. Penot: Natural closure, natural compositions and natural sums of monotone operators, J. Math. Pures Appl. 89(6) (2008) 523–537.

[28] J.-P. Penot: Monotonicities and dualities, in: Generalized Convexity and Related Topics, I. V. Konnor, D. T. Luc, A. M. Rubinov (eds.), Lecture Notes in Economics and Math. Systems 583, Springer, Berlin (2007) 300–414.

[29] J.-P. Penot, C. Zălinescu: Some problems about the representation of monotone operators by convex functions, ANZIAM J. 47 (2005) 1–20.

[30] J.-P. Penot, C. Zălinescu: On the convergence of maximal monotone operators, Proc. Amer. Math. Soc. 134(7) (2005) 1937–1946.

[31] S. Rolewicz: Convex analysis without linearity, Control Cybern. 23 (1994) 247–256.

[32] S. Rolewicz: Duality and convex analysis in the absence of linear structure, Math. Jap. 44(1) (1996) 165–182.

[33] S. Simons: Positive sets and monotone sets, J. Convex Analysis 14 (2007) 297–317.

[34] S. Simons: From Hahn-Banach to Monotonicity, Lecture Notes in Math. 1693, Springer, Berlin (2008).

[35] S. Simons, C. Zălinescu: A new proof for Rockafellar's characterization of maximal monotone operators, Proc. Amer. Math. Soc. 132 (2004) 2969–2972.

[36] S. Simons, C. Zălinescu: Fenchel duality, Fitzpatrick functions and maximal monotonicity, J. Nonlinear Convex Anal. 6(1) (2005) 1–22.

[37] C. Zălinescu: A new proof of the maximal monotonicity of the sum using the Fitzpatrick function, in: Variational Analysis and Applications, F. Giannessi, A. Maugeri (eds.), Springer, New York (2005) 1159–1172.