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An Existence Result for Equilibrium Problems with Some Surjectivity Consequences

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We present conditions for existence of solutions of equilibrium problems, which are sufficient in finite dimensional spaces, without making any monotonicity assumption on the bifunction which defines the problem. As a consequence we establish surjectivity of set-valued operators of the form $T + \lambda I$, with $\lambda > 0$, where T satisfies a property weaker than monotonicity, which we call pre-monotonicity. We study next the notion of maximal pre-monotonicity. Finally we adapt our condition for non-convex optimization problems, obtaining as a by-product an alternative proof of Frank-Wolfe's Theorem.

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1. Introduction

Let H be a Hilbert space. Take a nonempty, closed and convex set $K \subset H$ and $f : K \times K \to \mathbb{R}$ such that

P1: $f(x, x) = 0$ for all $x \in K$,
P2: $f(\cdot, y) : K \to \mathbb{R}$ is upper semicontinuous for all $y \in K$,
P3: $f(x, \cdot) : K \to \mathbb{R}$ is pseudo-convex and lower semicontinuous for all $x \in K$.

We remind that $h : H \to \mathbb{R}$ is *pseudo-convex* if, given $x, y \in H$ and $\alpha \in (0, 1)$, whenever $h(\alpha x + (1 - \alpha)y) \geq h(x)$ it holds that $h(\alpha x + (1 - \alpha)y) \leq h(y)$.

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We will consider the following alternative to P3:

P3': $f(x, \cdot) : K \to \mathbb{R}$ is quasi-convex and lower semicontinuous for all $x \in K$.

We remind that $h : H \to \mathbb{R}$ is *quasi-convex* if, given $x, y \in H$ and $\alpha \in (0, 1)$, it holds that

$$h\left(\alpha x + (1 - \alpha)y\right) \leq \max\{h(x), h(y)\}. \tag{1}$$

It follows easily that P3 implies P3'.

The equilibrium problem EP(f, K) consists of finding $x^* \in K$ such that $f(x^*, y) \geq 0$ for all $y \in K$. The set of solutions of EP(f, K) will be denoted as $S(f, K)$.

Equilibrium problems include, as particular cases, Nash equilibria problems, complementarity problems, fixed point problems, minimax problems, variational inequality problems (both monotone and non-monotone), and both scalar and vectorial convex optimization problems (see e.g., [8]).

This problem was considered in the past, with slight variations in the assumptions (e.g., convexity of $f(x, \cdot)$ substituting for pseudo-convexity in P3), and under various headings. The issue of necessary and/or sufficient conditions for existence of solutions of EP(f, K) was the starting point in the study of the problem. In 1972, Ky Fan proved in [12] existence of solutions assuming compactness of K (see Lemma 2.2 below), and a short time afterward the same result was established in [9] assuming instead some form of coerciveness of f.

EP(f, K) has been extensively studied in recent years, with emphasis on existence results (e.g. [5], [6], [7], [13], [14], [19], [20]). Recently, new necessary (and in some cases also sufficient) conditions for existence of solutions in infinite dimensional spaces were proposed in [17], and later on simplified and furtherly analyzed in [16]. These conditions play a significant role in our analysis, and appear as conditions P5, P5', P5" in Section 2.

We outline next the contents of this paper. In Section 2 we prove that in a finite dimensional setting P1–P3 and P5" are sufficient for ensuring existence of solutions of EP(f, K), while in the infinite dimensional case additional conditions are needed (P4 or its variants, as defined in Section 2). The same result holds when P3 is weakened to P3' and P5" is strenghtened to P5'. In Section 3 we introduce a class of set-valued operators, which we call pre-monotone, larger than the class of monotone operators. In Section 4 we exploit the existence result proved in Section 2 for extending Minty's Surjectivity Theorem to the class of pre-monotone operators. In Section 5 we sketch the first steps in the study of maximal pre-monotone operators. In Section 6 we reformulate conditions P5, P5" for the case of optimization problems, and show that any of them is necessary and sufficient for existence of minimizers of lower semicontinuous functions, obtaining as a consequence an alternative proof of Frank-Wolfe's Theorem. We close the paper with some final remarks presented in Section 7.

We end this section by recalling some standard concepts related to set-valued operators.

Given $T : H \to \mathcal{P}(H)$, its *domain* $D(T)$ is defined as $D(T) = \{x \in H : T(x) \neq \emptyset\}$, its *range* $R(T)$ as $R(T) = \cup_{x \in H} T(x)$ and its *graph* $G(T)$ as $G(T) = \{(x, u) \in H \times H : u \in T(x)\}$.

T is said to be *single-valued* if $T(x)$ is a singleton for all $x \in D(T)$, *bounded-valued* if $T(x)$ is bounded for all $x \in D(T)$ and *convex-valued* if $T(x)$ is convex for all $x \in D(T)$. T is *locally bounded* in a set $S \subset D(T)$ if for all $x \in S$ there exists a neighborhood V of x such that $\cup_{x \in V \cap S} T(x)$ is bounded; T is *globally bounded* if $R(T)$ is bounded. T is *closed* if $G(T)$ is closed; T is *monotone* if $\langle u - v, x - y\rangle \geq 0$ for all $(x, u), (y, v) \in G(T)$, and *maximal monotone* if it is monotone and additionally $T = T'$ for all monotone T' such that $G(T) \subset G(T')$. T is *coercive* if

$$\lim_{\|x\| \to \infty} \frac{\inf_{u \in T(x)} \langle u, x\rangle}{\|x\|} = \infty. \tag{2}$$

2. An existence result for finite dimensional equilibrium problems

Our departure point is the following celebrated result, due to Ky Fan.

Lemma 2.1. *Take a non-empty set $V \subset H$ and a closed-valued $G : V \to \mathcal{P}(H)$. If the following two conditions hold:*

C1) the convex hull of any finite subset $\{x^1, \dots, x^p\}$ of V is contained in $\cup_{i=1}^{p} G(x^i)$,
C2) $G(x)$ is compact for at least one $x \in V$,

then $\cap_{x \in V} G(x) \neq \emptyset$.

Proof. See Lemma 1 in [11]. □

This result holds in fact in an arbitrary real Hausdorff topological vector space. The following lemma, also due to Ky Fan, is a consequence of Lemma 2.1.

Lemma 2.2. *If f, K satisfy* P1, P2 *and* P3', *and K is compact, then $S(f, K) \neq \emptyset$.*

Proof. See [12]. □

We mention that Lemma 2.2 holds when P1 is replaced by the following weaker assumption:

P1': $f(x, x) \geq 0$ for all $x \in K$.

Also, H can be a real Hausdorff topological vector space, rather than a Hilbert one.

In the following existence results, for which finite dimensionality is essential, we will replace compactness of K by one of the following properties:

P5: For any sequence $\{x^k\} \subset K$ such that $\lim_{k \to \infty} \|x^k\| = \infty$, there exists $u \in K$ such that $f(x^k, u) \leq 0$ for large enough k.

P5': For any sequence $\{x^k\} \subset K$ such that $\lim_{k \to \infty} \|x^k\| = \infty$, there exists a sequence $\{u^k\} \subset K$ such that, for large enough k, $f(x^k, u^k) < 0$ and $\|u^k\| < \|x^k\|$.

P5": For any sequence $\{x^k\} \subset K$ such that $\lim_{k \to \infty} \|x^k\| = \infty$, there exists a sequence $\{u^k\} \subset K$ such that, for large enough k, $f(x^k, u^k) \leq 0$ and $\|u^k\| < \|x^k\|$.

We observe that, under P1–P3, P5' implies the compactness of $S(f, K)$ in the finite dimensional case, which is not the case for either P5 or P5". We comment next on the relations among these properties. Both P5 and P5' trivially imply P5" (for the first implication, take $u^k = u$ for all k), but the converse implications do not hold, as the following example shows.

Example 2.3. Take $H = \mathbb{R}$, $K = \mathbb{R}_+$ and define f as

$$f(x,y) = \begin{cases} x-y-1 & \text{if } y \le x-1 \\ 0 & \text{if } x-1 \le y \le x+1 \\ y-x-1 & \text{if } x+1 \le y. \end{cases}$$

This f satisfies P1, P2, P3 and P3' (indeed $f(x,\cdot)$ is convex for all $x \in K$), and also P5", taking $u^k = \max\{0, x^k - 1\}$. It is easy to check that it does not satisfy P5. In connection with P5', note that $S(f,K) = K$, because $f(x,y) \ge 0$ for all $(x,y) \in K \times K$. Since K is unbounded, P5' does not hold, in view of the comment above. Next we present our first finite dimensional existence result.

Theorem 2.4. *Assume that* $H = \mathbb{R}^n$. *If* f, K *satisfy* P1–P3 *and additionally* P5 *or* P5", *then* $S(f,K) \neq \emptyset$.

Proof. Assume first that P5" holds. Observe that Lemma 2.2 holds under P1–P3, because P3 implies P3'. For $k \in \mathbb{N}$, let B_k be the closed ball of radius k centered at 0, and $K_k = K \cap B_k$. For large enough k, K_k is nonempty. Being closed and bounded, K_k is also compact, and hence, in view of Lemma 2.2, $S(f, K_k) \neq \emptyset$ for large enough k. Take $x^k \in S(f, K_k)$. We consider now two cases. If $\{x^k\}$ has cluster points, let $\bar{x}$ be one of them. We claim that $\bar{x}$ belongs to $S(f,K)$. Take any $y \in K$. For $k \ge \|y\|$, y belongs to K_k, so that $f(x^k, y) \ge 0$, because x^k solves EP(f, K_k). Taking limits along a subsequence of $\{x^k\}$ converging to $\bar{x}$ and using P2, we get $f(\bar{x}, y) \ge 0$. Since this inequality holds for all $y \in K$, $\bar{x}$ belongs to $S(f,K)$. Assume now that $\{x^k\}$ does not have cluster points, i.e., $\lim_{k\to\infty} \|x^k\| = \infty$. By P5", there exists $\{u^k\} \subset K$ such that

$$f(x^k, u^k) \le 0 \tag{3}$$

and

$$\|u^k\| < \|x^k\| \tag{4}$$

for large enough k. Take any m large enough so that the above conditions hold for $k \ge m$. We claim that x^m solves EP(f,K). Consider any $y \in K$. If y belongs to K_m then $f(x^m, y) \ge 0$ because x^m solves EP(f, K_m). Otherwise, $\|y\| > m$ and, since y belongs to K, u^m belongs to K_m by (4), and $\|u^m\| < k$, there exists $\alpha \in (0,1)$ such that $y_\alpha := \alpha y + (1-\alpha)u^m$ belongs to K_m, so that

$$f(x^m, y_\alpha) \ge 0 \ge f(x^m, u^m), \tag{5}$$

using (3) in the second inequality. Since $f(x^m, \cdot)$ is pseudo-convex by P3, we get from (5) that

$$0 \le f(x^m, y_\alpha) \le f(x^m, y)$$

for all $y \in K$, so that $x^m \in S(f,K)$. Since P5 implies P5", the result holds under P5 as well. □

Next we consider a variation in the assumptions of Theorem 2.4. Instead of P3, we will use the weaker condition P3'. On the other hand, we will use assumption P5', which is stronger than P5".

Corollary 2.5. *Assume that $H = \mathbb{R}^n$. If f, K satisfy* P1, P2, P3' *and additionally* P5', *then $S(f, K) \neq \emptyset$.*

Proof. We follow the proof of Theorem 2.4, but now, instead of (3) and (5), we get, using P5',

$$f(x^k, u^k) < 0$$

and

$$f(x^m, y_\alpha) \geq 0 > f(x^m, u^m)$$

respectively. We then conclude, using the quasi-convexity of $f(x^m, \cdot)$, that $0 \leq f(x^m, y_\alpha) \leq f(x^m, y)$, and hence x^m solves EP(f, K) as in the theorem. □

We emphasize that Theorem 2.4 establishes non-emptiness of $S(f, K)$ in cases in which $S(f, K)$ fails to be compact, which is not the case of Corollary 2.5 and other results demanding coercivity of f (e.g. [9]).

A natural question arises related to P5, P5": are they also necessary for non-emptiness of $S(f, K)$, under P1, P2 and P3? The following example shows that the answer is negative.

Example 2.6. Take $n = 1$, $K = \mathbb{R}_+$ and $f(x, y) = x(x - y)$. It is easy to check that P1, P2 and P3 are satisfied and that 0 solves EP(f, K), but neither P5 nor P5" hold: $f(x^k, u^k) \leq 0$ with $x^k, u^k \in K$ implies $0 \leq x^k \leq u^k$, so that $\|u^k\| \geq \|x^k\|$. This f fails to satisfy P4*, P4' and P4", defined below.

It is worthwhile to mention that Theorem 2.4 remains valid when condition P1' replaces P1.

We comment now on the validity of these results in infinite dimensional spaces, and incidentally, on the reason for having jumped from P3 to P5 in our assumptions on EP(f, K). A variant of Corollary 2.5 which holds in reflexive Banach spaces appears as Theorem 4.2 in [16]. The main difficulty in extending the proof of Theorem 2.4 to infinite dimensional spaces is that K_k is not compact any more, at least in the norm topology. One way to overcome this difficulty is to consider the weak topology, for which K_k, being closed, bounded and convex, is indeed compact, but then there is trouble with Lemma 2.2: in order to obtain Lemma 2.2 from Lemma 2.1, one uses the map $G : K \to \mathcal{P}(H)$ defined as $G(y) = \{x \in K : f(x, y) \geq 0\}$, which is closed-valued in the strong topology, as a consequence of P2, but not in the weak topology. One introduces instead $\widehat{G}(y) = \{x \in K : f(y, x) \leq 0\}$. Now P3 implies that $\widehat{G}(y)$ is convex, because it is the sublevel set of the quasi-convex function $f(y, \cdot)$. Hence, $\widehat{G}(y)$, being closed and convex, is weakly closed, but the trouble is not over, because in principle $\widehat{G}(y)$ fails to satisfy condition *C1* in Lemma 2.1. In order to overcome this difficulty, one needs some additional property of f and K. Several alternatives are considered in [16], among them:

P4: For all $x^1, \dots, x^p \in K$ and all $\alpha_1, \dots, \alpha_p$ such that $\sum_{j=1}^p \alpha_j = 1$ it holds that

$$\min_{1 \leq i \leq p} f\left(x^i, \sum_{j=1}^m \alpha_j x^j\right) \leq 0. \tag{6}$$

P4*: If $f(x,y) \geq 0$ for some $x, y \in K$, then $f(y,x) \leq 0$.

P4': f satisfies (6) with strict inequality if $\{\alpha_1, \ldots, \alpha_p\} \subset (0,1)$ and $x^1, \ldots x^p$ are pairwise distinct.

P4": For all $x^1, \ldots, x^p \in K$ and all $\alpha_1, \ldots, \alpha_p$ such that $\sum_{j=1}^p \alpha_j = 1$ it holds that

$$\sum_{i=1}^p \alpha_i f\left(x^i, \sum_{j=1}^m \alpha_j x^j\right) \leq 0.$$

All these properties are weaker than *monotonicity* of f, defined as $f(x,y) + f(y,x) \leq 0$ for all $x, y \in K$. It has been proved in [16] that under P1–P3 and any one among P4, P4*, P4' and P4", property P5 is sufficient for existence of solutions of EP(f,K); under P1–P3 and any one among P4*, P4' and P4", P5 is also necessary. It has also been established in [16] that P4*, P4' and P4" are mutually independent, and all of them strictly stronger than P4. As shown by Theorem 2.4, in the finite dimensional case none of them is needed for ensuring existence of solutions of EP(f,K). This allows us to obtain from Theorem 2.4 some interesting consequences, in the form of new surjectivity results for a certain family of set-valued operators. These results are developed in the following two sections.

3. Pre-monotone operators

Minty's Theorem (see [18]) states that if H is a Hilbert space and $T : H \to \mathcal{P}(H)$ is maximal monotone, then $T + \lambda I$ is onto, where I is the identity operator and λ is any positive real number.

From now on, unless otherwise stated, we will deal with the finite dimensional case ($H = \mathbb{R}^n$). Next we introduce a class of operators for which we will prove a similar result.

Definition 3.1. An operator $T : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is said to be *pre-monotone* if there exists $\sigma : D(T) \to \mathbb{R}_+$ such that

$$\langle u - v, x - y\rangle \geq -\min\{\sigma(x), \sigma(y)\} \, \|x - y\| \quad \forall\, (x,u), (y,v) \in G(T). \tag{7}$$

We emphasize that no continuity assumption is imposed on σ; only finiteness and nonnegativity at all points of $D(T)$. It is easy to check that relation (7) is equivalent to

$$\langle u - v, x - y\rangle \geq -\sigma(y) \, \|x - y\| \quad \forall\, (x,u), (y,v) \in G(T). \tag{8}$$

For technical reasons we will quote (8) in most of our proofs concerning pre-monotonicity.

We continue with some elementary properties of pre-monotone operators.

Proposition 3.2.

i) *If T is pre-monotone then there exists $\rho : D(T) \to \mathbb{R}_+$ such that*

$$\sup_{u \in T(x)} \langle u, y - x\rangle \leq \rho(y) \, \|x - y\| \quad \forall x, y \in D(T). \tag{9}$$

ii) *If T is bounded-valued and there exists $\rho : D(T) \to \mathbb{R}_+$ such that* (9) *holds, then T is pre-monotone.*

Proof. *i)* Take $\rho(y) = \sigma(y) + \inf_{v\in T(y)} \|v\|$. In view of (8), we have

$$\langle u, y-x\rangle \le \langle v, y-x\rangle + \sigma(y)\,\|x-y\| \le [\|v\| + \sigma(y)]\,\|x-y\|$$

for all $(x,u),(y,v) \in G(T)$, so that

$$\langle u, y-x\rangle \le \left[\inf_{v\in T(y)} \|v\| + \sigma(y)\right] \|x-y\| = \rho(y)\,\|x-y\|$$

for all $(x,u) \in G(T)$ and all $y \in D(T)$. Hence, $\sup_{u\in T(x)} \langle u, y-x\rangle \le \rho(y)\,\|x-y\|$ for all $x, y \in D(T)$.

ii) Take $\sigma(y) = \rho(y) + \sup_{v\in T(y)} \|v\|$. Finiteness of σ follows from bounded-valuedness of T. Then

$$\begin{aligned}\langle u-v, x-y\rangle = \langle u, x-y\rangle + \langle v, y-x\rangle &\ge -\rho(y)\,\|x-y\| - \|v\|\,\|x-y\| \\ &\ge -\left[\rho(y) + \sup_{v\in T(y)} \|v\|\right] \|x-y\| = -\sigma(y)\,\|x-y\|\,.\end{aligned}$$

□

Proposition 3.3. *If T_1 and T_2 are pre-monotone, $D(T_1) \cap D(T_2) \neq \emptyset$, and $\alpha_1, \alpha_2 \in \mathbb{R}$ are positive then $T = \alpha_1 T_1 + \alpha_2 T_2$ is pre-monotone.*

Proof. If (8) holds for T_i with a function σ_i $(i = 1, 2)$, then it holds for T with $\sigma : D(T_1) \cap D(T_2) \to \mathbb{R}_+$ defined as $\sigma(y) = \alpha_1\sigma_1(y) + \alpha_2\sigma_2(y)$. □

Proposition 3.4.

i) *If T is monotone then T is pre-monotone.*
ii) *If T is globally bounded then T is pre-monotone.*
iii) *If $T = Q + R$, with Q monotone and R globally bounded, then T is pre-monotone.*

Proof. For *i)* take $\sigma(y) = 0$ for all $y \in \mathbb{R}^n$; for *ii)*, if $\|u\| \le \theta$ for all $(x,u) \in G(T)$, then take $\sigma(y) = 2\theta$ for all $y \in D(T)$; for *iii)* apply *i)*, *ii)* and Proposition 3.3. □

Observe also that pre-monotonicity with $\sigma(y) = 0$ for all $y \in \mathbb{R}^n$ coincides with monotonicity.

Proposition 3.5. *If T is pre-monotone then it is locally bounded in the interior of $D(T)$.*

Proof. Assume that T is not locally bounded. In such a case, there exists $x \in \mathrm{int}(D(T))$, a sequence $\{x^k\} \subset D(T)$ such that $\lim_{k\to\infty} x^k = x$, and a sequence $\{u^k\}$ such that $u^k \in T(x^k)$ for all k and $\lim_{k\to\infty} \left\|u^k\right\| = \infty$. Take $\delta > 0$ such that the closed ball $B(x,\delta)$ is contained in $D(T)$. Define $\bar{u}^k = \left\|u^k\right\|^{-1} u^k$, and let $\bar{u}$ be a cluster point of the bounded sequence $\{\bar{u}^k\}$. Take $y = x + \delta\bar{u}$. Note that $y \in B(x,\delta) \subset D(T)$. In view of Proposition 3.2 *i)* there exists $\rho : D(T) \to \mathbb{R}_+$ such that $\langle u^k, y - x^k\rangle \le \rho(y)\left\|y - x^k\right\|$, i.e.,

$$\langle u^k, x + \delta\bar{u} - x^k\rangle \le \rho(x+\delta\bar{u})\left\|x - x^k + \delta\bar{u}\right\|. \tag{10}$$

Dividing both sides of (10) by $\left\|u^k\right\|$, we get

$$\langle \bar{u}^k, x + \delta\bar{u} - x^k\rangle \le \frac{\rho(x+\delta\bar{u})\left\|x - x^k + \delta\bar{u}\right\|}{\|u^k\|}. \tag{11}$$

Taking limits in (11) along a subsequence of $\{\bar{u}^k\}$ converging to $\bar{u}$, and using that $\lim_{k\to\infty} \|u^k\| = \infty$, $\lim_{k\to\infty} x^k = x$, we get

$$0 < \delta = \delta\langle \bar{u}, \bar{u}\rangle \leq 0, \tag{12}$$

because the numerator in the right hand side of (11) converges to the finite limit $\delta\rho(x+\delta\bar{u})$ as $k \to \infty$. Since (12) is clearly a contradiction, the result holds. □

We observe now that the operators dealt with in Proposition 3.4 admit a constant σ. In this case, there exists $\theta \in \mathbb{R}_{++}$ such that

$$\langle u - v, x - y\rangle \geq -\theta \|x - y\| \quad \forall\, (x, u), (y, v) \in G(T).$$

Such operators are indeed *sub-monotone* (see e.g. [21] and [2]). We give next an example of a pre-monotone operator which is not sub-monotone.

Example 3.6. Define $\varphi : \mathbb{R} \to \mathbb{R}$ as

$$\varphi(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x - (m-1)m & \text{if } (m-1)m \leq x \leq m^2 \quad (m \in \mathbb{N}) \\ m(m+1) - x & \text{if } m^2 \leq x \leq m(m+1) \quad (m \in \mathbb{N}). \end{cases}$$

It is clear that

$$\varphi(x) \geq 0 \quad \forall x \in \mathbb{R}. \tag{13}$$

We claim that

$$\varphi(x) \leq x \quad \forall x \in \mathbb{R}_+. \tag{14}$$

This is obvious if $(m-1)m \leq x \leq m^2$. Otherwise $m^2 \leq x \leq m(m+1)$ for some $m \in \mathbb{N}$ and

$$\varphi(x) = m(m+1) - x \leq m(m+1) - m^2 = m \leq m^2 \leq x,$$

so that the claim holds. Define

$$\sigma(y) = \max\{\varphi(y), y - \varphi(y)\}. \tag{15}$$

σ is non-negative by (13). We claim that φ is pre-monotone. It suffices to check (8), which in this situation becomes:

$$(\varphi(x) - \varphi(y))(x - y) \geq -\sigma(y)\,|x - y| \quad \forall x, y \in \mathbb{R},$$

which holds trivially when $x = y$, and otherwise is equivalent to

$$(\varphi(x) - \varphi(y))\mathrm{sg}(x - y) \geq -\sigma(y) \quad \forall x, y \in \mathbb{R}. \tag{16}$$

Note that (16) holds trivially if any one among x and y is non-positive, in which case the left hand side of (16) is non-negative. Assume that $x > 0, y > 0, x \neq y$. If $x > y$ then

$$(\varphi(x) - \varphi(y))\mathrm{sg}(x - y) = \varphi(x) - \varphi(y) \geq -\varphi(y) \geq -\sigma(y),$$

using (13) and (15). If $x < y$ then

$$(\varphi(x) - \varphi(y))\mathrm{sg}(x - y) = \varphi(y) - \varphi(x) \geq \varphi(y) - x \geq \varphi(y) - y = -(y - \varphi(y)) \geq -\sigma(y),$$

using (14) in the first inequality, the fact that $x < y$ in the second one and (15) in the third one. The claim holds, and φ is pre-monotone.

Fix now $m \in \mathbb{N}$, and take $x_m = m(m+1)$, $y_m = m^2$, so that $\varphi(x_m) = 0$, $\varphi(y_m) = m$ and $\mathrm{sg}(x_m - y_m) = 1$. Thus $(\varphi(x_m) - \varphi(y_m))\mathrm{sg}(x_m - y_m) = -m$ so that

$$(\varphi(x_m) - \varphi(y_m))(x_m - y_m) = -m\,|x_m - y_m|\,.$$

It follows that there exists no $\theta > 0$ such that

$$(\varphi(x) - \varphi(y))(x - y) \geq -\theta\,|x - y|$$

for all $x, y \in \mathbb{R}$, i.e., no constant σ satisfies (8). We conclude that φ is pre-monotone but not sub-monotone.

4. The surjectivity theorem for pre-monotone operators

We start with a preliminary result, of some interest on its own.

Theorem 4.1. *Assume that $T : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is locally bounded, coercive, convex-valued, closed, and that $D(T) = \mathbb{R}^n$. Then T is onto.*

Proof. Take any $b \in \mathbb{R}^n$. We must prove that there exists $x^* \in \mathbb{R}^n$ such that $b \in T(x^*)$. Define

$$f_1(x, y) = \sup\nolimits_{u\in T(x)} \langle u, y - x\rangle.$$

Clearly, f_1 satisfies P1 and P3, because the supremum of affine functions is convex, hence pseudo-convex. We check P2. Suppose that there exist $\bar{x}, \bar{y} \in \mathbb{R}^n$ such that $f_1(\cdot, \bar{y})$ is not upper semicontinuous at $\bar{x}$. Hence, there exists a sequence $\{x^k\}$ such that $\lim_{k\to\infty} x^k = \bar{x}$ and $f_1(\bar{x}, \bar{y}) < \lim_{k\to\infty} f_1(x^k, \bar{y})$; say $f_1(\bar{x}, \bar{y}) \leq f_1(x^k, \bar{y}) - \delta$ for some $\delta > 0$ and large enough k. Since $f_1(x^k, \bar{y}) = \sup_{u\in T(x^k)} \langle u, \bar{y} - x^k\rangle$, there exists $u^k \in T(x^k)$ such that

$$f_1(\bar{x}, \bar{y}) \leq \langle u^k, \bar{y} - x^k\rangle - \frac{\delta}{2} \tag{17}$$

for large enough k. By local boundedness of T, we get that $\{u^k\}$ is bounded, because $\lim_{k\to\infty} x^k = x$, and hence it has a convergent subsequence, say $\{u^{j_k}\}$, with limit $\bar{u}$. Since $\lim_{k\to\infty}(x^{j_k}, u^{j_k}) = (\bar{x}, \bar{u})$ and $\{(x^k, u^k)\} \subset G(T)$, we obtain from the closedness of T that $(\bar{x}, \bar{u}) \in G(T)$. Taking limits in (17) with $k \to \infty$ along the subsequence, we get

$$\langle \bar{u}, \bar{y} - \bar{x}\rangle \leq \sup_{u\in T(\bar{x})} \langle u, \bar{y} - \bar{x}\rangle = f_1(\bar{x}, \bar{y}) \leq \langle \bar{u}, \bar{y} - \bar{x}\rangle - \frac{\delta}{2},$$

a contradiction which establishes that P2 holds for f_1. Define now

$$f(x, y) = f_1(x, y) - \langle b, y - x\rangle. \tag{18}$$

It follows easily that f inherits P1–P3 from f_1 (for P3, observe that both terms in the right hand side of (18) are convex functions of y). We claim now that f satisfies P5. Consider a sequence $\{x^k\} \subset \mathbb{R}^n$ with $\lim_{k\to\infty} \|x^k\| = \infty$ and take $u = 0$ in P5. Without loss of generality, assume that $\|x^k\| > 0$ for all k. Then

$$f(x^k, 0) = \sup_{u\in T(x^k)} \langle u, -x^k\rangle + \langle b, x^k\rangle \leq -\inf\nolimits_{u\in T(x^k)}\langle u, x^k\rangle + \|b\|\,\|x^k\|\,. \tag{19}$$

Dividing both sides of (19) by $\|x^k\|$, we get

$$\frac{f(x^k,0)}{\|x^k\|} \le -\frac{\inf_{u\in T(x^k)}\langle u, x^k\rangle}{\|x^k\|} + \|b\| . \tag{20}$$

Since T is coercive and $\lim_{k\to\infty}\|x^k\| = \infty$, we conclude from (20) that the right hand side of (20) is negative for large k and hence P5 holds. It follows from Theorem 2.4 that $\mathrm{EP}(f,\mathbb{R}^n)$ has solutions, i.e. there exists $x^* \in \mathbb{R}^n$ such that $f(x^*,y) \ge 0$ for all $y \in \mathbb{R}^n$, so that

$$\sup_{u\in T(x^*)} \langle u-b, y-x^*\rangle \ge 0 \quad \forall y \in \mathbb{R}^n.$$

Note that the set $T(x^*)$ is closed by closedness of T and bounded by assumption, hence compact. It follows that for each $y \in \mathbb{R}^n$ $\sup_{u\in T(x^*)}\langle u-b, y-x^*\rangle$ is attained at some $u_y \in T(x^*)$. Therefore, for all $y \in \mathbb{R}^n$ there exists $u_y \in T(x^*)$ such that $\langle u_y - b, y - x^*\rangle \ge 0$, or equivalently, for all $z \in \mathbb{R}^n$ there exists $v \in T(x^*) - b$ such that

$$\langle v, z\rangle \ge 0. \tag{21}$$

Since the set $T(x^*) - b$ is closed and convex by closedness and convex-valuedness of T, and cannot be separated from $\{0\}$ by (21), we get that $0 \in T(x^*) - b$, i.e., we have proved that T is onto. □

We mention that Theorem 4.1 can also be proved using, instead of Theorem 2.4, the coercivity result in [9]. Now we state and prove our surjectivity result for pre-monotone operators.

Theorem 4.2. *Assume that $T : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is pre-monotone, convex-valued and closed, and that $D(T) = \mathbb{R}^n$. Then $T + \lambda I$ is onto for all $\lambda > 0$.*

Proof. It is elementary that $T + \lambda I$ is also pre-monotone, convex valued and closed, and that $D(T + \lambda I) = \mathbb{R}^n$ (note that $T + \lambda I$ satisfies the pre-monotonicity inequality with the same σ as T). Since $D(T + \lambda I) = \mathbb{R}^n$, it follows from Proposition 3.5 that $T + \lambda I$ is locally bounded. We claim now that $T + \lambda I$ is coercive, i.e. that

$$\lim_{\|x\|\to\infty} \frac{\inf_{u\in T(x)}\langle u + \lambda x, x\rangle}{\|x\|} = \infty. \tag{22}$$

Take $v \in T(0)$. Then

$$\begin{aligned} \inf_{u\in T(x)}\langle u + \lambda x, x\rangle &= \inf_{u\in T(x)}\langle u, x\rangle + \lambda\|x\|^2 \\ &= \inf_{u\in T(x)}\langle u - v, x - 0\rangle + \langle v, x\rangle + \lambda\|x\|^2 \\ &\ge \; -\sigma(0)\|x\| - \|v\|\,\|x\| + \lambda\|x\|^2 , \end{aligned} \tag{23}$$

using (8) in the last inequality. Dividing both sides of (23) by $\|x\|$, we obtain (22). We have shown that $T + \lambda I$ satisfies all the assumptions of Theorem 4.1. It follows that $T + \lambda I$ is onto. □

Corollary 4.3. *Take $T = Q + R$ where $Q : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is maximal monotone, $R : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is globally bounded, closed and convex-valued, and $D(Q) = D(R) = \mathbb{R}^n$. Then $T + \lambda I$ is onto for all $\lambda > 0$.*

Proof. T is pre-monotone by Proposition 3.4 *iii*). Maximal monotone operators are closed and convex-valued (see e.g. Chapter 4 of [10]). Since $D(Q) = D(R) = \mathbb{R}^n$, both Q and R satisfy the remaining hypotheses of Theorem 4.2 and so does their sum T. □

Corollary 4.4. *Assume that $T : \mathbb{R}^n \to \mathbb{R}^n$ is single-valued, continuous and pre-monotone. Then $T + \lambda I$ is onto for all $\lambda > 0$.*

Proof. Continuity and single-valuedness of T take care of the closedness and convex-valuedness required as assumptions for Theorem 4.2. □

Corollary 4.5. *Take $T : \mathbb{R}^n \to \mathbb{R}^n$ of the form $T = Q + R$, where Q and R are single-valued and continuous, Q is maximal monotone and R is globally bounded. Then $T + \lambda I$ is onto for all $\lambda > 0$.*

Proof. T is pre-monotone by Proposition 3.4 *iii*). Apply Corollary 4.4. □

Remark 4.6. Corollary 4.4 implies that $\psi(x) = \varphi(x) + \lambda x$, with φ as in Example 3.6, is onto for all $\lambda > 0$. We have already established that φ is not submonotone, and hence it cannot be writen as the sum of a maximal monotone operator and a globally bounded one. Of course, surjectivity of ψ is an easy consequence of its definition, but this is a one-dimensional effect. Similar n-dimensional pre-monotone operators T can be constructed, for which the surjectivity of $T + \lambda I$ is not immediate, and which are not of the types considered in Proposition 3.4.

5. Maximal pre-monotone operators

In Minty's Theorem, the hypotheses of convex-valuedness and closedness of T are avoided by assuming that T is maximal monotone, in which case they are automatically satisfied. This suggests the introduction of a notion of maximal pre-monotonicity. One could be tempted to define a maximal pre-monotone operator as a pre-monotone one whose graph is not properly contained in the graph of another pre-monotone operator, but some caution is needed. If we take any pre-monotone operator T, which satisfies (8) for a given σ, and define $T_k : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ as $T_k(x) = T(x) + B(0, k)$ with $k \in \mathbb{N}$, then it follows from Propositions 3.3 and 3.4 *ii*) that T_k is pre-monotone whith $\sigma_k(x) = \sigma(x) + 2k$, and $\{T_k\}$ is an increasing chain of pre-monotone operators (with respect to the inclusion of their graphs), which is not bounded above by any pre-monotone operator, because $\cup_{k\in\mathbb{N}} G(T_k) = \mathbb{R}^n \times \mathbb{R}^n$, and the operator with this graph is not pre-monotone. Thus, with this notion of maximal pre-monotonicity there would be no maximal pre-monotone operators. It becomes clear that an adequate notion of maximal pre-monotonicity must refer to a given σ, as the following one does.

Definition 5.1. An operator $T : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is said to be *σ-maximal pre-monotone* if it satisfies (8) for a certain σ and additionally $T = T'$ for any operator T' wich satisfies (8) with the same σ and such that $G(T) \subset G(T')$.

With this definition, we have several desirable properties for a maximal object.

Proposition 5.2. *If T is a pre-monotone operator which satisfies* (8) *for a given σ, then there exists a σ-maximal pre-monotone operator $\bar{T}$ such that $G(T) \subset G(\bar{T})$.*

Proof. In view of Zorn's Lemma, it suffices to check that an increasing chain $\{T_j\}_{j\in J}$ of σ-pre-monotone operators whose graph contain $G(T)$ is bounded above. It is clear that $\widetilde{T}(x) := \cup_{j\in J} T_j(x)$ is such an upper bound, because it satisfies (8) with the given σ, since checking this inequality for $\widetilde{T}$ demands simultaneous consideration of only a pair of indices in J. □

Definition 5.3. Given $A \subset \mathbb{R}^n$ and $\sigma : A \to \mathbb{R}_+$, two pairs $(x,u), (y,v) \in A \times \mathbb{R}^n$ are *σ-pre-monotonically related* if

$$\langle u - v, x - y\rangle \geq -\min\{\sigma(x), \sigma(y)\} \|x - y\| . \tag{24}$$

We emphasize that the above definition can also be given using

$$\langle u - v, x - y\rangle \geq -\sigma(y) \|x - y\| ,$$

due to the equivalence between (7) and (8), as underlined before.

Proposition 5.4. *T is σ-maximal pre-monotone if and only if whenever a pair $(\bar{x}, \bar{u}) \in \mathbb{R}^n \times \mathbb{R}^n$ is σ-pre-monotonically related to all pairs $(y,v) \in G(T)$, it holds that $\bar{u} \in T(\bar{x})$.*

Proof. For the "only if" statement it suffices to observe that under the assumptions of the proposition the operator $\widehat{T}$ defined as

$$\widehat{T}(x) = \begin{cases} T(\bar{x}) \cup \{\bar{u}\} & \text{if } x = \bar{x} \\ T(x) & \text{otherwise,} \end{cases}$$

satisfies (8) with the given σ and also $G(T) \subset G(\widehat{T})$. The conclusion follows from Definition 5.1.

We prove now the "if" statement. Given an operator T' which satisfies (8) with σ, and such that $G(T) \subset G(T')$, observe that any pair $(x', u') \in G(T')$ is σ-pre-monotonically related to any pair $(x,u) \in G(T)$. From the assumption, we get that $(x', u') \in G(T)$, so that $G(T') \subset G(T)$, i.e., $T = T'$, and hence T is σ-maximal pre-monotone. □

We have observed that the concept of maximal pre-monotone operator must be connected to the function σ. We prove next that it is possible to properly choose a specific minimal σ which allows us to get an absolute notion of maximal pre-monotonicity.

Given $T : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ define $\sigma_T : D(T) \to \mathbb{R} \cup \{\infty\}$ as

$$\sigma_T(y) = \max\left\{ \sup_{v\in T(y)} \sup_{(x,u)\in G(T), x\neq y} \frac{\langle u - v, y - x\rangle}{\|x - y\|},\ 0 \right\}. \tag{25}$$

Proposition 5.5. *An operator T is σ_T-maximal pre-monotone and σ_T is finite everywhere if and only if it is σ-maximal pre-monotone for some $\sigma : D(T) \to \mathbb{R}_+$.*

Proof. The "only if" statement is obvious. We prove the "if" one. Assume that T satisfies (8) for some $\sigma : D(T) \to \mathbb{R}_+$. Multiply both sides of (8) by $-\|x - y\|^{-1}$, take the supremum of the left hand side first with respect to $(x, u) \in G(T)$ $(x \neq y)$ and then with respect to $v \in T(y)$, and conclude that

$$\sigma_T(y) \leq \sigma(y) \quad \forall y \in D(T). \tag{26}$$

Therefore, $\sigma_T(y) < \infty$ for all $y \in D(T)$. It follows from (26) and Definition 5.3 that if a pair (x, u) is σ-pre-monotonically related to all points in $G(T)$, then it is also σ_T-pre-monotonically related to all such pairs. The conclusion follows from Proposition 5.4. □

Proposition 5.5 allows us to get rid of the "$\sigma-$" in the definition of maximal pre-monotonicity.

Definition 5.6. An operator T is *maximal pre-monotone* if it is σ_T-maximal pre-monotone, with σ_T as in (25).

At this point, it is clear that we could have used σ_T from the beginning, i.e. even in the definition of pre-monotonicity. We have not done so because in general computing σ_T is not an easy task, and in most cases it is enough to consider a function σ greater than σ_T, e.g. in Propositions 3.3, 3.4 *ii*) and 3.4 *iii*).

Propositions 5.4 and 5.5 allow us to prove the following properties of maximal pre-monotone operators.

Proposition 5.7. *If T is maximal pre-monotone, then it is convex-valued and closed.*

Proof. Take any σ such that T is σ-maximal premonotone. Take $u^1, u^2 \in T(x)$, $\alpha \in (0, 1)$ and let $u = \alpha u^1 + (1 - \alpha)u^2$. We claim that (x, u) is σ-pre-monotonically related to any $(y, v) \in G(T)$. Note that, since (u^1, x) and (u^2, x) belong to $G(T)$, we have

$$\begin{aligned}\langle u - v, x - y\rangle &= \alpha\langle u^1 - v, x - y\rangle + (1 - \alpha)\langle u^2 - v, x - y\rangle \\ &\geq \alpha(-\sigma(y)) \|x - y\|) + (1 - \alpha)(-\sigma(y)) \|x - y\| = -\sigma(y) \|x - y\|,\end{aligned}$$

and so the claim holds. By Proposition 5.4, $u \in T(x)$. Thus $T(x)$ is convex for all $x \in D(T)$. For closedness, take a sequence $\{(x^k, u^k)\}$ contained in $G(T)$ and convergent to (x, u). We claim that (x, u) is σ-pre-monotonically related to any $(y, v) \in G(T)$. Since $u^k \in T(x^k)$ for all k, we have

$$\langle u^k - v, x^k - y\rangle \geq -\sigma(y) \left\|x^k - y\right\|. \tag{27}$$

Taking limits in (27) with $k \to \infty$ we get

$$\langle u - v, x - y\rangle \geq -\sigma(y) \|x - y\|,$$

establishing the claim. In view of Proposition 5.4, $(x, u) \in G(T)$ and hence T is closed. □

At this point, it is appropriate to identify some maximal pre-monotone operators. To begin with, it is clear that maximal monotone operators are precisely those operators which are σ-maximal pre-monotone with $\sigma(y) = 0$ for all y. An example of an operator which is maximal pre-monotone but not monotone is given in the next proposition.

Proposition 5.8. *Let $Q : \mathbb{R}^n \to \mathbb{R}^n$ be maximal monotone and single-valued. Take $\theta \in \mathbb{R}_{++}$. Then $T = Q + B(0,\theta)$ is maximal pre-monotone.*

Proof. Consider $\sigma(y) = 2\theta$ for all $y \in \mathbb{R}^n$. As we have seen in Proposition 3.4 *iii*), T satisfies (8) with this σ. Take any T' which satisfies (8) for this σ and such that $G(T) \subset G(T')$. Pick up any pair $(x,u) \in G(T')$. In view of Propositions 5.4 and 5.5, it suffices to prove that $u \in T(x)$. Since $G(T) \subset G(T')$,

$$\langle u - v, x - y \rangle \geq -2\theta \|x - y\| \tag{28}$$

for all $(y,v) \in G(T)$. If $u = Q(x)$ then $u \in T(x)$ and we are done. Otherwise, let

$$w = u - Q(x), \qquad \bar{w} = \|w\|^{-1} w, \qquad y = x + \frac{1}{k}\bar{w} \ \ (k \in \mathbb{N}), \qquad v = Q(y) - \theta\bar{w}.$$

Note that $\|x - y\| = 1/k$ and that $v \in T(y)$, because $\|\bar{w}\| = 1$. Replacing these values of u, v and y in (28) we get

$$\left\langle w + Q(x) - Q\left(x + \frac{1}{k}\bar{w}\right) + \theta\bar{w}, \frac{-1}{k}\bar{w} \right\rangle \geq \frac{-2\theta}{k}. \tag{29}$$

It follows from (29) that

$$2\theta \geq \left\langle w + Q(x) - Q\left(x + \frac{1}{k}\bar{w}\right) + \theta\bar{w}, \bar{w} \right\rangle = \left\langle Q(x) - Q\left(x + \frac{1}{k}\bar{w}\right), \bar{w} \right\rangle + \|w\| + \theta,$$

and therefore

$$\|w\| \leq \theta - \left\langle Q(x) - Q\left(x + \frac{1}{k}\bar{w}\right), \bar{w} \right\rangle. \tag{30}$$

It is well known that maximal monotone single-valued operators are continuous (see e.g. Chapter 4 in [10]). Taking limits in (30) with $k \to \infty$ we get $\|w\| \leq \theta$. Hence, $u = Q(x) + w \in Q(x) + B(0,\theta) = T(x)$, completing the proof. □

This example is also good to illustrate that, at variance with monotone operators, the inverses of pre-monotone operators need not be pre-monotone. For $T(x) = B(0,\theta)$ for all $x \in \mathbb{R}^n$, we get T^{-1} defined as

$$T^{-1}(x) = \begin{cases} \mathbb{R}^n & \text{if } \|x\| \leq \theta \\ \emptyset & \text{otherwise,} \end{cases}$$

and it can be easily seen that T^{-1} is not pre-monotone.

Now we rephrase our surjectivity result for the maximal pre-monotone case.

Theorem 5.9. *If $T : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is maximal pre-monotone and $D(T) = \mathbb{R}^n$, then $T + \lambda I$ is onto for all $\lambda > 0$.*

Proof. By Proposition 5.7, T is convex-valued and closed. The result follows from Theorem 4.2. □

At this point one could ask why we did present in Section 4 the surjectivity result without maximality, when the statement of Theorem 5.9 looks more elegant than the one of Theorem 4.2. The reason is that maximal pre-monotonicity is not only hard to establish but also rather scarce. For instance, single-valued and continuous pre-monotone operators in general fail to be maximal pre-monotone, as the following example shows, and therefore the results of Corollaries 4.3, 4.4 and 4.5 cannot be obtained as a consequence of Theorem 5.9. Before presenting the example, we observe that for a single-valued one-dimensional $T : \mathbb{R} \to \mathbb{R}$, the function σ_T defined in (25) becomes

$$\sigma_T(y) = \max \left\{ \sup_{x<y} \{T(x) - T(y)\}, \sup_{x>y} \{T(y) - T(x)\}, 0 \right\}. \tag{31}$$

Example 5.10. Define $T : \mathbb{R} \to \mathbb{R}$ as $T(x) = \sin x$. It follows easily from (31) that $\sigma_T(y) = 1 + |\sin y|$. It is easy to check that the pair $(0, 1)$ is σ_T-monotonically related to all pairs $(y, \sin y)$, but $1 \neq \sin 0$, establishing that T, though it is pre-monotone by Proposition 3.4 *ii*), is not maximal pre-monotone.

Regarding the difficulty in establishing maximal pre-monotonicity, up to now we know very few maximal pre-monotone operators. We even ignore whether an operator of the form $Q + B(0, \theta)$, where Q is maximal monotone but not single-valued, is maximal pre-monotone, as it happens in the single-valued case (cf. Proposition 5.8).

6. Existence results for optimization problems

In this section we reformulate conditions P5 and P5", introduced in Section 2, for the case of optimization problems. Our departure point is the fact that $\mathrm{EP}(f, K)$ includes optimization problems as particular cases: if we take $f(x, y) = h(y) - h(x)$, where $h : K \to \mathbb{R}$ is pseudo-convex and lower semicontinuous, then assumptions P1–P3 are satisfied, and it is easy to check that $S(f, K)$ is precisely the set of solutions of the problem consisting of minimizing $h(x)$ subject to $x \in K$. In this setting, both P5 and P5" are sufficient for existence of solutions of this problem, but in fact we can do without the pseudo-convexity assumption on h, and also both conditions turn out to be also necessary for existence of solutions (and hence equivalent). We give next a formal statement of the problem and of both conditions, suitably reformulated.

Take a proper and lower semicontinuous function $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ (we remind that h is *proper* if $h(x) < \infty$ for some $x \in \mathbb{R}^n$). Let $S(h) \subset \mathbb{R}^n$ be the (possibly empty) set of minimizers of h. Consider the following three properties:

H1: For all sequence $\{x^k\} \subset K$ such that $\lim_{k\to\infty} \|x^k\| = \infty$, there exists $u \in K$ such that $h(u) \leq h(x^k)$ for large enough k.

H2: For all sequence $\{x^k\} \subset K$ such that $\lim_{k\to\infty} \|x^k\| = \infty$, there exists a sequence $\{u^k\} \subset K$ such that, for large enough k, $h(u^k) \leq h(x^k)$ and $\|u^k\| < \|x^k\|$.

H3: For all sequence $\{x^k\} \subset K$ such that $\lim_{k\to\infty} \|x^k\| = \infty$, and $\lim_{k\to\infty} \|x^k\|^{-1} x^k$ exists, there exists a sequence $\{u^k\} \subset K$ such that, for large enough k, $h(u^k) \leq h(x^k)$ and $\|u^k\| < \|x^k\|$.

Conditions H1, H2 are clearly the specializations of P5, P5" respectively to the optimization setting. Condition H3 has been included mainly for simplifying the proof of

Theorem 6.1 below. Conditions related to these can be found in [1], [3] and [4]. We present next our existence theorem for the optimization case.

Theorem 6.1. *If* $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *is proper and lower semicontinuous then the four following statements are equivalent.*

i) H1 *holds,*
ii) H2 *holds,*
iii) H3 *holds,*
iv) $S(h) \neq \emptyset$.

Proof. *i)* $\Rightarrow$ *ii)* Let u be the vector whose existence is ensured by H1, take $u^k = u$ for all k, and note that H2 holds for the sequence $\{u^k\}$.

ii) $\Rightarrow$ *iii)* Immediate.

iii) $\Rightarrow$ *iv)* Fix any $k \in \mathbb{N}$ and consider the problem P_k defined as min $h(x)$ subject to $\|x\| \leq k$. Let S_k be the solution set of this problem. Since the feasible set for this problem is compact and the objective function is lower semicontinuous, S_k is non-empty and indeed closed, being the sublevel set of a lower semicontinuous function, so that there exist vectors of minimum norm in S_k. Now, for each $k \in \mathbb{N}$ let x^k be a minimum norm vector in S_k. We claim that $\{x^k\}$ is bounded. Otherwise, consider a subsequence $\{x^{j_k}\}$ of $\{x^k\}$ such that $\lim_{k\to\infty} \|x^{j_k}\| = \infty$, and $\lim_{k\to\infty} \|x^{j_k}\|^{-1} x^{j_k}$ exists. Denote $\tilde{x}^k = x^{j_k}$. H3 ensures the existence of a sequence $\{u^k\} \subset \mathbb{R}^n$ such that

$$h(u^k) \leq h(\tilde{x}^k), \tag{32}$$

$$\left\|u^k\right\| < \left\|\tilde{x}^k\right\|. \tag{33}$$

Since $\tilde{x}^k$ belongs to S_{j_k}, we have $\|x^{j_k}\| \leq j_k$. By (33), u^k is feasible for P_{j_k}, so that $h(\tilde{x}^k) \leq h(u^k)$. In view of (32), we have that $h(u^k) = h(\tilde{x}^k)$, so that u^k belongs to S_{j_k}, and hence, since $\tilde{x}^k$ is an element of minimum norm in S_{j_k}, we get $\left\|\tilde{x}^k\right\| \leq \left\|u^k\right\|$, which contradicts (33), establishing the claim. Thus $\{x^k\}$ is bounded, and hence it has a convergent subsequence, say $\{x^{\ell_k}\}$, with limit $\bar{x}$. We claim that $\bar{x}$ belongs to $S(h)$. Take any $y \in \mathbb{R}^n$. For any k such that $\ell_k \geq \|y\|$, we have that y is feasible for P_{ℓ_k}, and hence

$$h(x^{\ell_k}) \leq h(y), \tag{34}$$

because $x^{\ell_k} \in S_{\ell_k}$. Taking limits in (34) with $k \to \infty$ and using the lower semicontinuity of h, we get $h(\bar{x}) \leq h(y)$. It follows that $\bar{x} \in S(h)$.

iv) $\Rightarrow$ *i)* Take $\bar{x} \in S(h)$, and $u = \bar{x}$ in condition H1. Given any sequence $\{x^k\}$ with $\lim_{k\to\infty} \left\|x^k\right\| = \infty$, it is immediate that $h(u) \leq h(x^k)$ for all k, and $\|u\| < \left\|x^k\right\|$ for large k, so that H1 holds. □

Note that the results of Theorem 6.1 apply also to the constrained optimization problem consisting of minimizing $h(x)$ subject to $x \in C$, where $h : \mathbb{R}^n \to \mathbb{R}$ is proper and lower semicontinuous and $C \subset \mathbb{R}^n$ is closed, because, defining $\bar{h} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ as

$$\bar{h}(x) = \begin{cases} h(x) & \text{if } x \in C \\ +\infty & \text{otherwise,} \end{cases}$$

$\bar{h}$ turns out to be lower semicontinuous.

We will use Theorem 6.1 for obtaining an alternative proof of Frank-Wolfe's Theorem. We start with a technical result, akin to Proposition 2.3 of [3].

Lemma 6.2. *Let* $C = \{x \in \mathbb{R}^n : Bx \le b\}$ *be an unbounded polyhedron with* $B \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$. *Consider a sequence* $\{x^k\} \subset C$ *with* $\|x^k\| \to \infty$, *such that* $\{\|x^k\|^{-1} x^k\}$ *converges, say to* $u \in \mathbb{R}^n$. *Let* $I := \{i : (Bu)_i \ne 0\}$ *and define* $\eta_k \in \mathbb{R}$ *as*

$$\eta_k = \min\left\{\frac{\|x^k\|}{2} - \max_{i\in I}\left\{\frac{b_i}{(Bu)_i}\right\}, 2u^T x^k\right\} \tag{35}$$

if $I \ne \emptyset$, *and* $\eta_k = 2u^T x^k$ *otherwise. Then*

i) $\eta_k > 0$ *for large enough* k,
ii) $Bu \le 0$.
iii) $\|x^k - tu\| < \|x^k\|$ *and* $x^k - tu \in C$ *for all* $t \in (0, \eta_k)$ *and large enough* k,

Proof. Let $\bar{x}^k = \|x^k\|^{-1} x^k$. Note that $2u^T x^k = 2\|x^k\| u^T \bar{x}^k$. Since $\lim_{k\to\infty} u^T \bar{x}^k = \|u\|^2 = 1$ and $\lim_{k\to\infty}\|x^k\| = \infty$, we conclude that $\lim_{k\to\infty} 2u^T x^k = +\infty$. It follows that both terms in the rightmost expression in (35) go to $+\infty$ as $k \to \infty$, and hence the minimum between them is positive for large enough k, so that *i)* holds. Note that, since $\{x^k\} \subset C$, we have $B\bar{x}^k \le \|x^k\|^{-1} b$. Taking limits as $k \to \infty$, we get $Bu \le 0$, because $\lim_{k\to\infty}\|x^k\| = \infty$, which establishes *ii)*. The inequality $\|x^k - tu\| < \|x^k\|$ follows easily from the fact that $t < \eta_k \le 2u^T x^k$. We prove next that $x^k - tu$ belongs to C. It follows that $(Bu)_i < 0$ for all $i \in I$.

In view of (35), for $t < \eta_k$ we have, if $I \ne \emptyset$,

$$t < \frac{\|x^k\|}{2} - \max_{i\in I}\left\{\frac{b_i}{(Bu)_i}\right\}. \tag{36}$$

Take now $\ell \in I$. By (36),

$$\frac{b_\ell}{(Bu)_\ell} \le \max_{i\in I}\left\{\frac{b_i}{(Bu)_i}\right\} \le \frac{\|x^k\|}{2} - t,$$

so that, since $(Bu)_\ell < 0$ for $\ell \in I$, we get, for large enough k,

$$\begin{aligned} b_\ell &\ge \frac{\|x^k\|}{2}(Bu)_\ell - [B(tu)]_\ell \ge \|x^k\|(B\bar{x}^k)_\ell - [B(tu)]_\ell \\ &= (Bx^k)_\ell - [B(tu)]_\ell = [B(x^k - tu)]_\ell, \end{aligned} \tag{37}$$

using the fact that $\lim_{k\to\infty} B(\bar{x}^k)_\ell = (Bu)_\ell < 0$ in the last inequality. Thus $[B(x^k - tu)]_\ell \le b_\ell$ for $\ell \in I$. On the other hand, for $\ell \notin I$ we have $(Bu)_\ell = 0$, so that $[B(x^k - tu)]_\ell = (Bx^k)_\ell \le b_\ell$, because $x^k \in C$. We conclude that $B(x^k - tu) \le b$ (both when $I = \emptyset$ and when $I \ne \emptyset$), i.e., $x^k - tu \in C$ for large enough k and $t \in (0, \eta_k)$, which establishes the second statement in *iii)*. □

Lemma 6.2 allows us to prove the following result.

Theorem 6.3. *A quadratic function* $h : \mathbb{R}^n \to \mathbb{R}$ *is bounded below on a polyhedron* $C \subset \mathbb{R}^n$ *if and only if condition* $H1$ *above holds for the function* $\bar{h} : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ *defined as*

$$\bar{h}(x) = \begin{cases} h(x) & \text{if } x \in C \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. If C is bounded, then the result is obvious. So, we may assume that C is unbounded. Note that $\bar{h}$ is lower semicontinuous, because h is continuous and C is closed. Take C as in Lemma 6.2. For the "if" statement, by Theorem 6.1, if H1 holds then $\bar{h}$ has minimizers, i.e. h attains its minimum on C and "a fortiori" it is bounded below on C. We prove now the "only if" statement. Assume that h is bounded below on C. We claim first that condition H3 holds. Take a sequence $\{x^k\}$ such that $\lim_{k\to\infty} \|x^k\| = \infty$ and $\|x^k\|^{-1} x^k$, converges, say to $u \in \mathbb{R}^n$. If $x^k \notin C$, then $\bar{h}(x^k) = \infty$ and any vector u^k with $\|u^k\| < \|x^k\|$ satisfies $\bar{h}(u^k) \leq \bar{h}(x^k)$, so that we may assume, without loss of generality, that $\{x^k\} \subset C$. We consider now two cases: if $\lim_{k\to\infty} h(x^k) = \infty$, then we take any $u \in C$, define $u^k = u$ for all k, and since $\lim_{k\to\infty} \|x^k\| = \infty$, we have $\|u^k\| < \|x^k\|$, $h(u^k) < h(x^k)$ for large enough k, and so H3 holds for the sequence $\{x^k\}$. Thus we may assume, without loss of generality, that $\{h(x^k)\}$ is bounded above, and also below, by hypothesis. Let $h(x) = x^T Ax + a^T x + \alpha$, with $A \in \mathbb{R}^{n\times n}$ symmetric, $a \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Boundedness of $\{h(x^k)\}$ implies that there exist β, γ such that

$$\beta \leq x^T Ax + a^T x + \alpha \leq \gamma \tag{38}$$

for all $x \in C$. By Lemma 6.2 *iii*), $Bu \leq 0$. Thus, for all $z \in C$ and all $t > 0$,

$$B(z + tu) = Bz + tBu \leq Bz \leq b,$$

and hence $z + tu$ belongs to C for all $t \geq 0$. Substituting $z + tu$ with $z \in C$ in (38) and letting $t \to \infty$, one gets easily that $u^T Au = 0$, and so $Au = 0$ by symmetry of A. An easy consequence of the fact that $Au = 0$ is that $a^T u = 0$.

Take now η_k as in (35), choose $t \in (0, \eta_k)$, and define, for k large enough so that Lemma 6.2 holds, $u^k = x^k - tu$. By Lemma 6.2, u^k belongs to C and $\|u^k\| < \|x^k\|$ for large enough k. Also, the facts that $Au = 0$, $a^T u = 0$ imply easily that $h(u^k) = h(x^k - tu) = h(x^k)$. It follows that H3 holds, and hence, in view of Theorem 6.1, H1 holds. □

Now we obtain Frank-Wolfe's Theorem (see [15]), as a corollary of the last two theorems.

Corollary 6.4. *If a quadratic function is bounded below on a polyhedron* C*, then it attains its minimum on* C*.*

Proof. The result follows from Theorems 6.1 and 6.3. □

7. Final remarks

When one compares Theorem 5.9 with Minty's result, two limitations of the former become evident. Minty's result holds in Hilbert spaces (and in fact also in reflexive Banach spaces with the duality operator J substituting for the identity; see e.g. Chapter

4 in [10]), while our theorem demands finite dimensionality. Also, Minty's theorem holds without requiring that $D(T)$ be the whole space. Surjectivity of $T+\lambda I$ in Hilbert spaces for maximal pre-monotone operators with arbitrary domains is a reasonable conjecture, but our proof technique, based upon the existence result for equilibrium problems given in Theorem 2.4, is not good enough for the general case. To begin with, all known existence results for $\mathrm{EP}(f,K)$, either in finite or infinite dimension, require that K be closed and convex, and when we transpose the existence result to the context of operators, we are forced to require that $D(T)$ be closed and convex. It is well known that maximal monotone operators may have non-closed domains; e.g. $\phi(t) = \mathrm{tg}(t)$ is maximal monotone with domain $(-\pi/2, \pi/2)$. Our technique does not allow us to cover such cases. Regarding convexity of the domain, it holds automatically for maximal monotone operators (see e.g. Chapter 4 in [10]), and perhaps also for maximal pre-monotone ones, but we do not have a proof of this fact. Even if we assume that T is maximal pre-monotone and that $D(T)$ is closed and convex, we cannot get our surjectivity result: the proof of upper semicontinuity of $f_1(\cdot, y)$, as established in Theorem 4.1, requires local boundedness of T, which holds in the interior of the domain but not on the boundary; in fact we know that the image of a point of the boundary of the domain of a maximal monotone operator, if non-empty, is unbounded, because it contains a half-line. Upper semicontinuity of $\sup_{u\in T(x)}\langle u, y-x\rangle$ as a function of x for a maximal pre-monotone T is for the time being also a reasonable but unproved conjecture.

In connection with the surjectivity result in infinite dimensional spaces, we mention that, as commented upon in Section 2, the existence result for equilibrium problems in such spaces (see [16], [17]), requires some monotonicity-like property of f, like P4, P4*, P4' or P4" (see the definitions in Section 2), which translates into a similar property of T. Unfortunately none of such properties of T is inherited by $T+\lambda I$, unless we strenghten them up to monotonicity, but in such a case we are back to Minty's hypotheses, and the equilibrium approach does not improve upon previously known surjectivity results.

Independently of its use for establishing surjectivity results, the concept of pre-monotone operators, either in finite or infinite dimension, seems interesting enough as to justify the attempt to extend to them as much as posible of the theory of monotone operators. In this respect, the content of this paper is very preliminary; almost all the issues are open, specially in infinite dimension: local boundedness, demi-closedness of the graph in the maximal pre-monotone case, etc.

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