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Random Products of Quasi-Nonexpansive Mappings in Hilbert Space

Arkady Aleyner
Department of Mathematics,
The Technion – Israel Institute of Technology, 32000 Haifa, Israel
aaleyner@tx.technion.ac.il

Simeon Reich
Department of Mathematics,
The Technion – Israel Institute of Technology, 32000 Haifa, Israel
sreich@tx.technion.ac.il

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An algorithmic framework based on either random (unrestricted) or quasi-cyclic products for finding a point in the intersection of the fixed point sets of a finite collection of quasi-nonexpansive mappings is considered and two convergence theorems are established.

Keywords: Common fixed point, infinite product, quasi-nonexpansive mapping, relaxation method

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1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$, $I = \{1, 2, ..., m\}$ a finite set of indices, and $\{T_i\}_{i \in I}$ a finite collection of nonexpansive self-mappings of H. (Recall that a mapping T of a subset C of H into H is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.) Suppose further that the intersection F of the fixed point sets of these mappings is nonempty. Our aim is to find an element of F, that is, a common fixed point of the collection $\{T_i\}_{i \in I}$. We denote by $\mathbb{N}$ and $\mathbb{R}_+$ the set of positive integers and nonnegative real numbers, respectively.
A frequently employed approach to the solution of this problem is the following one. Let σ be a mapping from $\mathbb{N}$ onto I that takes each value in I infinitely often and let $\{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{R}_+$ be any bounded sequence. Then a sequence $\{x_n\}_{n=1}^{\infty}$ is generated by

$$x_1 \text{ is arbitrary}, \quad x_{n+1} = x_n + \lambda_n (T_{\sigma(n)} x_n - x_n), \quad \text{for all } n \in \mathbb{N}. \tag{1}$$

Comparing such sequences with the relaxed products generated by some form of control (for example, cyclic control [4]), we speak of random (unrestricted) products. Such products find application in different fields: decomposition methods for the numerical solution of partial differential equations [6], systems of linear equalities and inequalities [8], approximation theory [22], population biology [13], mathematical programming [17],

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and image recovery (in particular, computer tomography [16, 18]).

The sequence defined by (1) does not converge, in general, to a common fixed point as the simple example $H = \mathbb{R}$, $I = \{1\}$, $\lambda_n = 1$ for all $n \in \mathbb{N}$, and $T_1 = -Id$ shows (here Id stands for the identity operator). But in the case where $I = \{1\}$, T_1 is firmly nonexpansive and $0 < \epsilon \leq \lambda_n = \lambda \leq 2 - \epsilon$ for all $n \in \mathbb{N}$, it follows from a result of Opial [20] that the sequence $\{x_n\}_{n=1}^{\infty}$ does converge weakly to a point in F. Dye and Reich [12, Theorem 1] show that if T_1 and T_2 are nonexpansive mappings satisfying condition (W) and $\lambda_n = 1$ for all $n \in \mathbb{N}$, then any random product drawn from T_1 and T_2 converges weakly to a common fixed point. But what about the case where the collection of mappings consists of more than two mappings? Does the sequence $\{x_n\}_{n=1}^{\infty}$ still converge? This question is motivated by a long-standing question about the convergence of the successive projections method where each T_i is the nearest point projection onto some closed and convex subset C_i of H. In this case the problem is reduced to that of finding a point in the intersection of these sets. In most of the literature it is usually assumed that the sets are chosen either in an essentially cyclic (quasi-periodic) order (that is, every set is chosen at least once every K iterations, for some $K \geq m$) or according to the maximal distance rule (i.e., one chooses a set that is in some sense farthest away from the current iterate). Under such assumptions one can prove weak convergence of the iterates [5]. The notion of a step size (called relaxation parameter) λ_n was introduced in [1] and [19]. It has been observed that a value of λ_n different of 1 (i.e., under- or over-relaxation) can significantly improve the convergence rate (see [9, 15, 18]). The values of λ_n are usually restricted to the interval $[\epsilon, 2 - \epsilon]$. Thus the following questions arise naturally in this connection. The first one is, can the above assumptions on the order of projections be weakened? In particular, if no assumptions on the order of projections is made, would the iterates still converge? Secondly, is it possible to set the values of the relaxation parameters far away from this interval while preserving the convergence property of the sequence $\{x_n\}_{n=1}^{\infty}$? The third interesting question is whether a broader class of mappings, that properly contains the class of projection mappings, can be used and what is the connection between such a class of mappings and the range of relaxation parameters? The answer to the first question was shown by Prager [21] to be yes if H is finite dimensional and the C_i's are closed linear subspaces of H. Prager's result was later extended by Amemiya and Ando [2] to the case where the C_i's are closed linear subspaces of any Hilbert space H. However, they established only weak convergence in the infinite-dimensional case. Bruck [7] proved a weak convergence result assuming that $m = 3$ and $C_i = -C_i$ for all i. Youla [25] provided a general result which proves weak convergence assuming only that the C_i's share a so called "inner point". Dye and Reich [12] showed that if the sets C_i have a common "weak internal point", or $m = 3$ and T_1, T_2, T_3 are nonexpansive retractions onto C_1, C_2, C_3, respectively, then any random product converges weakly to some point in $\bigcap_{i \in I} C_i$. In [24] Tseng established two convergence theorems for products drawn from a finite collection of firmly nonexpansive mappings (recall that a nearest point projection mapping is, in fact, firmly nonexpansive [4, 14]).

In the present paper we extend and improve upon Tseng's theorems as follows. We prove our results in the very general context of products of mappings belonging to a recently defined class of mappings called ν-quasi-nonexpansive [10, 11]. This class of mappings properly contains the class of averaged and thus, in turn, of firmly nonexpansive mappings. When H is finite dimensional, no additional assumption on the order control is needed to establish convergence of our method (see Theorem 3.1 below). Moreover, we propose a new range of relaxation parameters which at each stage of the iteration depends, naturally enough, on the specific mapping employed at this particular stage. Finally, we establish a weak convergence result (Theorem 3.2) for infinite dimensional Hilbert spaces H under the so called quasi-cyclic order [24]. The quasi-cyclic order control may be considered an extension of the essentially cyclic order in which the lengths of the quasi-cycles are allowed to increase without bound, but not too fast.

2. Preliminaries

In this short section we recall the notions of quasi-nonexpansive and ν-quasi-nonexpansive mappings. We denote the fixed point set of a mapping T by Fix T.

Definition 2.1. Let C be a nonempty subset of a Hilbert space H and let $T : C \to H$ be a mapping with a fixed point. We say that T is quasi-nonexpansive if for every $x \in C$ and $f \in \text{Fix } T$,

$$\|Tx - f\| \leq \|x - f\|. \tag{2}$$

A more quantitative and stronger version of this definition has recently been introduced by Crombez [10, 11].

Definition 2.2. Let C be a nonempty subset of H and let $T : C \to H$ be a mapping with Fix $T \neq \emptyset$. Given $\nu \geq 0$, we say that T is a ν-quasi-nonexpansive mapping if for every $x \in C$ and $f \in \text{Fix } T$,

$$\|Tx - f\|^2 \leq \|x - f\|^2 - \nu\|x - Tx\|^2. \tag{3}$$

The class of ν-quasi-nonexpansive mappings properly contains all averaged mappings and thus all firmly nonexpansive mappings. There are several useful consequences of (3) (see [10, Theorem 3.2]). In the sequel we will only need the following one.

Proposition 2.3 ([10, Theorem 3.2(iii)]). *Let C be a nonempty subset of H and ν a nonnegative number. A mapping $T : C \to H$ with a nonempty fixed point set Fix T is ν-quasi-nonexpansive if and only if*

$$\langle f - x, Tx - x\rangle \geq \left(\frac{1+\nu}{2}\right)\|Tx - x\|^2 \tag{4}$$

for all $x \in C$ and $f \in$ Fix T.

3. Convergence Theorems

We introduce the following assumptions on the mappings, order control and relaxation parameters.

Assumptions on the mappings. $(T_i)_{i\in I}$ is a finite collection of continuous ν_i-quasi-nonexpansive self-mappings of a real Hilbert space H with a common fixed point, that is, $F = \bigcap_{i\in I} \text{Fix } T_i \neq \emptyset$.

Assumptions on the order control. The mapping $\sigma : \mathbb{N} \to I$ is such that each element of I appears in the sequence $\{\sigma(n)\}_{n=1}^{\infty}$ an infinite number of times.

Assumptions on the relaxation parameters. For a given $\epsilon > 0$, the sequence $\{\lambda_n\}_{n=1}^{\infty}$ is such that $\epsilon \le \lambda_n \le \nu_{\sigma(n)} + 1 - \epsilon$ for all $n \in I\!N$.

Given $x_1 \in H$, we define the sequence $\{x_n\}_{n=1}^{\infty}$ by

$$x_{n+1} = x_n + \lambda_n (T_{\sigma(n)} x_n - x_n). \tag{5}$$

Theorem 3.1. *Suppose that the Hilbert space H is finite dimensional, and that the above assumptions on the mappings, order control and relaxation parameters hold. Then the sequence $\{x_n\}_{n=1}^{\infty}$ generated by (5) converges to a point in F.*

Proof. We begin our proof by showing that $\{x_n\}$ is Fejér monotone [4, Definition 2.15] with respect to F and hence bounded. Indeed, from Proposition 2.3 it follows that for all $f \in F$,

$$\begin{aligned}
& \|x_n - f\|^2 - \|x_{n+1} - f\|^2 \\
= {} & \|x_n - f\|^2 - \|x_{n+1} - x_n + x_n - f\|^2 \\
= {} & \|x_n - f\|^2 - (\|x_{n+1} - x_n\|^2 + \|x_n - f\|^2 + 2\langle x_n - f, x_{n+1} - x_n\rangle) \\
= {} & 2\langle f - x_n, x_{n+1} - x_n\rangle - \|x_{n+1} - x_n\|^2 \\
= {} & 2\lambda_n \langle f - x_n, T_{\sigma(n)} x_n - x_n\rangle - \lambda_n^2 \|T_{\sigma(n)} x_n - x_n\|^2 \\
\ge {} & 2\lambda_n \left(\frac{1 + \nu_{\sigma(n)}}{2}\right) \|T_{\sigma(n)} x_n - x_n\|^2 - \lambda_n^2 \|T_{\sigma(n)} x_n - x_n\|^2 \\
= {} & \lambda_n (1 + \nu_{\sigma(n)} - \lambda_n) \|T_{\sigma(n)} x_n - x_n\|^2 \\
\ge {} & \epsilon^2 \|T_{\sigma(n)} x_n - x_n\|^2 \qquad\qquad (6) \\
\ge {} & 0,
\end{aligned}$$

that is, $\|x_n - f\| \ge \|x_{n+1} - f\|$. From the boundedness of $\{x_n\}$ we obtain that $\{T_{\sigma(n)} x_n\}$ is bounded too. Therefore it has a cluster point in H. We denote by Y the set of all cluster points of the sequence $\{T_{\sigma(n)} x_n\}$ and claim that $Y \bigcap F \ne \emptyset$. We prove this by contradiction. Suppose that $Y \bigcap F = \emptyset$. Let $y \in Y$ and let $\{n_k\}_{k=1}^{\infty}$ be any subsequence of $I\!N$ such that $\{T_{\sigma(n_k)} x_{n_k}\}_{k=1}^{\infty}$ converges to y. We can assume (after passing to another subsequence if necessary) that, for some $i \in I$, $\sigma(n_k) = i$ for all $k \in I\!N$. Then $T_{\sigma(n_k)} x_{n_k} = T_i x_{n_k}$ for all $k \in I\!N$. The continuity of T_i, when combined with (6), implies that $y = T_i y$, or equivalently, $y \in F_i := \operatorname{Fix} T_i$. Since $y \notin F$, there exists some nonempty subset $J \subset I$ such that $y \in \bigcap_{j \in J} F_j$ and $y \notin F_p$ for all $p \in I \setminus J$. For each $k \in I\!N$, let $\delta(n_k)$ denote the smallest integer $\tau > n_k$ such that $\sigma(\tau) \in I \setminus J$. Note that $\delta(n_k)$ is monotonically increasing and tends to ∞ as $k \to \infty$. From the definition of $\delta(n_k)$ we have that $\sigma(\tau) \in J$ for all $\tau \in \{n_k, n_k + 1, ..., \delta(n_k) - 1\}$. Since $y \in \bigcap_{j \in J} F_j$, it follows that $\|y - x_\tau\| \ge \|y - x_{\tau+1}\|$ for all $\tau \in \{n_k, n_k + 1, ..., \delta(n_k) - 1\}$, so that

$$\|y - x_{n_k}\| \ge \|y - x_{\delta(n_k)}\|. \tag{7}$$

Since our choice of n_k was arbitrary, inequality (7) holds for all $k \in I\!N$. Furthermore, since $\{T_{\sigma(\delta(n_k))} x_{\delta(n_k)}\}_{k=1}^{\infty}$ is bounded, it has some cluster point $\tilde{y}$. Similarly, we can assume (by passing to another subsequence if necessary) that there exists some $\tilde{i} \in I \setminus J$ such that $\sigma(\delta(n_k)) = \tilde{i}$ for all $k \in I\!N$ and that $\{T_{\sigma(\delta(n_k))} x_{\delta(n_k)}\}$ converges to $\tilde{y}$. Thus we obtain $T_{\sigma(\delta(n_k))}x_{\delta(n_k)} = T_{\tilde{i}}x_{\delta(n_k)}$ for all $k \in \mathbb{N}$. Since $T_{\tilde{i}}$ is continuous, this implies that $\tilde{y} = T_{\tilde{i}}\tilde{y}$, or equivalently, $\tilde{y} \in F_{\tilde{i}}$. Thus $\tilde{y} \neq y$. We also have

$$\begin{aligned} & \|y - T_{\sigma(\delta(n_k))}x_{\delta(n_k)}\|^2 \\ = \ & \|y - \tilde{y}\|^2 + 2\langle y - \tilde{y}, \tilde{y} - T_{\sigma(\delta(n_k))}x_{\delta(n_k)}\rangle + \|\tilde{y} - T_{\sigma(\delta(n_k))}x_{\delta(n_k)}\|^2, \end{aligned}$$

for all $k \in \mathbb{N}$. Hence

$$\lim_{k\to\infty} \|y - T_{\sigma(\delta(n_k))}x_{\delta(n_k)}\|^2 = \|y - \tilde{y}\|^2.$$

From (6) and (7) we obtain that

$$\begin{aligned} \lim_{k\to\infty} \|y - T_{\sigma(n_k)}x_{n_k}\|^2 &= \lim_{k\to\infty} \|y - x_{n_k}\|^2 \\ &\geq \lim_{k\to\infty} \|y - x_{\delta(n_k)}\| \\ &= \lim_{k\to\infty} \|y - T_{\sigma(\delta(n_k))}x_{\delta(n_k)}\|^2 \\ &= \|y - \tilde{y}\|^2. \end{aligned}$$

But $\{T_{\sigma(n_k)}x_{n_k}\}$ converges to y. Hence $y = \tilde{y}$, which contradicts the fact that $y \neq \tilde{y}$. Consequently, the sequence $\{x_n\}$ has a cluster point in F. By [4, Theorem 2.16], since the Fejér monotone sequence $\{x_n\}$ has a cluster point in F, it converges to a point in F. □

Next we consider another order control (namely, the quasi-cyclic order control [24]) which is more restrictive than the previous one. With this order control we are able to establish a weak convergence result in an infinite dimensional setting.

Assumptions on the order control. There exists a sequence of positive integers $\{\tau_1, \tau_2, ...\}$, satisfying $\tau_1 = 1, \tau_{k+1} - \tau_k \geq m$ for all $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} \frac{1}{\tau_{k+1}-\tau_k} = \infty$, such that $I \subset \{\sigma(\tau_k), \sigma(\tau_k + 1), ..., \sigma(\tau_{k+1} - 1)\}$. In other words, this order means that every mapping T_i is applied at least once between the τ_k-th and the $(\tau_{k+1} - 1)$-th iteration (called the k-th quasi-cycle) for all k, and that the length of the k-th quasi-cycle, namely, $\tau_{k+1} - \tau_k$, cannot grow too fast with k. The special case where $\tau_k = m(k-1)+1$ for all k gives rise to the well known cyclic order for which $\sigma(n) = n \pmod m$ for all $n \in \mathbb{N}$ (thus the length of each quasi-cycle is exactly m). Another interesting choice of the τ_k's is given by $\tau_{k+1} = \tau_k + km$ for all $k \in \mathbb{N}$. In this case the length of the k-th quasi-cyclic increases linearly with k.

Theorem 3.2. *Suppose that the assumptions on the mappings and the relaxation parameters hold, and let the sequence $\{x_n\}_{n=1}^{\infty}$ be generated by (5) under the quasi-cyclic order control. Then the sequence $\{x_n\}_{n=1}^{\infty}$ has a unique weak cluster point in F.*

Proof. First we assert that there exists a subsequence $\{n_k\} \subset \mathbb{N}$ such that

$$\lim_{k\to\infty} \sum_{t=\tau_{n_k}+1}^{\tau_{n_k+1}} \|x_{t+1} - x_t\| = 0. \tag{8}$$

Suppose that such a subsequence does not exist. Then there exist some $\delta > 0$ and an integer $\bar{k}$ such that

$$\delta \leq \sum_{n=\tau_k+1}^{\tau_{k+1}} \|x_{n+1} - x_n\|$$

for all $k \geq \bar{k}$.
By the Cauchy-Bunyakovskii-Schwarz inequality, we know that

$$\sum_{n=\tau_k+1}^{\tau_{k+1}} \|x_{n+1} - x_n\| \leq \sqrt{\sum_{n=\tau_k+1}^{\tau_{k+1}} \|x_{n+1} - x_n\|^2} \cdot \sqrt{\tau_{k+1} - \tau_k}.$$

This inequality implies, in turn, that

$$\delta^2 \leq \sum_{n=\tau_k+1}^{\tau_{k+1}} \|x_{n+1} - x_n\|^2 (\tau_{k+1} - \tau_k)$$

for all $k \geq \bar{k}$. Thus

$$\delta^2 \sum_{k=\bar{k}}^{\infty} \frac{1}{\tau_{k+1} - \tau_k} \leq \sum_{k=\bar{k}}^{\infty} \sum_{n=\tau_k+1}^{\tau_{k+1}} \|x_{n+1} - x_n\|^2 = \sum_{n=\tau_{\bar{k}}+1}^{\infty} \|x_{n+1} - x_n\|^2. \tag{9}$$

But from (6) it follows that

$$\|x_n - f\|^2 - \|x_{n+1} - f\|^2 \geq \left(\frac{\epsilon}{\lambda_n}\right)^2 \|x_{n+1} - x_n\|^2.$$

Since $\{\|x_n - f\|^2\}_{n=1}^{\infty}$ is monotonically decreasing and

$$\sum_{k=1}^{n} \left(\|x_k - f\|^2 - \|x_{k+1} - f\|^2\right) \leq \|x_1 - f\|^2,$$

we obtain that

$$\sum_{n=1}^{\infty} \left(\|x_n - f\|^2 - \|x_{n+1} - f\|^2\right) < \infty,$$

which, in turn, implies that

$$\sum_{n=1}^{\infty} \left(\frac{\epsilon}{\lambda_n}\right)^2 \|x_{n+1} - x_n\|^2 < \infty.$$

Since the index set I is finite, we see that while the right-hand side of (9) is finite, the left-hand side is infinite, thereby obtaining a contradiction. Thus there exists some subsequence $\{n_k\} \subset I\!N$ satisfying (8).
Since $\{x_n\}_{n=1}^{\infty}$ is bounded, there exist some $\bar{x} \in H$ and some subsequence of $\{n_k\}_{k=1}^{\infty}$, which we again denote by $\{n_k\}_{k=1}^{\infty}$, such that $\{x_{\tau_{n_k}+1}\}_{k=1}^{\infty}$ converges weakly to $\bar{x}$. We

assert that $\bar{x} \in F$. To see this, fix any $i \in I$. Since the mappings are applied in quasi-cyclic order, for each $k \in \mathbb{N}$, there exists some $\rho_{n_k} \in \{\tau_{n_k}, \tau_{n_k}+1, ..., \tau_{n_k+1}-1\}$ such that $\sigma(\rho_{n_k}) = i$. Using the fact that

$$\begin{aligned} \|x_{\rho_{n_k}+1} - T_{\sigma(\rho_{n_k})} x_{\rho_{n_k}}\| &= \|x_{\rho_{n_k}+1} - x_{\rho_{n_k}} + x_{\rho_{n_k}} - T_{\sigma(\rho_{n_k})} x_{\rho_{n_k}}\| \\ &\leq \|x_{\rho_{n_k}+1} - x_{\rho_{n_k}}\| + \|x_{\rho_{n_k}} - T_{\sigma(\rho_{n_k})} x_{\rho_{n_k}}\| \\ &\leq \|x_{\rho_{n_k}+1} - x_{\rho_{n_k}}\| + \frac{1}{\lambda_{\rho_{n_k}}} \|x_{\rho_{n_k}+1} - x_{\rho_{n_k}}\| \\ &\leq (1 + \frac{1}{\epsilon}) \|x_{\rho_{n_k}+1} - x_{\rho_{n_k}}\| \end{aligned}$$

for all $k \in \mathbb{N}$, we have

$$\begin{aligned} \|x_{\tau_{n_k}+1} - T_{\sigma(\rho_{n_k})} x_{\rho_{n_k}}\| &\leq \sum_{t=\tau_{n_k}+1}^{\tau_{n_k+1}} \|x_{t+1} - x_t\| + \|x_{\rho_{n_k}+1} - T_{\sigma(\rho_{n_k})} x_{\rho_{n_k}}\| \\ &\leq \sum_{t=\tau_{n_k}+1}^{\tau_{n_k+1}} \|x_{t+1} - x_t\| + (1 + \frac{1}{\epsilon}) \|x_{\rho_{n_k}+1} - x_{\rho_{n_k}}\| \end{aligned}$$

for all $k \in \mathbb{N}$. Thus from (8) we obtain that $\lim_{k\to\infty} \|x_{\tau_{n_k}+1} - T_{\sigma(\rho_{n_k})} x_{\rho_{n_k}}\| = 0$, so that $\{T_{\sigma(\rho_{n_k})} x_{\rho_{n_k}}\}_{k=1}^{\infty}$ also converges weakly to $\bar{x}$. Since $T_{\sigma(\rho_{n_k})} x_{\rho_{n_k}} = T_i x_{\rho_{n_k}}$ for all $k \in \mathbb{N}$, this together with $T_i x_{\rho_{n_k}} - x_{\rho_{n_k}} \to 0$ implies that $\{x_{\rho_{n_k}}\}$ converges weakly to $\bar{x}$. Using the Demiclosedness Principle [4, Fact 1.2], we may now conclude that $\bar{x} \in F_i$. Since the choice of $i \in I$ was arbitrary, we obtain that $\bar{x} \in F$. Finally, invoking [4, Theorem 2.16], we deduce that $\bar{x}$ is the unique weak cluster point of the sequence $\{x_n\}_{n=1}^{\infty}$ in F, as claimed. □

Remark. When each mapping T_i, $1 \leq i \leq m$, is firmly nonexpansive we may assume, in both Theorems 3.1 and 3.2, that each T_i is just a self-mapping of a given closed and convex subset C of H. This is a consequence of the extension theorem in [3] which is proved by using the convex-analytic approach of [23] to the Kirszbraun-Valentine extension theorem.

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