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New Properties of the Variational Sum of Monotone Operators*

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Dedicated to Stephen Simons on the occasion of his 70th birthday.

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We study the Variational Sum of monotone operators, in particular its relationship with the Extended Sum of monotone operators. First, we establish some new properties of the Variational Sum, among them that this sum has closed graph and convex values. Then, we show that the graph of the Variational Sum always contains the graph of the Extended Sum, and hence, it contains also the graph of the usual sum. An example is given showing that the latter inclusions are proper in general.

Keywords: Monotone operator, extended sum, variational sum

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1. Introduction and Preliminaries

Throughout this article X denotes a reflexive real Banach space, X^* its continuous dual and $\langle \cdot, \cdot \rangle$ the pairing between X and X^*. In $X \times X^*$ we will consider the product topology generated by the strong topologies in X and X^*.

For a sequence $\{x_n\} \subset X$ and $x \in X$, as usual, $x_n \rightharpoonup x$ denotes convergence in the weak topology.

Given a (single or set-valued) operator $T : X \rightrightarrows X^*$ its *inverse* is the operator $T^{-1} : X^* \rightrightarrows X$ defined by $T^{-1}(x^*) = \{x \in X : x^* \in T(x)\}$.

The *graph* of T is the set $\mathrm{Gr}(T) := \{(x, x^*) \in X \times X^* : x^* \in T(x)\}$, and its projection onto X is called the *domain* of T, denoted by $\mathrm{Dom}\, T$.

For an operator $T : X \rightrightarrows X^*$, we denote by $\overline{T}$ the operator defined by $\overline{T}(x) = \overline{T(x)}$, where the notation $\overline{A}$ means the closure of the set $A \subset X^*$ with respect to the strong topology in X^*; and by $\overline{T}^G$ the operator whose graph is the closure of $\mathrm{Gr}(T)$ in $X \times X^*$.

A set-valued operator $T : X \rightrightarrows X^*$ is said to be *monotone* if it satisfies:

$$\langle y^* - x^*, y - x \rangle \geq 0 \ \text{ for every } (x, x^*), (y, y^*) \in \mathrm{Gr}(T).$$

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Such an operator is called *maximal monotone* if its graph is not properly contained in the graph of any other monotone operator from X to X^*.

Equivalently, a monotone operator T is maximal monotone if every pair (y, y^*) which is *monotonically related* to $\mathrm{Gr}(T)$ (i.e. $\langle x^* - y^*, x - y\rangle \geq 0$ for every $(x, x^*) \in \mathrm{Gr}(T)$) belongs to $\mathrm{Gr}(T)$. It is well known that every maximal monotone operator has convex closed values and that its graph is closed in $X \times X^*$.

Recently, a particular type of monotone operator, the so-called premaximal operator, has attracted the attention of several authors, see for example [9, 22]. A monotone operator $T : X \rightrightarrows X^*$ is called *premaximal* if it has a unique maximal monotone extension.

Recall that an extended real-valued function $f : X \to \mathbb{R} \cup \{+\infty\}$ is *proper* if its *effective domain* $\mathrm{dom}\, f = \{x \in X : f(x) < +\infty\}$ is nonempty.

Given a convex lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$, the *subdifferential* $\partial f : X \rightrightarrows X^*$ of f is defined by

$$\partial f(x) = \big\{x^* \in X^* : f(y) \geq f(x) + \langle x^*, y - x\rangle \text{ for every } y \in X\big\},$$

if $x \in \mathrm{dom}\, f$ and $\partial f(x) = \emptyset$ elsewhere.

A classical result of R. T. Rockafellar [17, Theorem A] asserts that the subdifferential of any proper convex lower semicontinuous function on a Banach space is a maximal monotone operator.

We denote by J the *duality mapping* between X and X^*. This mapping is the subdifferential of the continuous convex function $\frac{1}{2}\|\cdot\|^2 : X \to \mathbb{R}$, $x \mapsto \frac{1}{2}\|x\|^2$, and it can be given by

$$J(x) = \big\{x^* : \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\big\}.$$

As a consequence of the result of R. T. Rockafellar mentioned above, J is a maximal monotone operator.

The following proposition gives us an idea of the importance of J in the study of Banach spaces.

Proposition 1.1 ([13, Proposition 2.17]). *Let X be a real Banach space (not necessarily reflexive). Then:*

(*1*) *The norm on X is everywhere Gâteaux differentiable (except at the origin) if and only if J is single valued.*
(*2*) *The mapping J is one-to-one (that is, $J(x) \cap J(y) = \emptyset$ whenever $x \neq y$) if and only if the norm on X is strictly convex.*
(*3*) *The surjectivity of J is equivalent to the reflexivity of X.*

E. Asplund [1] showed that when X is reflexive, there exists an equivalent norm such that this norm and its polar norm on X^* are both everywhere Gâteaux differentiable (except at the origin). With such norms, it is easy to see that the duality mapping J^* between X^* and X is equal to J^{-1}, so from Proposition 1.1 both norms are strictly convex. We will assume from now on that the norms on X and X^* have these properties.

This paper is organized as follows. In Section 2 we recall the definitions of the Extended and Variational Sums of monotone operators, and we survey some fundamental results about these sums. We begin Section 3 establishing new properties of the Variational Sum. It turns out that the graph of this operator is closed in $X \times X^*$ and it has convex closed values. We also show that in the setting of Euclidean spaces, the Variational Sum coincides with the intersection of all its maximal monotone extensions. We then state the main result of this work, *that the graph of the Extended Sum is contained in the graph of the Variational Sum*, and we provide an example showing that this inclusion may be proper. This allows us to give a positive answer to the long-standing question of whether the graph of the usual sum is contained in the graph of the Variational Sum.

2. The Extended and Variational Sums

2.1. The Extended Sum

Given a monotone operator $T : X \rightrightarrows X^*$ and $\varepsilon > 0$, the $\varepsilon-$*enlargement* of T is the operator $T^\varepsilon : X \rightrightarrows X^*$ defined by

$$T^\varepsilon(x) = \big\{x^* \in X^* : \langle y^* - x^*, y - x\rangle \geq -\varepsilon \ \text{ for every } (y, y^*) \in \mathrm{Gr}(T)\big\}.$$

It is easy to see from the definition that T^ε has convex closed values.

This notion has been introduced independently by J.-E. Martínez-Legaz and M. Théra in [10], and by R. Burachik et al. in [4], and then systematically studied and extended in a more general framework by B. F. Svaiter et al. in [5, 6, 21].

With this notion in mind, and motivated by a well known result of J.-B. Hiriart-Urruty and R. R. Phelps [8, Theorem 2.1] dealing with ε-subdifferentails, J. Revalski and M. Théra introduced in [16] the concept of Extended Sum.

Definition 2.1 ([16]). Let $T_1, T_2 : X \rightrightarrows X^*$ be two monotone operators. The *Extended Sum* of T_1 and T_2 is the operator $T_1 \underset{\text{ext}}{+} T_2 : X \rightrightarrows X^*$ defined by

$$(T_1 \underset{\text{ext}}{+} T_2)(x) = \bigcap_{\varepsilon>0} \overline{T_1^\varepsilon(x) + T_2^\varepsilon(x)}, \quad \text{for every } x \in X. \tag{1}$$

Remark 2.2. The concept of Extended Sum was introduced in arbitrary Banach spaces in [14], where the closure on the right-hand side of (1) was taken with respect to the weak star topology in X^*.

Notice that in our setting of reflexive Banach spaces, since for any $\varepsilon > 0$, $T_1^\varepsilon + T_2^\varepsilon$ has convex values, these two closures coincide.

It is clear from the definition that the Extended Sum is an extension of the usual one in the sense of graph inclusion. The following theorem gives us some of the most important properties of this sum.

Theorem 2.3. *Let $T_1, T_2 : X \rightrightarrows X^*$ be two maximal monotone operators. Then:*

(*i*) *The Extended Sum $T_1 \underset{\text{ext}}{+} T_2$ is a monotone operator.*

(*ii*) *If $\overline{T_1 + T_2}$ is a maximal monotone operator, then $\overline{T_1 + T_2} = T_1 \underset{\text{ext}}{+} T_2$.*

(*iii*) *If f and g are two proper lower semicontinuous convex functions with* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$*, then*

$$\partial(f + g)(x) = (\partial f \underset{\text{ext}}{+} \partial g)(x), \ \textit{for all } x \in X.$$

Y. García, M. Lassonde and J. Revalski have recently shown (i) in [7, Proposition 3.4], while (ii) and (iii) were shown by J. Revalski and M. Théra in [16, Theorem 4.1 and Theorem 4.4]. It was also shown in [7] that, in general, the Extended Sum is not a maximal monotone operator. All these results were established for arbitrary Banach spaces.

2.2. The Variational Sum

Remember that the norms on X and X^* are assumed to be both Gâteaux differentiable (except at the origin), so that $J : X \rightrightarrows X^*$ is bijective and $\|\cdot\| \times w$-continuous, i.e., continuous from $(X, \|\cdot\|)$ to (X^*, w).

Given a maximal monotone operator $T : X \rightrightarrows X^*$, for $\lambda > 0$, the *Yosida resolvent* of T of order λ is the operator $J_\lambda^T : X \to X$ which assigns to every $x \in X$ the unique solution $x_\lambda = J_\lambda^T(x)$ of the inclusion

$$0 \in J(x_\lambda - x) + \lambda T(x_\lambda). \tag{2}$$

The fact that x_λ exists and is unique was shown by R. T. Rockafellar in [18, Proposition 1].

The *Yosida regularization* of T of order $\lambda > 0$ is the operator $T_\lambda : X \to X^*$ defined by

$$T_\lambda(x) = \frac{1}{\lambda} J(x - x_\lambda), \quad x \in X. \tag{3}$$

Notice that from (2) and (3), for any $\lambda > 0$

$$T_\lambda(x) \in T(J_\lambda^T x) = T\big(x - \lambda J^{-1} T_\lambda(x)\big), \quad \text{for all } x \in X. \tag{4}$$

By convention we put $T_0 = T$.

By [18, Proposition 1] for every $\lambda > 0$, the operator T_λ is maximal monotone, single valued, everywhere defined and $\|\cdot\| \times w$-continuous (see also the work of H. Brézis, M. G. Crandall and A. Pazzy [3] for more properties of this notion).

Given two maximal monotone operators defined in an arbitrary Banach space, several sufficient conditions for the maximality of the sum of these operators had been introduced in the literature; see for example the works of R. R. Phelps [12], S. Simons et al. [19, 20], M. D. Voisei [22] and the references therein.

Let us observe that, given two maximal monotone operators $T_1, T_2 : X \rightrightarrows X^*$ and $\lambda, \mu \geq 0$ such that $\lambda + \mu > 0$, $T_{1,\lambda} + T_{2,\mu}$ is a maximal monotone operator because at least one of the operators concerned is everywhere defined. The idea behind the Variational Sum is to take advantage from this fact doing some kind of "approximation" with these maximal monotone operators.

The notion of Variational Sum was introduced in the setting of Hilbert spaces by H. Attouch, J.-B. Baillon and M. Théra in [2], and later extended to the setting of reflexive Banach spaces by Revalski and Théra in [15].

Definition 2.4 ([2, 15]). Let $T_1, T_2 : X \rightrightarrows X^*$ be two maximal monotone operators. The Variational Sum of T_1 and T_2 is the operator $T_1 \underset{\mathrm{v}}{+} T_2 : X \rightrightarrows X^*$ whose graph is defined by

$$\operatorname{Gr}(T_1 \underset{\mathrm{v}}{+} T_2) = \liminf_{\mathcal{F}} (T_{1,\lambda} + T_{2,\mu}), \tag{5}$$

where $\liminf_{\mathcal{F}}$ is taken in the sense of Painlevé-Kuratowski (see [15] for details).

In other words, $(x, x^*) \in \operatorname{Gr}(T_1 \underset{v}{+} T_2)$ if and only if for every sequence

$$\{(\lambda_n, \mu_n)\} \in I = \Big\{\{(\lambda_n, \mu_n)\}\colon \lambda_n,\, \mu_n \geq 0,\, \lambda_n + \mu_n > 0, (\lambda_n, \mu_n) \to (0,0)\Big\},$$

there exists $\{(x_n, x_n^*)\} \subset X \times X^*$ such that

$$(x_n, x_n^*) \in \operatorname{Gr}(T_{1,\lambda_n} + T_{2,\mu_n}), \quad \text{for all } n \in \mathbb{N}, \text{ and}$$

$$(x_n, x_n^*) \to (x, x^*).$$

Next, we list some properties of the Variational Sum, which where established in the context of Hilbert spaces in [2] and in the case of reflexive Banach spaces considering Fréchet differentiable (except at the origin) norms in X and X^* in [15, Proposition 4.6 and Theorem 5.1].

Proposition 2.5 (cfr. [2, 15]). *Let $T_1, T_2 : X \rightrightarrows X^*$ be two maximal monotone operators. Then:*

(*i*) *The Variational Sum $T_1 \underset{v}{+} T_2$ is a monotone operator.*

(*ii*) $\operatorname{Dom}(T_1 + T_2) \subset \operatorname{Dom}(T_1 \underset{v}{+} T_2)$.

(*iii*) *If $T_1 \underset{v}{+} T_2$ is a maximal monotone operator, then* $\operatorname{Gr}(T_1 + T_2) \subset \operatorname{Gr}(T_1 \underset{v}{+} T_2)$.

(*iv*) *If f and g are two proper lower semicontinuous convex functions with* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$, *then*

$$\partial(f + g)(x) = (\partial f \underset{v}{+} \partial g)(x), \quad \textit{for all } x \in X.$$

While (*i*) follows directly from the definition, (*ii*)–(*iv*) will be proven later.

3. Main result

Given two maximal monotone operators $T_1, T_2 : X \rightrightarrows X^*$ and $\{(\lambda_n, \mu_n)\} \in I$, for each $(x, x^*) \in X \times X^*$ and $n \in \mathbb{N}$, we denote by $\psi_{\lambda_n,\mu_n}(x, x^*) = x_n$ the unique solution of

$$x^* \in J(x_n - x) + T_{1,\lambda_n}(x_n) + T_{2,\mu_n}(x_n). \tag{6}$$

Lemma 3.1. *Let $T_1, T_2 : X \rightrightarrows X^*$ be two maximal monotone operators and let $(y, y^*) \in X \times X^*$. The following statements are equivalent:*

(*i*) $(y, y^*) \in \operatorname{Gr}(T_1 \underset{v}{+} T_2)$;

(*ii*) *for every $(x, x^*) \in X \times X^*$ and $\{(\lambda_n, \mu_n)\} \in I$, the sequence $\{x_n\}$, where $x_n = \psi_{\lambda_n,\mu_n}(x, x^*)$ for each $n \in \mathbb{N}$, is bounded and for every subsequence $x_{n_k} \rightharpoonup \overline{x}$*

$$\frac{1}{2}\|y - x\|^2 + \big\langle x^* - y^*, \overline{x} - y\big\rangle \geq \frac{1}{2}\limsup \|x_{n_k} - x\|^2; \tag{7}$$

(*iii*) *for every $\{(\lambda_n, \mu_n)\} \in I$, $\psi_{\lambda_n,\mu_n}(y, y^*) \to y$.*

Proof. *(i)* $\Rightarrow$ *(ii)* Let $(x,x^*) \in X \times X^*$ and $\{(\lambda_n,\mu_n)\} \in I$. Since $(y,y^*) \in \operatorname{Gr}(T_1 \underset{v}{+} T_2)$, from the definition of the Variational Sum, there exists a sequence $\{(y_n,y_n^*)\} \subset X \times X^*$ such that $y_n^* \in (T_{1,\lambda_n} + T_{2,\mu_n})(y_n)$ and $(y_n,y_n^*) \to (y,y^*)$. From the monotonicity of $T_{1,\lambda_n} + T_{2,\mu_n}$ and (6), we have

$$\langle x^* - J(x_n - x) - y_n^*, x_n - y_n \rangle \geq 0, \tag{8}$$

hence, $\langle x^* - y_n^*, x_n - y_n\rangle \geq \langle J(x_n - x), x_n - y_n\rangle$. Since J is the subdifferential of $\frac{1}{2}\|\cdot\|^2$, we have

$$\langle J(x_n - x), y_n - x_n\rangle \leq \frac{1}{2}\|y_n - x\|^2 - \frac{1}{2}\|x_n - x\|^2.$$

Combining the last two inequalities we obtain that

$$\frac{1}{2}\|y_n - x\|^2 + \langle x^* - y_n^*, x_n - y_n\rangle \geq \frac{1}{2}\|x_n - x\|^2. \tag{9}$$

Since $\{y_n\}$ and $\{y_n^*\}$ are convergent sequences, we derive from (9) that there exist $M, K \geq 0$ such that

$$M + K\|x_n - x\| \geq \frac{1}{2}\|x_n - x\|^2,$$

proving that $\{x_n\}$ is bounded. Passing to the limit in (9) with any subsequence $x_{n_k} \rightharpoonup \overline{x}$ we obtain (7).

(ii) $\Rightarrow$ *(iii)* Let $\{(\lambda_n,\mu_n)\} \in I$. From *(ii)* with $(x,x^*) = (y,y^*)$, we deduce that the sequence $\{y_n\}$, where $y_n = \psi_{\lambda_n,\mu_n}(y,y^*)$, is bounded and that any weak-converging subsequence $\{y_{n_k}\}$ verifies

$$0 \geq \frac{1}{2}\limsup \|y_{n_k} - y\|^2,$$

so $y_{n_k} \to y$. This shows that in fact the whole sequence $\{y_n\}$ converges to y.

(iii) $\Rightarrow$ *(i)* Let $\{(\lambda_n,\mu_n)\}$ be an arbitrary element of I. By definition of $y_n = \psi_{\lambda_n,\mu_n}(y,y^*)$, there exists $u_n^* \in T_{1,\lambda_n}(y_n)$ and $v_n^* \in T_{2,\mu_n}(y_n)$ such that

$$y^* = J(y_n - y) + u_n^* + v_n^*.$$

Since $\|J(y_n - y)\| = \|y_n - y\| \to 0$, we deduce that $v_n^* + u_n^* \to y^*$. Since $\{(\lambda_n,\mu_n)\}$ is arbitrary, this shows that $(y,y^*) \in \operatorname{Gr}(T_1 \underset{v}{+} T_2)$ by definition of the Variational Sum. □

Proposition 3.2. *Let $T_1, T_2 : X \rightrightarrows X^*$ be two maximal monotone operators. Then:*

(i) *If $\{(y_n,y_n^*)\} \subset \operatorname{Gr}(T_1 \underset{v}{+} T_2)$, $y_n \to y$ and $y_n^* \rightharpoonup y^*$, then $(y,y^*) \in \operatorname{Gr}(T_1 \underset{v}{+} T_2)$.*

(ii) *$T_1 \underset{v}{+} T_2$ has closed graph and convex values.*

Proof. *(i)* Let $(x,x^*) \in X \times X^*$ and $\{(\lambda_n,\mu_n)\} \in I$. By Lemma 3.1*(ii)*, $\{x_n\}$ where $x_n = \psi_{\lambda_n,\mu_n}(x,x^*)$, is bounded and for each $n \in \mathbb{N}$, (y_n,y_n^*) verifies

$$\frac{1}{2}\|y_n - x\|^2 + \langle x^* - y_n^*, \overline{x} - y_n\rangle \geq \frac{1}{2}\limsup \|x_{n_k} - x\|^2,$$

for every subsequence $x_{n_k} \rightharpoonup \overline{x}$. Passing to the limit, we get

$$\frac{1}{2}\|y-x\|^2 + \langle x^* - y^*, \overline{x} - y\rangle \geq \frac{1}{2}\limsup \|x_{n_k} - x\|^2.$$

By the equivalence between (ii) and (i) in Lemma 3.1, we derive that $(y, y^*) \in \mathrm{Gr}(T_1 \underset{v}{+} T_2)$.

(ii) The fact that $T_1 \underset{v}{+} T_2$ has closed graph follows directly from (i). Let $(y, y_i^*) \in \mathrm{Gr}(T_1 \underset{v}{+} T_2)$, $i = 1, 2$ and $t \in]0,1[$. Let $\{(\lambda_n, \mu_n)\} \in I$. Applying the equivalence between (i) and (iii) in Lemma 3.1, for $(x, x^*) = \big(y, ty_1^* + (1-t)y_2^*\big)$ we obtain

$$\big\langle ty_1^* + (1-t)y_2^* - y_i^*, \overline{y} - y\big\rangle \geq \frac{1}{2}\limsup \|y_{n_k} - y\|^2, \quad \text{for } i = 1, 2,$$

where $y_n = \psi_{\lambda_n,\mu_n}(y, ty_1^* + (1-t)y_2^*)$ and $y_{n_k} \rightharpoonup \overline{y}$. Multiplying the relation with $i = 1$ by t, the one with $i = 2$ by $1 - t$ and summing up, we obtain

$$0 \geq \frac{1}{2}\limsup \|y_{n_k} - y\|^2,$$

so $y_{n_k} \to y$. This shows that in fact $\psi_{\lambda_n,\mu_n}(y, ty_1^* + (1-t)y_2^*) \to y$. By the equivalence between (i) and (iii) in Lemma 3.1, $\big(y, ty_1^* + (1-t)y_2^*\big) \in \mathrm{Gr}(T_1 \underset{v}{+} T_2)$. Hence $T_1 \underset{v}{+} T_2$ has convex values. □

Proposition 3.3. *Let X be an Euclidean space, and let $T_1, T_2 : X \rightrightarrows X^*$ be two maximal monotone operators. Then $T_1 \underset{v}{+} T_2$ coincides with the intersection of all its maximal monotone extensions.*

Proof. Let $\mathcal{M}$ be the intersection of all the maximal monotone extensions of $T_1 \underset{v}{+} T_2$, i.e., the operator whose graph is the intersection of the graphs of all the maximal monotone extensions of $T_1 \underset{v}{+} T_2$.

Since $\mathrm{Gr}(T_1 \underset{v}{+} T_2) \subset \mathrm{Gr}(\mathcal{M})$, it suffices to show the converse inclusion. We proceed as in the proof of Lemma 3.1 $(i) \Rightarrow (ii)$. Let $(x, x^*) \in \mathrm{Gr}(\mathcal{M})$ and fix $\{(\lambda_n, \mu_n)\} \in I$. Since we are assuming that X is Euclidean, J is the Identity map. Then, for $(y, y^*) \in \mathrm{Gr}(T_1 \underset{v}{+} T_2)$, (8) becomes

$$\big\langle x^* - (x_n - x) - y_n^*, x_n - y_n\big\rangle \geq 0. \tag{10}$$

From Lemma 3.1(ii), $\{x_n\}$ is bounded, so passing to the limit with any subsequence $x_{n_k} \to \overline{x}$ yields

$$\big\langle (x^* + x - \overline{x}) - y^*, \overline{x} - y\big\rangle \geq 0.$$

Notice that (y, y^*) is an arbitrary element of $\mathrm{Gr}(T_1 \underset{v}{+} T_2)$, which means that the pair $(x^* + x - \overline{x}, \overline{x})$ is monotonically related to $T_1 \underset{v}{+} T_2$; hence $(x^* + x - \overline{x}, \overline{x})$ belongs to some maximal monotone extension $\mathcal{M}'$ of $T_1 \underset{v}{+} T_2$. Since $(x, x^*) \in \mathrm{Gr}(\mathcal{M}')$, we must have

$$-\|x - \overline{x}\|^2 = \big\langle (x^* + x - \overline{x}) - x^*, \overline{x} - x\big\rangle \geq 0.$$

Therefore, $x = \overline{x}$ and it follows that the whole sequence $\{x_n\}$ converges to x. Since $\big\{(\lambda_n, \mu_n)\big\} \in I$ was arbitrary, we conclude from Lemma 3.1 that $(x, x^*) \in \mathrm{Gr}(T_1 \underset{\mathrm{v}}{+} T_2)$. This finishes the proof. □

Corollary 3.4. *Let X be an Euclidean space, and let $T_1, T_2 : X \rightrightarrows X^*$ be two maximal monotone operators. If $T_1 \underset{\mathrm{v}}{+} T_2$ is premaximal, then it is maximal.*

Proof. From Proposition 3.3, $T_1 \underset{\mathrm{v}}{+} T_2$ coincides with the intersection of all its maximal monotone extensions. Since there is only one extension, the Variational Sum must be itself a maximal monotone operator. □

In order to prove the main result we need the following lemma, which is an extension of [11, Lemma 3.1].

Lemma 3.5. *Let $T : X \rightrightarrows X^*$ be a maximal monotone operator. Given $\lambda, \varepsilon \geq 0$, if $w^* \in T^\varepsilon(w)$ and $u^* \in T_\lambda(u)$, then*

$$\langle u^* - w^*, u - w \rangle + \frac{\lambda}{4} \|w^*\|^2 \geq -\varepsilon. \tag{11}$$

Proof. Recall that for any $\lambda \geq 0$, $u^* \in T_\lambda(u)$ implies that $u^* \in T\big(u - \lambda J^{-1}(u^*)\big)$. From the definition of ε-enlargement it follows that

$$\big\langle u^* - w^*, u - \lambda J^{-1}(u^*) - w \big\rangle \geq -\varepsilon.$$

Thus,

$$\begin{aligned} \langle u^* - w^*, u - w \rangle &\geq \lambda \big\langle u^* - w^*, J^{-1}(u^*) \big\rangle - \varepsilon \\ &\geq \lambda \big(\|u^*\|^2 - \|w^*\| \|u^*\| \big) - \varepsilon \\ &\geq -\frac{\lambda}{4} \|w^*\|^2 - \varepsilon, \end{aligned}$$

hence the result. □

Now, we give the main result of this work.

Theorem 3.6. *Let $T_1, T_2 : X \rightrightarrows X^*$ be two maximal monotone operators. Then*

$$\mathrm{Gr}(T_1 \underset{\mathrm{ext}}{+} T_2) \subset \mathrm{Gr}(T_1 \underset{\mathrm{v}}{+} T_2).$$

Proof. Let $(y, y^*) \in \mathrm{Gr}(T_1 \underset{\mathrm{ext}}{+} T_2)$. The result is proved by showing that (y, y^*) verifies Lemma 3.1(*ii*). Consider $(x, x^*) \in X \times X^*$, $\big\{(\lambda_n, \mu_n)\big\} \in I$, $x_n = \psi_{\lambda_n, \mu_n}(x, x^*)$ and $u_n^* \in T_{1,\lambda_n}(x_n)$, $v_n^* \in T_{2,\mu_n}(x_n)$ such that

$$x^* = J(x_n - x) + u_n^* + v_n^*. \tag{12}$$

For $\varepsilon > 0$, let $(y, y_1^* + y_2^*) \in \mathrm{Gr}(T_1^\varepsilon + T_2^\varepsilon)$ with $y_i^* \in T_i^\varepsilon(y)$ for $i = 1, 2$; and $\|y^* - (y_1^* + y_2^*)\| < \varepsilon$. By applying Lemma 3.5 to $(y, y_1^*), (x_n, u_n^*)$ and $(y, y_2^*), (x_n, v_n^*)$ respectively, we have

$$\langle u_n^* - y_1^*, x_n - y \rangle + \frac{\lambda_n}{4} \|y_1^*\|^2 \geq -\varepsilon$$

and

$$\langle v_n^* - y_2^*, x_n - y\rangle + \frac{\mu_n}{4}\|y_2^*\|^2 \geq -\varepsilon.$$

Summing up these inequalities, we obtain

$$\big\langle (u_n^* + v_n^*) - (y_1^* + y_2^*), x_n - y\big\rangle + \frac{\lambda_n}{4}\|y_1^*\|^2 + \frac{\mu_n}{4}\|y_2^*\|^2 \geq -2\varepsilon.$$

Equivalently, by (12)

$$\big\langle x^* - J(x_n - x) - (y_1^* + y_2^*), x_n - y\big\rangle + \frac{\lambda_n}{4}\|y_1^*\|^2 + \frac{\mu_n}{4}\|y_2^*\|^2 \geq -2\varepsilon.$$

Now, using the fact that $\big\langle J(x_n - x), y - x_n\big\rangle \leq \frac{1}{2}\|y - x\|^2 - \frac{1}{2}\|x_n - x\|^2$, it follows that

$$\frac{1}{2}\|y - x\|^2 + \big\langle x^* - (y_1^* + y_2^*), x_n - y\big\rangle + \lambda_n\frac{\|y_1^*\|^2}{4} + \mu_n\frac{\|y_2^*\|^2}{4} \geq \frac{1}{2}\|x_n - x\|^2 - 2\varepsilon. \quad (13)$$

Observe first that (13) implies that the sequence $\{x_n\}$ is bounded. Now, consider a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \rightharpoonup \overline{x}$. Passing to the limit in (13) as $n_k \to +\infty$, we get

$$\frac{1}{2}\|y - x\|^2 + \big\langle x^* - (y_1^* + y_2^*), \overline{x} - y\big\rangle \geq \limsup \frac{1}{2}\|x - x_{n_k}\|^2 - 2\varepsilon.$$

Now, passing to the limit as $\varepsilon \searrow 0$, we obtain

$$\frac{1}{2}\|y - x\|^2 + \langle x^* - y^*, \overline{x} - y\rangle \geq \limsup \frac{1}{2}\|x - x_{n_k}\|^2.$$

This shows that (y, y^*) verifies Lemma 3.1(ii). The proof is complete. □

Corollary 3.7. *Let $T_1, T_2 : X \rightrightarrows X^*$ be two maximal monotone operators. Then*

$$\operatorname{Gr}(T_1 + T_2) \subset \operatorname{Gr}(T_1 \underset{\mathrm{v}}{+} T_2).$$

Proof. Since $\operatorname{Gr}(T_1 + T_2) \subset \operatorname{Gr}(T_1 \underset{\mathrm{ext}}{+} T_2)$, the result follows from Theorem 3.6. □

Corollary 3.8. *Let $T_1, T_2 : X \rightrightarrows X^*$ be two maximal monotone operators. Then*

$$\operatorname{Gr}\big(\overline{T_1 \underset{\mathrm{ext}}{+} T_2}^G\big) \subset \operatorname{Gr}(T_1 \underset{\mathrm{v}}{+} T_2).$$

Proof. Recall that $\overline{T}^G$ is the operator whose graph is the closure of $\operatorname{Gr}(T)$. From Theorem 3.6

$$\operatorname{Gr}(T_1 \underset{\mathrm{ext}}{+} T_2) \subset \operatorname{Gr}(T_1 \underset{\mathrm{v}}{+} T_2),$$

and from Proposition 3.2(ii), $\operatorname{Gr}(T_1 \underset{\mathrm{v}}{+} T_2)$ is closed, hence the result. □

Corollary 3.9. *Let $T_1, T_2 : X \rightrightarrows X^*$ be two maximal monotone operators. If $\overline{T_1 \underset{\mathrm{ext}}{+} T_2}^G$ is a maximal monotone operator, then*

$$\overline{T_1 \underset{\mathrm{ext}}{+} T_2}^G = T_1 \underset{\mathrm{v}}{+} T_2.$$

Proof. Since both the Variational and the Extended Sum are monotone operators this result follows directly from Corollary 3.8. □

Corollary 3.10 ([14, Corollary 3.7]). *Let $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be two proper lower semicontinuous convex functions such that $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$. Then*

$$\partial(f+g) = \partial f \underset{\text{ext}}{+} \partial g = \partial f \underset{\text{v}}{+} \partial g.$$

Proof. Since $\partial(f+g) = \partial f \underset{\text{ext}}{+} \partial g$, in particular the Extended sum is a maximal monotone operator and then its graph is closed. The result now follows applying Corollary 3.9. □

Corollary 3.11 ([14, Theorem 3.5]). *Let $T_1, T_2 : X \rightrightarrows X^*$ be two maximal monotone operators. If $\overline{T_1 + T_2}$ is a maximal monotone operator, then*

$$\overline{T_1 + T_2} = T_1 \underset{\text{ext}}{+} T_2 = T_1 \underset{\text{v}}{+} T_2.$$

Proof. Since the Variational, the Extended and the usual sums are monotone and $\operatorname{Gr}(T_1 + T_2) \subset \operatorname{Gr}(T_1 \underset{\text{ext}}{+} T_2)$, this follows from Corollary 3.8. □

Corollary 3.12. *Let X be an Euclidean space, and let $T_1, T_2 : X \rightrightarrows X^*$ be two maximal monotone operators. If $T_1 \underset{\text{ext}}{+} T_2$ (or $T_1 + T_2$) is premaximal, then its only maximal monotone extension is $T_1 \underset{\text{v}}{+} T_2$.*

Proof. From Theorem 3.6, we know that $\operatorname{Gr}(T_1 \underset{\text{ext}}{+} T_2) \subset \operatorname{Gr}(T_1 \underset{\text{v}}{+} T_2)$. It follows that $T_1 \underset{\text{v}}{+} T_2$ must be premaximal, thus from Corollary 3.4 the Variational Sum is maximal. □

We finish giving an example which shows that in general the Extended and the Variational Sum are different.

Example 3.13 ([7, Example 3.11]). Let X be the Hilbert space $l_2 \times l_2$ endowed with the usual inner product and induced norm, and let us identify X^* with X. Let us define $\operatorname{Dom} T = D \times D$, where

$$D = \big\{\{x_n\} \subset l_2 : \{2^n x_n\} \in l_2\big\},$$

and $T : \operatorname{Dom} T \to l_2 \times l_2$ by

$$T\big(\{x_n\}, \{y_n\}\big) = \big(\{2^n y_n\}, -\{2^n x_n\}\big).$$

Then $\langle Tu, u\rangle = 0$, for all $u \in D$. For $T_1 = T$ and $T_2 = -T$ it follows that:

(i) T_1 and T_2 are maximal monotone operators with $\operatorname{Dom} T_1 = \operatorname{Dom} T_2 = D \times D$.

(ii) $\operatorname{Dom}(T_1 \underset{\text{ext}}{+} T_2) = D \times D$ and $(T_1 \underset{\text{ext}}{+} T_2)(\overline{u}) = 0$, for all $\overline{u} \in D \times D$.

(iii) $\operatorname{Dom}(T_1 \underset{\text{v}}{+} T_2) = l_2 \times l_2$ and $(T_1 \underset{\text{v}}{+} T_2)(\overline{u}) = 0$, for all $\overline{u} \in l_2 \times l_2$.

Indeed, (i) and (ii) were established in [7]. From Theorem 3.6, $\mathrm{Gr}(T_1 \underset{\text{ext}}{+} T_2) \subset \mathrm{Gr}(T_1 \underset{\text{v}}{+} T_2)$, and from Proposition 3.2(*ii*) we know that $\mathrm{Gr}(T_1 \underset{\text{v}}{+} T_2)$ is closed; since $D \times D$ is dense in X we obtain (iii).

In particular, we have $T_1 \underset{\text{ext}}{+} T_2 \neq \overline{T_1 \underset{\text{ext}}{+} T_2}^{G} = T_1 \underset{\text{v}}{+} T_2$. In this case $T_1 \underset{\text{v}}{+} T_2$ is the zero operator, which is maximal monotone.

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