Journal of Convex Analysis
Volume 16 (2009), No. 3&4, 1003–1009

Regular Maximal Monotone Multifunctions and Enlargements

Andrei Verona
Department of Mathematics, California State University, Los Angeles, CA 90032, USA
averona@calstatela.edu

Maria Elena Verona
Department of Mathematics, University of Southern California, Los Angeles, CA 90089-1113, USA
verona@usc.edu

Dedicated to Stephen Simons on the occasion of his 70th birthday.

Received: May 20, 2008
Revised manuscript received: December 9, 2008

In this note we use recent results concerning the sum theorem for maximal monotone multifunctions in general Banach spaces to find new characterizations and properties of regular maximal monotone multifunctions and then use these to describe the domain of certain enlargements.

Keywords: Maximal monotone multifunctions, regular maximal monotone multifunctions, enlargements of maximal monotone multifunctions

2000 Mathematics Subject Classification: 47H05, 49J52, 47N10

1. Introduction

Throughout this note X will denote a Banach space with topological dual X^*. For $(x, x^*) \in X \times X^*$, $\langle x, x^* \rangle$ (or $\langle x^*, x \rangle$) will denote the natural evaluation map. The unit ball in X^* is denoted B^*. If $T : X \rightrightarrows X^*$ is a multifunction then $G(T) = \{(x, x^*) \in X \times X^* : x^* \in T(x)\}$ is called its *graph*, while $D_T = \{x \in X : T(x) \neq \emptyset\}$ is called its *domain*. The multifunction T is called *monotone* if $\langle x^* - y^*, x - y \rangle \geq 0$ for all $(x, x^*), (y, y^*) \in G(T)$; T is called *maximal monotone* if there exists no monotone multifunction $S : X \rightrightarrows X^*$ such that $G(S)$ strictly contains $G(T)$. The theory of maximal monotone multifunctions in reflexive spaces is now more or less complete, due to Rockafellar's results from the 1960s and 1970s. However, in general Banach spaces, the theory is much more complicated. In order to extend Rockafellar's results to general Banach spaces, different authors have introduced several classes of maximal monotone operators and proved some of his results in these particular cases (see [9], [10] for a detailed description of these classes and results and also [13], [14] and [15] for properties of regular maximal monotone multifunctions).

In this note we use recent results concerning the sum theorem for maximal monotone multifunctions in general Banach spaces to find new characterizations for regular maximal monotone multifunctions (Theorem 1.2, Corollary 1.3, Theorem 1.4). We also prove that

ISSN 0944-6532 / $ 2.50

the sum of a regular maximal monotone multifunction and a bounded maximal monotone multifunction is maximal. In the last part we characterize a type of enlargements (Theorem 1.7), give a characterization of regularity in terms of certain enlargements (Theorem 1.8), and show that the domain of another type of enlargements is contained in the closure of the domain of the initial multifunction (Theorem 1.9).

It is worth mentioning that if a sum theorem for maximal monotone multifunction in general Banach spaces was proved then any maximal monotone multifunction would be regular and therefore all our results in this note would be true for any maximal monotone multifunction.

Regular maximal monotone multifunctions

In [13] we introduced the following number

$$L(x, x^*, T) = 0 \vee \sup \left\{ \frac{\langle x^* - y^*, y - x \rangle}{\|x - y\|} \,:\, y \neq x, (y, y^*) \in G(T) \right\}$$

(here $a \vee b = \max\{a, b\}$) and proved that $L(x, x^*, T) \leq \inf\{\|y^* - x^*\| : y^* \in T(x)\} = d(x^*, T(x))$. (When T is the subdifferential of a proper, convex, lower semicontinuous function this number was considered by Simons [11].) The maximal monotone multifunction T is called *regular* if $L(x, x^*, T) = \inf\{\|y^* - x^*\| : y^* \in T(x)\} = d(x^*, T(x))$ for any $(x, x^*) \in X \times X^*$. Here is a list with some relevant facts about regular maximal monotone multifunctions:

- If T is a regular maximal monotone multifunction then $\overline{D_T}$ is convex [13, Theorem 2].
- Any maximal monotone multifunction in a reflexive Banach space is regular [13, Corollary 1(2)].
- The subdifferential of any proper convex lower semicontinuous function is regular [13, Theorem 6].
- If T is a regular maximal monotone multifunction and $x \in \overline{D_T}$ then T is locally bounded at x if and only if $x \in \text{int} D_T$ [13, Corollary 3].
- If T is a linear (possibly discontinuous) and maximal monotone multifunction then T is regular [14, Proposition 3.2].
- If T is a strongly-representable maximal monotone multifunction then T is regular (see Remark 7 in [17]). Since any maximal monotone multifunction of type (NI) is strongly-representable (Proposition 26 in [17]) it follows that any maximal monotone multifunction of type (NI) is regular (recently it was proved [5, Theorem 1.2] that the class of strongly-representable monotone multifunctions coincides with the class of maximal monotone multifunction of type (NI)). In particular any maximal monotone multifunction of type (D) is regular (since any maximal monotone multifunction of type (D) is of type (NI), [10, Theorem 36.3(a)]).

We shall prove in this note that:

- If T is a maximal monotone multifunction and D_T is either closed and convex or has non-empty interior then T is regular (Corollary 1.3).
- If T is a maximal monotone multifunction then T is regular if and only if it is dually strongly maximal (Theorem 1.4).

Theorem 1.1. *Let $T : X \rightrightarrows X^*$ be a regular maximal monotone multifunction and $S : X \rightrightarrows X^*$ be a bounded maximal monotone multifunction. Then $T + S$ is maximal monotone.*

Proof. In view of the Debrunner-Flor Extension Theorem (see [7, Lemma 1.7]), $D_S = X$. Let M be such that $\|u^*\| \le M$ for any $(y, u^*) \in G(S)$. Let $(x, x^*) \in X \times X^*$ be monotonically related to $G(T + S)$, that is

$$\langle y^* + u^* - x^*, y - x\rangle \ge 0 \text{ for any } (y, y^*) \in G(T),\ (y, u^*) \in G(S).$$

Then

$$\langle y^* - x^*, x - y\rangle \le \langle u^*, y - x\rangle \le \|u^*\|\,\|y - x\| \le M\|y - x\|$$

from which it follows that $L(x, x^*, T) \le M$. Since T is regular, this implies that $d(x^*, T(x)) \le M < \infty$ and therefore $T(x) \ne \emptyset$. Thus $x \in D(T)$. Our assertion follows now from Theorem 24.1(c) in [10].

For any $x \in X$ and $\lambda > 0$ consider the following convex lower semicontinuous function: $g_{\lambda,x}(z) = \lambda\|z - x\|,\ z \in X$. It is known that

$$\partial g_{\lambda,x}(z) = \{z^* \in X^* : \|z^*\| \le \lambda \text{ and } \langle z^*, z - x\rangle = \lambda\|z - x\|\} \ne \emptyset$$

and in particular

$$\partial g_{\lambda,x}(x) = \lambda B^*.$$

Theorem 1.2. *A maximal monotone multifunction $T : X \rightrightarrows X^*$ is regular if and only if $T + \partial g_{\lambda,x}$ is maximal monotone for any $x \in X$ and $\lambda > 0$.*

Proof. The "only if" part follows from the previous theorem. The "if" part was essentially proved in [13]. Since it is quite short, we shall repeat it here. To this end, let $(x, x^*) \in X \times X^*$. We have to show that $L(x, x^*, T) = d(x^*, T(x))$. If $L(x, x^*, T) = \infty$ there is nothing to prove since, as mentioned earlier, $L(x, x^*, T) \le d(x^*, T(x))$. So, assume that $L(x, x^*, T) = \lambda < \infty$. A direct computation (see also Lemma 4 in [13]) shows that $L(x, x^*, T + \partial g_{\lambda,x}) = 0$ which means that (x, x^*) is monotonically related to $T + \partial g_{\lambda,x}$. Since, by hypothesis, $T + \partial g_{\lambda,x}$ is maximal, it follows that $x^* \in (T + \partial g_{\lambda,x})(x) = T(x) + \lambda B^*$ and therefore $d(x^*, T(x)) \le \lambda = L(x, x^*, T)$. This completes the proof.

Corollary 1.3. *If $T : X \rightrightarrows X^*$ is a maximal monotone multifunction and D_T is either closed and convex or has non-empty interior then T is regular.*

Proof. If D_T is closed and convex this follows immediately from Theorem 1.2 and Voisei's result (the sum of two maximal monotone multifunctions with closed convex domains that satisfy the usual constraint qualification is maximal, see [16]). If D_T has non-empty interior our assertion follows from our Theorem 1.2 and Theorem 9(i) in [1].

Strongly maximal monotone multifunctions

We begin this section by recalling the following definition, due to Simons (see [9], [10]): a monotone multifunction $T : X \rightrightarrows X^*$ is called *strongly maximal* if the following two conditions are satisfied

(SM1) "whenever $C \subset X$ is nonempty, convex and $w(X, X^*)$-compact, $x_0^* \in X^*$, and $(y, y^*) \in G(T)$ there exists $x = x_{y,y^*} \in C$ such that $\langle y^* - x_0^*, y - x\rangle \geq 0$" then there exists $x_0 \in C$ such that $x_0^* \in T(x_0)$;

(SM2) "whenever $C \subset X^*$ is nonempty, convex and $w(X^*, X)$-compact, $x_0 \in X$, and $(y, y^*) \in G(T)$ there exists $x^* = x^*_{y,y^*} \in C$ such that $\langle y^* - x^*, y - x_0\rangle \geq 0$" then there exists $x_0^* \in C$ such that $x_0^* \in T(x_0)$.

Theorem 1.4. *A maximal monotone multifunction $T : X \rightrightarrows X^*$ is regular if and only if it satisfies condition* (SM2).

Proof. It was proved in [13] (Proposition 1) that if T satisfies condition (SM2) then it is regular. Conversely, assumes that T is regular. We shall adapt a proof of Simons of this assertion in the case when T is a subdifferential (see for example [10]). Let $x_0 \in X$ and $C \subset X^*$ be nonempty, convex and $w(X^*, X)$-compact that satisfy the assumption in (SM2). Define a convex, lower semicontinuous function $f : X \to R$ by $f(x) = \max\langle x_0 - x, C\rangle = \max\langle x - x_0, -C\rangle$. It is known and not difficult to see that

$$(*) \qquad u^* \in \partial f(x) \text{ if and only if } u^* \in -C \text{ and } \langle u^*, x - x_0\rangle = f(x).$$

On the other hand, if $y^* \in T(y)$ and $x^* = x^*_{y,y^*} \in C$ is as in the assumption of (SM2), then $\langle y^* - x^*, y - x_0\rangle \geq 0$. It follows that

$$(**) \qquad \langle y^*, x_0 - y\rangle \leq \langle x^*, x_0 - y\rangle \leq f(y).$$

From $(*)$ and $(**)$ we get that for any $y^* \in T(y)$ and $u^* \in \partial f(y)$ we have

$$\langle y^* + u^*, y - x_0\rangle = \langle y^*, y - x_0\rangle - \langle u^*, x_0 - y\rangle = \langle y^*, y - x_0\rangle + f(y) \geq 0$$

which means that the pair $(x_0, 0)$ is monotonically related to $T + \partial f$. Since T is regular and ∂f is bounded, the maximality of $T + \partial f$ implies that $0 = x_0^* + u^*$ with $x_0^* \in T(x_0)$ and $u^* \in \partial f(x_0) = -C$. Thus $x_0^* \in T(x_0) \cap C$ and the theorem is proved.

Corollary 1.5. *Any strongly maximal monotone multifunction is regular. In particular, any maximal monotone multifunction whose graph is convex is regular.*

Proof. The first assertion is obvious while the second one follows from the previous theorem and Theorem 46.1 in [10].

Enlargements of regular maximal monotone multifunctions

Let $T : X \rightrightarrows X^*$ be a monotone multifunction. Recall that an *enlargement* of T is a multifunction $E : [0, \infty) \times X \rightrightarrows X^*$ such that $T(x) \subseteq E(\varepsilon, x)$ for any x and any $\varepsilon \geq 0$. An enlargement $E : [0, \infty) \times X \rightrightarrows X^*$ is called a *full enlargement* of T if for any $x \in D_T$ and for any $\varepsilon > 0$ there exists $\delta = \delta(x, \varepsilon) > 0$ such that $T(x) + \delta B^* \subseteq T^\varepsilon$. Enlargements were first considereded in [4] (especially in the case of subdifferentials) and [6] and later systematically studied in [2], [3] (these papers contain further references); see also [8].

Basically, there are two types of enlargements that are considered. The first one is defined as follows:

$$E(\varepsilon, x) = T^\varepsilon(x) = \{x^* \in X^* : \langle x^* - y^*, x - y\rangle \geq -\varepsilon\|x - y\|,\ (y, y^*) \in G(T)\}.$$

Lemma 1.6. *With the above notation:* $T(x) + \varepsilon B^* \subseteq T^\varepsilon(x)$, *for any* $x \in D_T$. *In particular,* $\{T^\varepsilon\}_{\varepsilon \geq 0}$ *is a full enlargement of* T.

Proof. Let $(x, x^*) \in G(T)$ and $u^* \in \varepsilon B^*$. Then, for any $(y, y^*) \in G(T)$ we have

$$\langle x^* + u^* - y^*, x - y\rangle = \langle x^* - y^*, x - y\rangle + \langle u^*, x - y\rangle \geq -\varepsilon\|x - y\|$$

which shows that $x^* + u^* \in T^\varepsilon(x)$.

Theorem 1.7. *Let* $T : X \rightrightarrows X^*$ *be a regular maximal monotone multifunction,* $x \in X$, $\varepsilon \geq 0$, *and* $x^* \in T^\varepsilon(x)$. *Then*

(*a*) $x \in D_T$.

(*b*) $T^\varepsilon(x) = T(x) + \varepsilon B^*$.

(*c*) *The enlargement* $\{T^\varepsilon\}_{\varepsilon \geq 0}$ *satisfies the following monotonicity property:*

$$\langle x^* - y^*, x - y\rangle \geq -(\varepsilon + \delta)\|x - y\|, \quad x^* \in T^\varepsilon(x),\ y^* \in T^\delta(y).$$

(*d*) *The enlargement* $\{T^\varepsilon\}_{\varepsilon \geq 0}$ *is maximal in the following sense: if* $(x, x^*) \in X \times X^*$ *and*

$$\langle x^* - y^*, x - y\rangle \geq -(\varepsilon + \delta)\|x - y\| \text{ for any } \delta \geq 0 \text{ and } y^* \in T^\delta(y)$$

then $x^* \in T^\varepsilon(x)$.

Proof. It is easily seen that

$$x^* \in T^\varepsilon(x) \text{ if and only if } L(x, x^*, T) \leq \varepsilon$$

Thus, since T is regular, if $x^* \in T^\varepsilon(x)$ it follows that $d(x^*, T(x)) = L(x, x^*, T) \leq \varepsilon$ and therefore $x \in D_T$ and $x^* \in T(x) + \varepsilon B^*$ (since $T(x)$ is weak* closed and εB^* is weak* compact). This proves (a) and in view of Lemma 1.6, also (b).

To prove (c), let $x^* \in T^\varepsilon(x)$ and $y^* \in T^\delta(y)$. In view of part (b), $x^* = x_1^* + u^*$ and $y^* = y_1^* + v^*$ with $x_1^* \in T(x)$, $y_1^* \in T(y)$, $u^* \in \varepsilon B^*$, $v^* \in \delta B^*$. Then $\|u^* - v^*\| \leq \varepsilon + \delta$ and

$$\begin{aligned}\langle x^* - y^*, x - y\rangle &= \langle x_1^* + u^* - y_1^* - v^*, x - y\rangle \\ &= \langle x_1^* - y_1^*, x - y\rangle + \langle u^* - v^*, x - y\rangle \geq -(\varepsilon + \delta)\|x - y\|\end{aligned}$$

which proves the assertion. Finally, (d) follows from the definition of $T^\varepsilon(x)$.

Remark. One can generalize the definition of $L(x, x^*, T)$ as follows:

$$L(x, x^*, T^\varepsilon) = 0 \vee \sup\left\{\frac{\langle x^* - y^*, y - x\rangle}{\|x - y\|} - \varepsilon - \delta : \delta \geq 0, y \neq x, y^* \in T^\delta(y)\right\} \quad \varepsilon \geq 0$$

It is not difficult to see that if T is regular then $L(x, x^*, T^\varepsilon) = d(x^*, T^\varepsilon(x))$. Indeed,

$$\begin{aligned} &L(x, x^*, T^\varepsilon) \\ &= 0 \vee \sup\left\{ \frac{\langle x^* - y^*, y - x\rangle}{\|x - y\|} - \varepsilon - \delta : \delta \geq 0, y \neq x, y^* \in T^\delta(y) \right\} \\ &= 0 \vee \sup\left\{ \frac{\langle x^* - z^* - u^*, y - x\rangle}{\|x - y\|} - \varepsilon - \delta : \delta \geq 0, y \neq x, z^* \in T(y),\ u^* \in \delta B^* \right\} \\ &= 0 \vee \sup\left\{ \frac{\langle x^* - z^*, y - x\rangle}{\|x - y\|} - \varepsilon + \frac{\langle u^*, x - y\rangle}{\|x - y\|} - \delta : \delta \geq 0, y \neq x, z^* \in T(y),\ u^* \in \delta B^* \right\} \\ &= 0 \vee \sup\left\{ \frac{\langle x^* - z^*, y - x\rangle}{\|x - y\|} - \varepsilon : y \neq x, z^* \in T(y) \right\} \\ &= 0 \vee (L(x, x^*, T) - \varepsilon) = 0 \vee d(x^*, T(x)) - \varepsilon = d(x^*, T(x) + \varepsilon B^*) = d(x^*, T^\varepsilon(x)). \end{aligned}$$

We shall now give a characterization of regularity in terms of the enlargement $\{T^\varepsilon\}_{\varepsilon \geq 0}$.

Theorem 1.8. *A maximal monotone multifunction $T : X \rightrightarrows X^*$ is regular if and only if $D_{T^\varepsilon} = D_T$ for any $\varepsilon > 0$.*

Proof. The "only if" part follows from Theorem 1.7(a). The "if" part will follow from Theorem 1.2 once we show that $T + \partial g_{\lambda,x}$ is maximal monotone for any $x \in X$ and $\lambda > 0$. To this end, let $x \in X$ and $\lambda > 0$ and assume that $(z, z^*) \in X \times X^*$ is monotonically related to $T + \partial g_{\lambda,x}$. Let $(y, y^*) \in G(T)$ and $u^* \in \partial g_{\lambda,x}(y)$. We have

$$\langle z^* - y^*, z - y\rangle = \langle z^* - y^* - u^*, z - y\rangle + \langle u^*, z - y\rangle \geq 0 - \lambda\|z - y\| = -\lambda\|z - y\|$$

which shows that $z^* \in T^\lambda(z)$ and in particular $z \in D_{T^\lambda} = D_T$. Theorem 24.1(c) in [10] proves that $T + \partial g_{\lambda,x}$ is maximal monotone and this finishes the proof.

We shall now turn our attention to another type of enlargements which were studied in [3] and [8]. If $T : X \rightrightarrows X^*$ is a monotone multifunction and $x \in X$ define

$$E(\varepsilon, x) = T_\varepsilon(x) = \{x^* \in X^* : \langle x^* - y^*, x - y\rangle \geq -\varepsilon \text{ for any } (y, y^*) \in G(T)\}.$$

It is worth mentioning that this enlargement belongs to the class $\mathbb{E}(T)$ introduced in [12] while the enlargement $\{T^\varepsilon\}$ considered earlier does not.

Theorem 1.9. *If $T : X \rightrightarrows X^*$ is a regular maximal monotone multifunction, $\varepsilon > 0$ and $T_\varepsilon(x) \neq \emptyset$ then $x \in \overline{D_T}$, that is $D_{T_\varepsilon} \subseteq \overline{D_T}$.*

Proof. Assume not. Then there exists $\delta > 0$ such that $\|x - y\| > \delta$ for any $y \in D_T$. Let $x^* \in T_\varepsilon(x)$. Then

$$\langle x^* - y^*, x - y\rangle \geq -\varepsilon \geq -\frac{\varepsilon}{\delta}\|x - y\| \quad \text{for any } (y, y^*) \in G(T)$$

and therefore $x^* \in T^{\varepsilon/\delta}(x)$. By Theorem 1.7($a$), $x \in D_T$, which is a contradiction. It follows that $x \in \overline{D_T}$.

Remark. A particular case of this result (when X is reflexive) was proved in [3].

Acknowledgements. We would like to thank the referee for carefully reading the manuscript and for making several useful comments.

References

[1] J. M. Borwein: Maximality of sums of two maximal monotone operators in general Banach spaces, Proc. Amer. Math. Soc. 135 (2007) 3917–3924.

[2] R. S. Burachik, A. N. Iusem: On non-enlargeable and fully enlargeable monotone operators, J. Convex Analysis 13 (2006) 603-622.

[3] R. S. Burachik, B. F. Svaiter: ε-enlargements of maximal monotone operators in Banach spaces, Set-Valued Anal. 7 (1999) 117-132.

[4] A. Jofré, D. T. Luc, M Théra: ε-subdifferentials and ε-monotonicity, Set-Valued Anal. 4 (1996) 327–332.

[5] M. Marques Alves, B. F. Svaiter: A new old class of maximal monotone operators, J. Convex Analysis 16(3&4) (2009) 881–890.

[6] W. Oettli, M. Théra: On maximal monotonicity of perturbed mappings, Boll. Unione Mat. Ital., VII. Ser., A 9 (1995) 47–55.

[7] R. R. Phelps: Lectures on Maximal Monotone Operators, Extr. Math. 15 (1997) 193–230.

[8] J. P. Revalski, M. Théra: Enlargements and sums of monotone operators, Nonlinear Anal., Theory Methods Appl. 48A (2002) 505–519.

[9] S. Simons: Minimax and Monotonicity, Lecture Notes in Mathematics 1693, Springer, Berlin (1998).

[10] S. Simons: From Hahn-Banach to Monotonicity, 2nd Ed., Lecture Notes in Mathematics 1693, Springer, Berlin (2008).

[11] S. Simons: The least slope of a convex function and the maximal monotonicity of its subdifferential, J. Optimization Theory Appl. 71 (1991) 127–136.

[12] B. F. Svaiter: A family of enlargements of maximal monotone operators, Set-Valued Anal. 8 (2000) 311-328.

[13] A. Verona, M. E. Verona: Regular maximal monotone operators, Set-Valued Anal. 6 (1998) 302–312.

[14] A. Verona, M. E. Verona: Regular maximal monotone operators and the sum theorem, J. Convex Analysis 7 (2000) 115–128.

[15] A. Verona, M. E. Verona: Regularity and the Brøndsted-Rockafellar properties of maximal monotone operators, Set-Valued Anal. 14 (2006) 149–157.

[16] M. D. Voisei: A maximality theorem for the sum of maximal monotone operators in non-reflexive Banach spaces, Math. Sci. Res. J. 10 (2006) 36–41.

[17] M. D. Voisei, C. Zălinescu: Strongly-representable monotone operators, J. Convex Analysis 16(3&4) (2009) 1011–1033.