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When is a Convex Cone the Cone of all the Half-Lines Contained in a Convex Set?

Emil Ernst
Aix-Marseille Univ., UMR6632, Marseille, 13397, France
emil.ernst@univ-cezanne.fr

Michel Volle
Université d'Avignon et des Pays de Vaucluse,
74 rue Louis Pasteur - 84029 Avignon Cedex 1, France
michel.volle@univ-avignon.fr

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In this article we prove that every convex cone V of a real vector space X possessing an uncountable Hamel basis may be expressed as the cone of all the half-lines contained within some convex subset C of X (in other words, V is the infinity cone to C). This property does not hold for lower-dimensional vector spaces; more precisely, a convex cone V in a vector space X with a denumerable basis is the infinity cone to some convex subset of X if and only if V is the union of a countable ascending sequence of linearly closed cones, while a convex cone V in a finite-dimensional vector space X is the infinity cone to some convex subset of X if and only if V is linearly closed.

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1. Statement of the problem

It has been noticed for a long time (Steinitz, [11]) that each and every convex subset C of X which is closed with respect to some vector space topology on X possess the following geometric property that we refer hereafter as *exactness*: *For every two points $x, y \in C$ and every vector $d \in X$, the statements $(x + \mathbb{R}_+ d) \subset C$ and $(y + \mathbb{R}_+ d) \subset C$ hold or fail together.*

Steinitz's theorem (originally stated in the finite-dimensional setting, but valid with no modification for any topological real vector space) is arguably at the origin of the recession analysis, a basic tool in today convex analysis and optimization. For a detailed account of this field, as well as for an impressive list of applications to problems in convex geometry, optimization and PDE, the reader is refereed to the classical Rockafellar's monograph [10], or to the recent publication of Attouch, Buttazzo and Michaille, [1]; the non-convex setting is addressed by Auslender and Teboulle, [2].

Of course, exactness is not a property reserved to closed convex sets. Open half-spaces

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obviously have it, and since an easy to prove statement reads that the class of all the exact sets is closed with respect to nonvoid intersection, it follows that every *evenly convex* set, that is, (see [9], for a detailed analysis of the finite-dimensional case, and [5], for the study of this notion in the Banach setting) every set which may be expressed as the intersection of a family of open half-spaces, is also an exact set.

On the other hand, exactness may fail, even for convex sets in finite-dimensional spaces. Consider for instance the convex cone W of $\mathbb{R} \times \mathbb{R}$, defined as

$$W = \{(0,0)\} \cup \mathbb{R}_+^\star \times \mathbb{R}_+^\star,$$

and take $x = (0,0)$, $y = (1,1)$ and $d = (1,0)$. It is easy to see that the half-line $(y + \mathbb{R}_+ d) = [1, +\infty) \times \{1\}$ is completely included in W, while its translate $(x + \mathbb{R}_+ d) = \mathbb{R}_+ \times \{0\}$ is not.

In order to address the case of non-exact sets, it is customary to provide a more detailed analysis of the class of half-lines contained within a given set. To this respect, let us associate (we adopt the definitions and notations from [7, p. 327]) to every pair consisting of a nonempty subset A of a real vector space X, and of a point a of A, the *inscribed cone* at a to A, denoted $\mathcal{I}_0(a, A)$, and defined as the union between all the rays (*i.e.* half-lines issuing from θ, the null vector of X) whose translates stemming from a are completely contained in A, to which we add the singleton $\{\theta\}$:

$$\mathcal{I}_0(a, A) = \{v \in X : a + \mathbb{R}_+ v \subset A\}.$$

The *recession cone*, $\mathcal{R}(A)$, to the set A is the intersection of all the inscribed cones to A, that is,

$$\mathcal{R}(A) = \bigcap_{a \in A} \mathcal{I}_0(a, A);$$

their union,

$$\mathcal{I}(A) = \bigcup_{a \in A} \mathcal{I}_0(a, A),$$

is usually called the *infinity cone* to A. Using this notation, exactness requires that the recession and the infinity cones to a set coincide.

It is easy to prove that both the recession and the infinity cones to a non-empty convex set are convex. This article addresses the following problem:

(P) *Let X be a real vector space; characterize all the convex cones which can be expressed as the infinity cone to some convex subset of X.*

In more geometrical terms, this problem request to find, for a given convex cone V from a real vector space X, a way to rebuild all the rays from V into a convex subset C of X in such a manner that no half-line, other than the translates of rays from V, be contained within C.

Let us first remark that problem (P) is not trivial, since the infinity cone to a given convex cone V may be larger than V, and thus the cone V itself is not always an answer to (P).

Consider for instance the case of the previously defined cone $W \subset \mathbb{R}^2$, $W = \{(0,0)\} \cup \mathbb{R}^\star_+ \times \mathbb{R}^\star_+$. An easy calculation yields that the inscribed cone at $(0,0)$ to W amounts to W, while at any point (x,y) of W such that $(x,y) \neq (0,0)$, it holds that $\mathcal{I}_0((x,y),W) = \mathbb{R}_+ \times \mathbb{R}_+$. Accordingly, $\mathcal{I}(W) = \mathbb{R}_+ \times \mathbb{R}_+$, which is a convex cone properly containing V.

More precisely, if we denote by $\kappa(A)$ the *Klee envelope* of a set A, that is, the union between A and all the vectors $x \in X$ which are *linearly accessible* from A, that is, all the points $x \notin A$ such that an open segment emanating from x lies entirely within A (notion initially defined in [8]), it holds (see [6]) the following characterization of the infinity cone to a convex cone.

Proposition 1.1. *For every convex cone V we have $I(V) = \kappa(V)$.*

It results that a convex cone V coincides with its infinity cone, and thus that problem (P) is simple to prove, if and only if V is *linearly closed*, that is, $V = \kappa(V)$ (notice that in a finite dimensional setting, a convex set is linearly closed if and only if it is closed with respect to the Euclidean topology on X.)

The main result of our article (Theorem 4.1, Section 4) provides a complete characterisation of the convex cones which can be expressed as infinity cones to convex sets.

When X is of finite dimension, we prove that a cone V is the infinity cone to some convex set if and only if V is linearly closed, and thus closed with respect to the Euclidean topology on X (in other words, in finite dimensional spaces there is no other solution to problem (P) than the trivial one.)

In the setting of vector space of denumerable (that is, countable and infinite) dimension, the class of convex cones V for which $V = \mathcal{I}(C)$ for some convex set C is larger than the class of linearly closed convex cones. Namely, V is the infinity cone to some convex set if and only if V is the union of an ascending sequence of linealy closed and convex cones.

Finally, Theorem 4.1 shows that every convex cone V of a real vector space X of uncountable dimension may be expressed as the infinity cone to some convex subset of X.

The situation depicted by Theorem 4.1 goes along with our geometrical intuition: in lower-dimensional spaces there is less room to "correctly" spread the rays composing a given cone into a convex set with no half-lines "in excess", so the lower will be the dimension of the underlying space, the larger will be the class of cones for which problem (P) cannot be solved.

On the other hand, Theorem 4.1 proves that three apparently unrelated sets of convex cones, namely the class of linearly closed and convex cones of a finite-dimensional space, the class gathering all the cones which can be represented as the union of an ascending sequence of linearly closed and convex cones in a vector space of denumerable dimension, and the class of all the convex cones in a vector spaces with an uncountable dimension, are the only cones which can be expressed as the infinity cone to some convex set, being thereof closely connected.

The interplay between these three classes is made clear in Section 2, where we define and study the notion of spreading cover for a convex cone, and prove that (Proposition 2.7) the three above-defined classes are precisely the classes of all the convex cones admitting a spreading cover. Finally, Propositions 2.6 and 3.3 show that a convex cone is the infinity cone to some convex set if and only if it possess a spreading cover.

2. Spreading covering

The main object of this section is to provide necessary conditions ensuring that a convex cone is the infinity cone to some set. To this end, we must conduct a detailed analysis of the family of inscribed cones to a given convex set.

To begin with, let us collect several evident properties of the inscribed and infinity cones to a set, as well as of the Klee envelope of a convex set.

Lemma 2.1. *Let P and K be two convex sets. Then*

i) *for any $x \in P$ it holds that $\mathcal{I}_0(x, P) \subset \mathcal{I}_0(x, K)$ provided that $P \subset K$,*
ii) *if $P \subset K$, then $\mathcal{I}(P) \subset \mathcal{I}(K)$,*
iii) *for any vector $x \in X$, we have $\mathcal{I}(P) = \mathcal{I}(x + P)$,*
iv) *$\kappa(P \cap K) \subset \kappa(P) \cap \kappa(K)$.*

Let us now consider C a convex subset of X, and set V for its infinity cone. Since $V = \bigcup_{x \in C} \mathcal{I}_0(x, C)$, the class of all the inscribed cones to C form a cover of V composed by convex cones.

It worths to be noticed that this covering of V may admits sub-coverings composed from significantly fewer elements; indeed, one does not always need to unite *all* the inscribed cones to C in order to get V. To illustrate this fact, let us, once more, consider the cone $W = (0, 0) + \mathbb{R}_+^\star \times \mathbb{R}_+^\star$, and notice that

$$\mathcal{I}_0((1, 1), W) = \mathbb{R}_+ \times \mathbb{R}_+ = \mathcal{I}(W).$$

Even if such a spectacular gap between the cardinal number of C and the cardinal number of the class of points of C actually nedded to achieve its infinity cone cannot always be expected, we still can prove that the infinity cone to a convex set can be obtained by uniting a family of inscribed cones whose cardinal number does not exceed the dimension of X.

Proposition 2.2. *Let C be a non-empty convex subset of the real vector space X. Then there is D, a (non-necessarily convex) subset of C of cardinal number less or equal to the dimension of X, such that*

$$\mathcal{I}(C) = \bigcup_{x \in D} \mathcal{I}_0(x, C).$$

Proof of Proposition 2.2. When the dimension of X is greater or equal to the cardinal number of the real line, it is well-known that the cardinal number of the set X itself is equal to the dimension of X; the requirements of Proposition 2.2 can thus be fulfilled by simply takeing $D = C$.

By the Continuum Hypothesis it results that all what it remains to be addressed is the case when the dimension of X is countable.

Before considering this case, let us recall the notion of *pseudo relative interior* of a convex set A, $\mathrm{pri}(A)$, that is, the set of all points $x \in A$ such that the cone $\mathbb{R}_+(A-x)$ is in fact a linear space (for an ample discution of this notion, see [3]). Remark also that in the finite dimensional setting, the pseudo relative interior reduces to the familiar concept of *relative interior* of a convex set A, that is the interior of A with respect to the topology induced on the affine span of A by the Euclidean topology of the underlying space.

The following technical result will be used repeatedly.

Lemma 2.3. *Let D be a subset of the non-empty convex set C, and consider $\overline{x} \in \mathrm{pri}(\mathrm{co}(D))$. Then*

$$\mathcal{I}_0(x,C) \subset \mathcal{I}_0(\overline{x},C) \quad \forall x \in D. \tag{1}$$

In particular,

$$\mathcal{I}(C) = \mathcal{I}_0(\overline{w},C) \quad \forall \overline{w} \in \mathrm{pri}(C). \tag{2}$$

Proof of Lemma 2.3. Let $\overline{x} \in \mathrm{pri}(\mathrm{co}(D))$; all what we have to prove in order to establish relation (1) is that, for any $x \in D$ and $v \in \mathcal{I}_0(x,C)$, we have

$$\overline{x} + r\,v \in C \quad \forall r \geq 0. \tag{3}$$

The definition of the pseudo relative interior implies that there is a point $y \in \mathrm{co}(D)$ such that $\overline{x}$ lies in the open segment of extremities x and y:

$$\exists y \in \mathrm{co}(D),\ \lambda \in (0,1) \quad \overline{x} = \lambda\, x + (1-\lambda) y. \tag{4}$$

Recall that $v \in \mathcal{I}_0(x,C)$, that is, $x + \mathbb{R}_+ v \in C$, and in particular that

$$x + \frac{r}{\lambda} v \in C. \tag{5}$$

As an easy consequence of relation (4), we see that

$$\overline{x} + r\,v = \lambda \left(x + \frac{r}{\lambda} v \right) + (1-\lambda) y; \tag{6}$$

combine relations (5) and (6) to get relation (3), and hence relation (1).

Let us now consider a point $\overline{w} \in \mathrm{pri}(C)$. Applied for $D = C$, relation (1) implies that $\mathcal{I}_0(x,C) \subset \mathcal{I}_0(\overline{w},C)$ for any $x \in C$. Accordingly, it holds that $\mathcal{I}(C) \subset \mathcal{I}_0(\overline{w},C)$, and, as the reverse inclusion is obvious, relation (2) is proved. □

Let us return to the proof of Proposition 2.2. We address the first of the two remaining cases, namely the one assuming that the dimension of the space X is finite. We know ([10, Corollary 6.4]) that the relative interior and the pseudo relative interior of the non-empty finite-dimensional convex set C coincide, and that ([10, Corollary 6.2]) the relative interior of C is non-empty. It is hence possible to pick $\overline{w} \in \mathrm{pri}(C)$; Lemma 2.3 (relation (2)) establishes that $\mathcal{I}(C) = \mathcal{I}_0(\overline{w},C)$, and Proposition 2.2 is proved by simply takeing $D = \{\overline{w}\}$.

Finally, let us consider C, a non-empty convex subset of X, a space of denumerable dimension. When C reduces to the singleton $\{\theta\}$, there is nothing to prove, so let us assume that C contains a non-null vector, say w. Consider $B = \{b_i : i \in \mathbb{N}\}$ one of the Hamel basis of X such that $b_1 = w$, and set X_n for the subspace of X spanned by the first n vectors in B. Put $C_n = C \cap X_n$; as $b_1 \in C$, we infer that each and every set $(C_i)_{i\in\mathbb{N}}$ is non-empty and convex.

As already observed, it is possible to pick $\overline{x}_i \in \operatorname{pri}(C_i)$. We claim that the countable subset of C obtained by gathering all the points $\overline{x}_i$ meets the requirements of Proposition 2.2, that is,

$$\mathcal{I}(C) = \bigcup_{i\in\mathbb{N}} \mathcal{I}_0(\overline{x}_i, C). \tag{7}$$

By virtue of Lemma 2.3 (relation 2) applied with $C = C_i$, it follows that

$$\mathcal{I}(C_i) = \mathcal{I}_0(\overline{x}_i, C_i) \quad \forall\, i \in \mathbb{N}.$$

Apply $i)$ of Lemma 2.1 with $x = \overline{x}_i$, $P = C_i$ and $K = C$ to infer that

$$\mathcal{I}_0(\overline{x}_i, C_i) \subset \mathcal{I}_0(\overline{x}_i, C) \quad \forall\, i \in \mathbb{N};$$

combine the two previous relations to get

$$\bigcup_{i\in\mathbb{N}} \mathcal{I}(C_i) \subset \bigcup_{i\in\mathbb{N}} \mathcal{I}_0(\overline{x}_i, C) \subset \mathcal{I}(C). \tag{8}$$

Let us also remark that, for any finite-dimensional convex subset L of X, and in particular for any half-line, it holds that $M \subset X_n$ for some $n \in C$. Thus any half-line contained in C must lie within one of the sets C_i:

$$\mathcal{I}(C) \subset \bigcup_{i\in\mathbb{N}} \mathcal{I}(C_i). \tag{9}$$

From relations (8) and (9) it results that

$$\mathcal{I}(C) \subset \bigcup_{i\in\mathbb{N}} \mathcal{I}(C_i) = \bigcup_{i\in\mathbb{N}} \mathcal{I}_0(\overline{x}_i, C_i) \subset \mathcal{I}(C);$$

all the inclusions in the previous line are in fact equalities, thereof relation (7) holds true. □

We have thus proved that:

P1) *The covering* $V = \bigcup_{x\in C} \mathcal{I}_0(x, C)$ *always admits a sub-covering whose cardinal number does not exceed the dimension of* X.

To state another important property of this covering, let $\{x_i : 1 \le i \le n\}$ be a finite family of points in C. As V is convex and it contains all the cones $\mathcal{I}_0(x_i, C)$, it follows that V contains the set $\operatorname{co}\big(\bigcup_{1\le i\le n} \mathcal{I}_0(x_i, C)\big)$.

The following lemma shows that V actually contains more than the set $\operatorname{co}\big(\bigcup_{1\le i\le n} \mathcal{I}_0(x_i, C)\big)$, namely its Klee envelope.

Lemma 2.4. *Let C be a convex set, and let $(x_i)_{1\le i\le n} \subset C$ be a finite family of points. Then*

$$\kappa\left(\mathrm{co}\left(\bigcup_{1\le i\le n} \mathcal{I}_0(x_i, C)\right)\right) \subset \mathcal{I}(C). \tag{10}$$

Proof of Lemma 2.4. As already remarked, the pseudo relative interior of any convex set of finite dimension is non-empty; it is thus possible to pick $\overline{x} \in \mathrm{pri}(\mathrm{co}(\{x_i : 1 \le i \le n\}))$.

Apply Lemma 2.3 (relation 1) for the subset $D = \{x_i : 1 \le i \le n\}$ of the convex set C to deduce that

$$\mathcal{I}_0(x_i, C) \subset \mathcal{I}_0(\overline{x}, C) \quad \forall\, 1 \le i \le n.$$

The cone $\mathcal{I}_0(\overline{x}, C)$ is convex; we may use the previous inclusion to show that

$$\mathrm{co}\left(\bigcup_{1\le i\le n} \mathcal{I}_0(x_i, C)\right) \subset \mathcal{I}_0(\overline{x}, C),$$

whence we infer that

$$\kappa\left(\mathrm{co}\left(\bigcup_{1\le i\le n} \mathcal{I}_0(x_i, C)\right)\right) \subset \kappa\left(\mathcal{I}_0(\overline{x}, C)\right). \tag{11}$$

Applying Proposition 1.1 to the cone $\mathcal{I}_0(\overline{x}, C)$ we deduce that

$$\kappa\left(\mathcal{I}_0(\overline{x}, C)\right) = \mathcal{I}\left(\mathcal{I}_0(\overline{x}, C)\right); \tag{12}$$

use *iii*) Lemma 2.1 for $x = \overline{x}$ and $P = \mathcal{I}_0(\overline{x}, C)$ to obtain that

$$\mathcal{I}\left(\mathcal{I}_0(\overline{x}, C)\right) = \mathcal{I}\left(\overline{x} + \mathcal{I}_0(\overline{x}, C)\right), \tag{13}$$

and *ii*) Lemma 2.1 for $P = \overline{x} + \mathcal{I}_0(\overline{x}, C)$ and $K = C$, to infer that

$$\mathcal{I}\left(\overline{x} + \mathcal{I}_0(\overline{x}, C)\right) \subset \mathcal{I}(C). \tag{14}$$

Relation (10) follows from relations (11), (12), (13) and (14). □

Lemma 2.4 implies that:

P2) *The convex cone V contains the Klee envelope of the convex hull of any finite family of cones from the covering $(\mathcal{I}_0(x, C))_{x\in X}$.*

In view of Properties *P1*) and *P2*), statements which are fulfilled by each and every covering of form $V = \bigcup_{x\in C} \mathcal{I}_0(x, C)$, let us intoduce the main notion of this section.

Definition 2.5. Given V a convex cone in X, a family $(V_j)_{j\in J}$ of convex cones is called a spreading cover over V if the following three conditions hold true:

a) $(V_j)_{j\in J}$ is a cover of V,

b) V contains the Klee envelope of the convex hull of any finite collection of cones from the family $(V_j)_{j\in J}$,

c) The cardinal number of the set J is less or equal to the dimension of X.

The notion of spreading cover allows us to restate Proposition 2.2 and Lemma 2.4, and to establish the following result.

Proposition 2.6. *If a convex cone V is the infinity cone to a convex set, then it possesses a spreading cover.*

We conclude this section with a proposition which achieves a complete characterization of the convex cones admitting spreading covers.

Proposition 2.7. *The following three statements are true:*

i) *Any convex cone admits a spreading cover provided that the underlying space X has an uncountable dimension,*

ii) *When X has a denumerable dimension, a convex cone of X has a spreading cover if and only if it is the union of an ascending sequence of linearly closed and convex cones,*

iii) *A cone of a finite-dimensional space admits a spreading cover if and only if it is linearly closed (and thus closed with respect to the Euclidean topology on X.)*

Moreover, each time when a cone possesses a spreading cover, it also admits a spreading cover composed only of finite-dimensional convex cones.

Proof of Proposition 2.7. *i)* Let V be a convex cone from X, a real vector space with an uncountable dimension. Consider $V_x = \mathbb{R}_+ x$ for any $x \in V$, a family of convex cones which obviously fulfills conditions a) and b) from Definition 2.5. As already noticed, the Continuum Hypothesis implies that the set X has a cardinal number equal to its dimension; as C is a subset of X, the third condition of the definition is also satisfied. Also, all the members of this covering are finite-dimensional sets.

ii) Let X be a real vector space of denumerable dimension, and V the union of the ascending sequence of linearly closed and convex cones $(W_i)_{i\in\mathbb{N}}$. Fix $B = \{b_i : i \in \mathbb{N}\}$ a Hamel basis of X, and set, as above, X_n for the subspace of X spanned by the first n vectors in B, to define $Y_i = W_i \cap X_i$. It is obvious that $(Y_i)_{i\in\mathbb{N}}$ is an ascending sequence of finite-dimensional linearly closed and convex cones whose union amounts to V, and thus is a spreading cover over V composed by finite-dimensional convex cones.

Reciprocally, consider $(V_j)_{j\in J}$ a spreading cover over V; point c) from Definition 2.5 implies that J is countable (finite or denumerable), so there is a one-to-one mapping $f : J \to \mathbb{N}$. Define

$$U_i = \left(\bigcup_{f(j)\leq i} V_j \right) \cap X_i \quad \forall\, i \in \mathbb{N},$$

and

$$Z_i = \kappa\left(\mathrm{co}(U_i)\right) \quad \forall\, i \in \mathbb{N}.$$

Obviously, $Z_m \subset Z_n$ provided that $m \leq n$, and for each and every $i \in \mathbb{N}$, Z_i is a convex cone. Moreover, being the Klee envelope of the finite-dimensional convex cone $\mathrm{co}(U_i)$, Z_i is linearly closed ([8]).

It results that $(Z_i)_{i\in\mathbb{N}}$ is an ascending sequence of linearly closed and convex cones. All what it remains to be proved is that $(Z_i)_{i\in\mathbb{N}}$ is a cover for V, that is,

$$V = \bigcup_{i\in\mathbb{N}} Z_i. \tag{15}$$

As $V = \bigcup_{j\in J} V_j$ (point a) of Definition 2.5) and $X = \bigcup_{i\in\mathbb{N}} X_i$, it follows that V amounts to the union of all the cones from the family $(U_i)_{i\in\mathbb{N}}$, and since $U_i \subset V_i$ it results that

$$V = \bigcup_{i\in\mathbb{N}} U_i \subset \bigcup_{i\in\mathbb{N}} Z_i. \tag{16}$$

Finally, use *iv)* of Lemma 2.1 with $P = \bigcup_{f(j)\le i} V_j$ and $K = X_i$, and the obvious fact that $\kappa(X_i) = X_i$, in order to get

$$Z_i = \kappa\left(\mathrm{co}(U_i)\right) = \kappa\left(\mathrm{co}\left(\bigcup_{f(j)\le i} V_j\right)\cap X_i\right) \subset \kappa\left(\mathrm{co}\left(\bigcup_{f(j)\le i} V_j\right)\right)\cap X_i\,.$$

By virtue of point b) of Definition 2.5, it follows that

$$\kappa\left(\mathrm{co}\left(\bigcup_{f(j)\le i} V_j\right)\right) \subset V;$$

combine the last two relations to deduce that

$$Z_i \subset V \quad \forall\, i \in \mathbb{N}. \tag{17}$$

Relation (15) follows from relations (16) and (17).

iii) Let us address the case when the dimension of X is finite. For each and every linearly closed convex cone V of X, it is easy to see that the one-member collection consisting from V alone fulfills the conditions of a spreading cover, and its only member is of finite dimension.

Conversely, if the convex cone V admits a spreading cover $(V_i)_{j\in J}$, then, by combining condition a) of Definition 2.5 and the fact that V is convex, it results that

$$V = \bigcup_{j\in J} V_j \subset \mathrm{co}\left(\bigcup_{j\in J} V_j\right) = \mathrm{co}(V) = V.$$

Accordingly, $V = \mathrm{co}\left(\bigcup_{j\in J} V_j\right)$. By virtue of point c) of Definition 2.5, J is a finite set; we may thus apply condition b) from Definition 2.5 to the family $(V_j)_{j\in J}$ to infer that V contains $\kappa\left(\mathrm{co}\left(\bigcup_{j\in J} V_j\right)\right)$, and hence $\kappa(V)$. The cone V is consequently linearly closed. □

3. A theorem on finite type collections of convex sets

Once a spreading cover $(V_j)_{j\in J}$ over some convex cone V is given, it remains to describe a way to translate each and every cone from the covering $(V_j)_{j\in J}$ in such a way that the convex hull of the union of all these translates does not contain more half-lines than the rays of V.

In order to illustrate the major difficulty in achieving our objective, let us take $(A_i)_{i\in I}$, a family of convex sets. It is obvious that any point x from the convex hull of the union of sets A_i can be expressed as a convex combination of points from a finite collection of sets A_i, that is, $x \in \text{co}\left(\bigcup_{i\in K} A_i\right)$ for a finite subset K of I. This fact is no longer true for subsets of $\text{co}\left(\bigcup_{i\in I} A_i\right)$; indeed, a subset D of the convex hull of the union of sets A_i may not be contained in any of the sets of form $\text{co}\left(\bigcup_{i\in K} A_i\right)$, where K is a finite subset of I, even when D itself is of finite dimension.

Let, for instance, $X = \mathbb{R}$, and, for every $i \in \mathbb{N}$, define $A_i = [0, i]$. Then, the half-line $\mathbb{R}_+$ is contained in the convex hull of the union of sets A_i (in fact it coincides with this convex hull), but the convex hull of any union of a finite collection of sets A_i is a bounded segment of the real axis, and cannot thus contain any half-line.

We are thus lead to introduce the main two notions of this section.

Definition 3.1. The family $(A_i)_{i\in I}$ of convex sets is called *of finite type* if any subset D of finite dimension from the convex hull of the union of sets A_i is contained in a set of form $\text{co}\left(\bigcup_{i\in K} A_i\right)$, for some K, finite subset of I.

Definition 3.2. A collection $(v_j)_{j\in J}$ is called *spreading sole* of the spreading cover $(V_j)_{j\in J}$ over V, if the family of convex sets $(v_j + V_j)_{j\in J}$ is of finite type.

The following result proves that if a convex cone V possesses both a spreading cover and a spreading sole, then it can be represented as the infinity cone to some convex set.

Proposition 3.3. *Consider V a convex cone, $(V_j)_{j\in J}$ a spreading cover over V, and $(v_j)_{j\in J}$ a spreading sole of $(V_j)_{j\in J}$. Then*

$$\mathcal{I}(C) = V, \tag{18}$$

where C stands for the convex set $\text{co}\left(\bigcup_{j\in J}(v_j + V_j)\right)$.

Proof of Proposition 3.3. As, for any $j \in J$, it holds that $V_j \subset \mathcal{I}_0(v_j, C)$, it follows that

$$V_j \subset \mathcal{I}(C) \quad \forall\, j \in J,$$

and thus that

$$V = \bigcup_{j\in J} V_j \subset \mathcal{I}(C). \tag{19}$$

To the end of establishing the reverse inclusion, let us consider $v \in \mathcal{I}(C)$; accordingly, the half-line $L = \hat{x} + \mathbb{R}_+ v$ is completely contained within C for some $\hat{x} \in C$. As L is a set of finite dimension and v_j is a spreading sole, we deduce that there is a finite subset F of J such that

$$L = \hat{x} + \mathbb{R}_+ v \subset \mathrm{co}\left(\bigcup_{j\in F}(v_j + V_j)\right).$$

Accordingly,

$$v \in \mathcal{I}\left(\mathrm{co}\left(\bigcup_{j\in F}(v_j + V_j)\right)\right) \quad \forall\, v \in \mathcal{I}(C). \tag{20}$$

We need the following result, which computes the infinity cone to the convex hull of a family of translates of convex cones.

Lemma 3.4. *Let $(V_i)_{1\le i\le n}$ be a finite collection of convex cones, and consider $\{v_i : 1 \le i \le n\}$ a finite set of vectors. Set $C = \mathrm{co}\left(\bigcup_{1\le i\le n}(v_i + V_i)\right)$ and $V = \mathrm{co}\left(\bigcup_{1\le i\le n} V_i\right)$. Then*

$$\mathcal{I}(C) = \kappa\,(V)\,. \tag{21}$$

Proof of Lemma 3.4. As already noticed, it is always possible to pick a vector $\overline{x} \in \mathrm{pri}\,(\mathrm{co}\,(\{x_i : 1 \le i \le n\}))$. In view of Lemma 2.3 (relation 1) applied for $D = \{x_i : 1 \le i \le n\}$, it holds that

$$V_i \subset \mathcal{I}_0(x_i, C) \subset \mathcal{I}_0(\overline{x}, C) \;\; \forall\, i \in \{1, ..., n\};$$

as $\mathcal{I}_0(\overline{x}, C)$ is a convex cone, we can thus infer that $V \subset \mathcal{I}_0(\overline{x}, C)$. Accordingly,

$$\kappa(V) \subset \kappa\,(\mathcal{I}_0(\overline{x}, C))\,;$$

use Proposition 1.1 to prove that

$$\kappa\,(\mathcal{I}_0(\overline{x}, C)) = \mathcal{I}\,(\mathcal{I}_0(\overline{x}, C))\,,$$

and combine the conclusions of *iii*) of Lemma 2.1 with $x = \overline{x}$ and $P = \mathcal{I}_0(\overline{x}, C)$ and of *ii*) of the same Lemma, used for $P = \overline{x} + \mathcal{I}_0(\overline{x}, C)$ and $K = C$, to infer that

$$\mathcal{I}\,(\mathcal{I}_0(\overline{x}, C)) \subset \mathcal{I}(C).$$

From the three previous relations it results that

$$\kappa(V) \subset \mathcal{I}(C). \tag{22}$$

In order to establish the reverse inclusion, let us pick $v \in \mathcal{I}(C)$. Accordingly, the half-line $\hat{x} + \mathbb{R}_+ v$ is contained in C for some $\hat{x} \in C$, so

$$\hat{x} + r\,v = \left(\sum_{i=1}^{i=n} \lambda_{r,i}\, x_i\right) + \left(\sum_{i=1}^{i=n} v_{r,i}\right) \quad \forall\, r \ge 0, \tag{23}$$

where

$$\lambda_{r,i} \ge 0 \;\forall\, r \ge 0,\; i \in \{1, ..., n\}, \quad \sum_{i=1}^{i=n} \lambda_{r,i} = 1 \;\; \forall\, r \ge 0,$$

and $v_{r,i} \in V_i$ for any $r \geq 0$ and $i \in \{1, ..., n\}$. Set Y for the linear subspace of X spanned by $\hat{x}$, v and $\{x_i : 1 \leq i \leq n\}$, and remark that

$$\sum_{i=1}^{i=n} v_{r,i} \in \sum_{i=1}^{i=n} V_i = V, \qquad \hat{x} + r\,v - \left(\sum_{i=1}^{i=n} \lambda_{r,i}\, x_i \right) \in Y; \tag{24}$$

from relations (23) and (24) we deduce that

$$\hat{x} + r\,v - \left(\sum_{i=1}^{i=n} \lambda_{r,i}\, x_i \right) \in (V \cap Y).$$

Accordingly,

$$v(r) = v + \left(\frac{\hat{x}}{r} - \sum_{i=1}^{i=n} \frac{\lambda_{r,i}}{r}\, x_i \right) \in (V \cap Y);$$

but $\lambda_{r,1} \in [0, 1]$ for any $r \geq 0$ and $i \in \{1, ..., n\}$, so v is the limit when r goes to $+\infty$ of $v(r)$ (with respect to the Euclidean topology on Y). Recall that the Euclidean closure of a finite dimensional convex set and its Klee envelope coincide, and use *iv*) of Lemma 2.1 applied with $P = V$ and $K = Y$, to infer that

$$v \in \kappa(V \cap Y) \subset \kappa(V) \quad \forall\, v \in \mathcal{I}(C). \tag{25}$$

The desired relation (21) follows by combining relations (22) and (25). □

Let us now return to the proof of Proposition 3.3 and apply the result of the previous lemma to the set $\mathrm{co}\left(\bigcup_{j \in F}(v_j + V_j)\right)$, to deduce that

$$\mathcal{I}\left(\mathrm{co}\left(\bigcup_{j \in F}(v_j + V_j)\right)\right) = \kappa\left(\mathrm{co}\left(\bigcup_{j \in F} V_j\right)\right).$$

Point b) in Definition 2.5 implies that

$$\kappa\left(\mathrm{co}\left(\bigcup_{j \in F} V_j\right)\right) \subset V.$$

From the previous two relations, we conclude that

$$\mathcal{I}\left(\mathrm{co}\left(\bigcup_{j \in F}(v_j + V_j)\right)\right) \subset V; \tag{26}$$

finally, relations (20) and (26) prove that

$$\mathcal{I}(C) \subset V. \tag{27}$$

Relation (18) steems from relations (19) and (27). □

It remains to address the construction of a spreading sole for a given spreading cover. Let us first consider a more general problem.

Theorem 3.5. *Let $(A_j)_{j\in J}$ be a family of finite dimensional convex sets from the real vector space X. Assume that the cardinal number of J is less or equal to the dimension of X. Then, there is a collection $(a_j)_{j\in J}$ of vectors of X, such that the family $(a_j+A_j)_{j\in J}$ is of finite type.*

Prior to the proof of Theorem 3.5, let us consider several standard properties of ordinal and cardinal numbers nedeed in the sequel.

3.1. Initial well-ordenings: basic facts

Let us first recall that, for every set A it is possible to define a well-ordering $(A, \leq)$ of minimal ordinal number among all the ordinal numbers of the same cardinality (such ordinal numbers are usually called *initial*); for an overview of this topic, the reader is refeered to [4, Exercise 10, p. 125]. Let us collect some of the classical properties of the initial well-orderings nedded hereafter.

Lemma 3.6. *Let $(A, \leq)$ be an initial well-ordering on an infinite set of cardinal number $\mathfrak{a}$. As customary, for any pair of elements a, $b \in A$ such that $a \leq b$, the set $[a, b] = \{x \in A : a \leq x \leq b\}$ is caled the order interval of extremities a and b, and when $a \leq b$ and $a \neq b$, we write $a < b$.*

i) *For any element $a \in A$, the set $S_A(a) = \{x \in A : a < x\}$ is non-empty; as a consequence, we define the* succesor *mapping $\mathfrak{S} : A \to A$, associating to any element a the least element of $S_A(a)$,*

ii) *For any element $a \in A$, the set $I_A(a) = \{x \in A : x < a\}$ has a cardinal number strictly lower than $\mathfrak{a}$,*

iii) *The set*

$$\mathcal{F}_A(M) = \{b \in A : \exists a \in M \text{ s.t. } b \leq a \text{ and } [b, a] \text{ is finite}\}$$

is finite, provided that the subset M of A is finite.

The following standard result is proved here for the reader's convenience.

Lemma 3.7. *Let A and B be two infinite sets such that $\mathfrak{a}$, the cardinal number of A is lower or equal than $\mathfrak{b}$, the cardinal number of B, and endow B with an initial well-ordering $(B, \leq)$.*

Then every mapping $f : A \to B$ is dominated by some one-to-one mapping $g : A \to B$, in the sense that

$$f(a) < g(a) \quad \forall\, a \in A. \tag{28}$$

Proof of Lemma 3.7. We shall construct the desired one-to-one function $g : A \to B$ by (transfinite) induction. To this respect, let us endow A with an initial well-ordering; since no confusion risks to occur, we denote this ordering by $(A, \leq)$. Let a_0 be the least element of A; our first objective is to define $g(a_0) \in B$ such that $f(a_0) < g(a_0)$.

On one hand, it is obvious that the cardinal number of the singleton $\{f(a_0)\}$ (that is, 1) is strictly lower than the infinite cardinal number $\mathfrak{b}$. On the other hand, since the ordinal number of $(B, \leq)$ is initial, it follows (see *ii*) Lemma 3.6) that the set $I_B(f(a_0)) = \{b \in B : b < f(a_0)\}$ have a cardinal number strictly lower than $\mathfrak{b}$.

Remark that an infinite set never amounts to the union of a finite family of sets with strictly lower cardinal numbers than its own cardinal number. Let us apply this fact to the infinite set B, and to the sets $\{f(a_0)\}$ and $I_B(f(a_0))$ of cardinal numbers strictly lower than $\mathfrak{b}$, to conclude that the difference set $B \setminus (\{f(a_0)\} \cup I_B(f(a_0)))$ is non-empty. It is now sufficient to pick $g(a_0)$ within this difference to be assured that $f(a_0) < g(a_0)$.

Suppose now that, for some $w \in B$, we have already defined an one-to-one mapping $g : I_A(w) \to B$ such that relation $f(a) < g(a)$ holds for every $a \in I_A(w)$. We have to prove that it is possible to pick an element $g(w)$ in B laying both without $\{f(w)\} \cup I_B(f(w))$ (to ensure that $f(w) < g(w)$) and without $g\,(I_A(w))$ (to ensure the injectivity of g). In other words, we have to prove that

$$B \setminus (g\,(I_A(w)) \cup I_B(f(w)) \cup \{f(w)\}) \neq \emptyset. \tag{29}$$

Use once more time the fact that the ordinal number of $(B, \leq)$ is initial, to prove (*ii*) Lemma 3.6) that the cardinal number of the set $I_B(f(w))$ is strictly lower than $\mathfrak{b}$. Similarly, from the fact that the ordinal number of $(A, \leq)$ is initial, it results that the set $I_A(w)$ has a cardinal number strictly lower than $\mathfrak{a}$. Since the mapping $g : I_A(w) \to B$ is one-to-one, we deduce that the set $g\,(I_A(w)))$ has the same cardinal number as $I_A(w)$, being thus strictly lower than $\mathfrak{a}$. But $\mathfrak{a} \leq \mathfrak{b}$; hence, the cardinal number of the set $g\,(I_A(w)))$ is strictly lower than $\mathfrak{b}$.

Consequently, the cardinal number of each of the three subsets $I_B(f(w))$, $g\,(I_A(w)))$ and $\{f(w)\}$ of the set infinite set B is strictly lower than the cardinal number of B.

As already remarked, this fact proves relation (29). □

3.2. The proof of Theorem 3.5

Let us turn to the proof of Theorem 3.5, and consider a family of convex sets $(A_j)_{j\in J}$ of finite dimension from a real vector space X whose dimension is greater or equal to the cardinal number of J. Our aim is to construct a collection $(a_j)_{j\in J}$ of vectors such that the family of convex sets $(a_j + A_j)_{j\in J}$ is of finite type.

To begin with, fix $\{b_i : i \in I\}$, a (Hamel) basis of X, and endow I with the initial well-ordering $(I, \leq)$. As when the dimension of X is finite, the family $(A_j)_{j\in J}$ is finite, and thus of finite type, we shall assume that I is an infinite set.

Define the mapping $c : 2^X \to 2^I$ as:

$$c(A) = \{i \in I : \exists x \in A \text{ s.t. } x_i \neq 0\} \quad \forall\, A \subset X;$$

in order to simplify the notation, we will write $c(x)$ instead of $c(\{x\})$. Remark that, for any $j \in J$, the set $c(A_j)$ is finite, so it is possible to introduce the mapping $b : J \to I$,

$$b(j) = \max(c(A_j)) \quad \forall\, j \in J.$$

Apply Lemma 3.7 for $A = J$, $B = I$ and $f = b$, to infer the existence of a one-to-one mapping $g : J \to I$ such that $\max(c(A_j)) < g(j)$ for any $j \in J$. Finally, define

$$a_j = \left(\sum_{k \in c(A_j)} b_{\mathfrak{S}(k)} \right) + b_{g(j)} \quad \forall j \in J. \tag{30}$$

The following lemma captures the main technical feature of the proof.

Lemma 3.8. *Let F be a finite subset of J, and*

$$x = \sum_{j \in F} \lambda_j \, (a_j + u_j), \tag{31}$$

where

$$u_j \in A_j, \ \lambda_j > 0 \ \forall j \in F, \quad \text{and} \quad \sum_{j \in F} \lambda_j = 1. \tag{32}$$

Then, for any $j \in F$ it holds that $g(j) \in \mathcal{F}_I(c(x))$.

Proof of Lemma 3.8. Suppose, to the end of achieving a contradiction, that there is an element $w \in F$ such that $g(w) \notin \mathcal{F}_I(c(x))$. Our aim is to find an element $l \in I$ such that:

$$x_l = 0, \tag{33}$$

$$(u_j)_l = 0 \quad \forall j \in F, \ u_j \in A_j, \tag{34}$$

and

$$\exists k \in F \quad (a_k)_l = 1. \tag{35}$$

Indeed, relations (31), (32), (33), (34) and (35) contradict each other, proving in this way that our initial assumption is false.

To the purpose of constructing l, let us remark that, in view of the definition of the set $\mathcal{F}_I(c(x))$, saying that $g(w) \notin \mathcal{F}_I(c(x))$ means that there is some $z \in I$ such that the interval $[g(w); z[= \{i \in I : \ g(w) \le i < z\}$ is infinite and totally misses $c(x)$.

Define the set

$$P = \bigcup_{j \in F} \{i \in c(A_j) : \ [g(w); i] \text{ is finite}\},$$

and recall that the mapping g was defined such that $\max(c(A_w)) < g(w)$, that is, $j < g(w)$ for any in $j \in c(A_w)$. Hence, provided that $j \in c(A_w)$, the interval $[g(w), j]$ is void, thus finite, and so $j \in P$; thus $c(A_w) \subset P$.

Accordingly, $c(A_w) \subset P \subset \bigcup_{j \in F} c(A_j)$, so the set P is non-empty and finite. Set i_m for its maximum; we claim that the element

$$l = \max(g(w), \mathfrak{S}(i_m)) \tag{36}$$

is such that relations (33), (34) and (35) are verified.

To prove relation (33), let us first notice that, since $i_m \in P$, it follows that the interval $[g(w), i_m]$ is finite. It is easy to see that, for any $a, b \in I$, it holds that $[a, \mathfrak{S}(b)] \subset$

$[a, b] \cup \{\mathfrak{S}(b)\}$; accordingly, we deduce that the interval $[g(w), \mathfrak{S}(i_m)]$ is finite (it may be empty).

The way in which l is defined proves thus that the set $[g(w), l]$ is finite. As $[g(w), z[$ is an infinite interval, it follows that $l < z$; combine this fact with the obvious remark that $g(w) \leq l$ to deduce that $l \in [g(w), z[$. The interval $[g(w), z[$ misses $c(x)$; hence $l \notin c(x)$, whence follows relation (33).

Let us now address relation (34). We have already remarked that the interval $[g(w), l]$ is finite; if we assume that $l \in \bigcup_{j\in F} c(A_j)$, it yields that $l \in P$, which is impossible (l is larger or equal to the successor of the largest element from P). Thus $l \notin \bigcup_{j\in F} c(A_j)$, so relation (34) is proved.

Finally, consider relation (35). The definition (30) of the set $(a_j)_{j\in J}$ of vectors proves that

$$(a_j)_{g(j)} = 1 \quad \forall j \in J,$$

and

$$(a_j)_k = 1 \quad \forall k \in \mathfrak{S}(c(A_j)).$$

Since $i_m \in P \subset \bigcup_{j\in F} c(A_j)$, it follows that $i_m \in c(A_{j_m})$ for some $j_m \in F$; thus $l = g(w)$ or $l \in \mathfrak{S}(A_{j_m})$ (see relation (36)); it follows that

$$(a_w)_l = 1 \quad \text{or} \quad (a_{j_m})_l = 1,$$

fact which proves relation (35). □

Let us return to the proof of Theorem 3.5, and consider D, a finite dimensional subset of $\operatorname{co}\left(\bigcup_{j\in J}(a_j + A_j)\right)$. The set $c(D)$ is thus finite, and, by virtue of *iii*) Lemma 3.6, the same holds for the set $\mathcal{F}_I(c(D))$. The mapping $g : J \to I$ being one-to-one we see that the set $G = g^{-1}(\mathcal{F}_I(c(D)))$ is a finite subset of J. We claim that $D \subset \operatorname{co}\left(\bigcup_{j\in G}(a_j + A_j)\right)$.

Let $x \in D$; all we have to prove is that

$$x \in \operatorname{co}\left(\bigcup_{j\in G}(a_j + A_j)\right). \tag{37}$$

Since $x \in D \subset \operatorname{co}\left(\bigcup_{j\in J}(a_j + A_j)\right)$, we know that, for some finite subset F of J and elements $(u_j)_{j\in F} \in A_j$, it holds that

$$x = \sum_{j\in F} \lambda_j\,(a_j + u_j), \quad \sum_{j\in F} \lambda_j = 1 \quad \text{and} \quad \lambda_j > 0\ \forall j \in F. \tag{38}$$

Lemma 3.8 reads now that $g(F) \subset \mathcal{F}_I(c(x))$. As $x \in D$ implies that $c(x) \subset c(D)$, it follows that

$$F \subset g^{-1}(\mathcal{F}_I(c(x))) \subset g^{-1}(\mathcal{F}_I(c(D))) = G. \tag{39}$$

Relation (37) follows from relations (38) and (39).

4. The main result

We are now in a position to prove the main result of this article.

Theorem 4.1. *The following three statements are true:*

i) *Any convex cone is the infinity cone to a convex set provided that the underlying space X is of uncountable dimension,*

ii) *When X has denumerable dimension, a convex cone of X is the infinity cone to some convex subset of X if and only if it is the union of an ascending sequence of closed and convex finite-dimensional cones,*

iii) *A cone in a finite-dimensional space is the infinity cone to a convex set if and only if it is closed.*

Proof of Theorem 4.1. In view of Proposition 2.7, we only need to prove that a convex cone V is the infinity cone to some convex set if and only if it admits a spreading cover. Moreover, the *only if* part of this equivalence is proved by Proposition 2.6.

In order to establish the *if* part, consider V a convex cone possessing a spreading cover $(V_j)_{j\in J}$. In view of Proposition 2.7, we can assume, without altering the generality of our study, that all the convex cones composing the spreading covering $(V_j)_{j\in J}$ are of finite dimension. Apply now Theorem 3.5 to the collection $(V_j)_{j\in J}$ composed only of finite dimensional convex sets, to deduce that there is a family of vectors of X, say $(v_j)_{j\in J}$, such that the family $(v_j + V_j)_{j\in J}$ is of finite type. In other words, $(v_j)_{j\in J}$ is a spreading sole of $(V_j)_{j\in J}$, and Proposition 3.3 proves that V is the infinity cone to the convex set $\mathrm{co}\left(\bigcup_{j\in J}(v_j + V_j)\right)$. □

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