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Deville's Master Lemma and Stone's Discreteness in Renorming Theory*

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Dedicated to Stephen Simons on the occasion of his 70th birthday.

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Banach spaces X with an equivalent $\sigma(X,F)$-lower semicontinuous and locally uniformly rotund norm, for a norming subspace $F \subset X^*$, are those spaces X that admit countably many families of convex and $\sigma(X,F)$-lower semicontinuous functions $\{\varphi_i^n : X \to \mathbb{R}^+; i \in I_n\}_{n=1}^\infty$ such that there are open subsets

$$G_i^n \subset \{\varphi_i^n > 0\} \cap \{\varphi_j^n = 0 : j \neq i, j \in I_n\}$$

with $\{G_i^n : i \in I_n, n \in \mathbb{N}\}$ a basis for the norm topology of X

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1. Introduction

Let $(X, \|\cdot\|)$ be a normed space. The norm $\|\cdot\|$ in X is said to be locally uniformly rotund (**LUR** for short) if

$$\lim_n (2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2) = 0 \Rightarrow \lim_n \|x - x_n\| = 0$$

for any sequence (x_n) and x in X. The construction of this kind of norm in separable Banach spaces lead Kadets to the proof of the existence of homeomorphisms between all separable Banach spaces, ([1], Section VI.9). For a non separable Banach space is not always possible to have such an equivalent norm, for instance the space l^∞ does not have it, see for instance p. 74 in [2]. When such a norm exists its construction is usually

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based on a good system of coordinates that we must have on the normed space X from the very beginning, for instance a biorthogonal system

$$\{(x_i, f_i) \in X \times X^* : i \in I\}$$

with some additional properties such as being a strong Markushevich basis, [18], p. 21. Sometimes there is not such a system and the norm is constructed modelling enough convex functions on the given space X to add all of them up with the powerful lemma of Deville, see Lemma 3.1 in Section 3 and Lemma VII 1.1 in [2]. Deville's lemma has been extensively used by R. Haydon in his seminal papers [5], [6], as well as in [7]. It is based on the construction of an equivalent **LUR** norm on a weakly compactly generated Banach space by the second named author in [17], where the convex functions are measuring distances to suitable finite dimensional subspaces as well as evaluations on some coordinate functionals in the dual space X^*; see [18], Theorem 7.3. The method we have developped in [14] is mainly based on Stone's theorem about paracompactness of metric spaces. A σ-discrete basis for the norm topology of a normed space X can be refined to obtain a σ-slicely isolated network if, and only if, the normed space X admits an equivalent **LUR** norm, [14]. Recent contributions show an interplay between both methods, [6, 9, 10]. It is our intention here to show the connection between both approaches. The linking property will be the notion of slicely relatively discreteness, or slicely isolatedness, that glues the discreteness of Stone's theorem with the linear topological structure of the dual pair associated to X. Let us recall precise definitions and results:

Definition 1.1. Let X be a normed space and F be a norming subspace in the dual X^*. A family $\mathcal{B} := \{B_i : i \in I\}$ of subsets in X is called $\sigma(X, F)$-slicely isolated (or $\sigma(X, F)$-slicely relatively discrete) if for every

$$x \in \bigcup\{B_i : i \in I\}$$

there exist a $\sigma(X, F)$-open half space H and $i_0 \in I$ such that

$$H \cap \bigcup\{B_i : i \in I, i \neq i_0\} = \emptyset \quad \text{and} \quad x \in B_{i_0} \cap H.$$

Our approach for **LUR** renormings is also based on the topological concept of network. A family $\mathcal{N}$ of subsets in a topological space $(T, \mathcal{T})$ is called a *network* for the topology $\mathcal{T}$ if for every open set $W \in \mathcal{T}$ and every $x \in W$ there is some $N \in \mathcal{N}$ such that $x \in N \subset W$. A main result proved with our approach is the following:

Theorem 1.2 ([14], Chapter III). *Let X be a normed space and F a norming subspace in the dual X^*. X admits an equivalent $\sigma(X, F)$-lower semicontinuous and locally uniformly rotund norm if, and only if, the norm topology has a network $\mathcal{N}$ that can be written as $\mathcal{N} = \bigcup_{n=1}^{\infty} \mathcal{N}_n$ where each family $\mathcal{N}_n$ is $\sigma(X, F)$-slicely isolated.*

The known proofs of the sufficiency part of this result show a difficult task when they arrive to a convexification process of the sets $\bigcup \mathcal{N}_n$ needed to construct a countable family of seminorms, see [14, 16]. We are going to present here a different approach where the convexification process is not nedeed any more. We shall do it by developping a connection between Deville's master lemma and the $\sigma(X, F)$-slicely isolated families in our connection Lemma 3.2. A straightforward consequence will be a new proof of the sufficiency part of Theorem 1.2 presented in Corollary 3.3. Another result we shall present here is that we can replace the network with a basis of the norm topology in the former theorem to obtain:

Theorem 1.3. *Let X be a normed space with a norming subspace $F \subset X^*$. Then X admits an equivalent $\sigma(X, F)$-lower semicontinuous and locally uniformly rotund norm if, and only if, the norm topology admits a basis $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ such that every $\mathcal{B}_n$ is $\sigma(X, F)$-slicely isolated and norm discrete.*

The notion of slicely discreteness we are using lead us to replace the diameter by the distance to a weakly compact set in the network hypothesis of Theorem 1.2; i.e. the fact that for every $\varepsilon > 0$ and every $x \in X$ there is $n \in \mathbb{N}$, $N \in \mathcal{N}_n$, with $x \in N$ and $\| \cdot \| - \operatorname{diam}(N) \leq \varepsilon$ by a measure of the ε-weak compactness of the set N . In order to do it we shall deal with spaces where the dual space has a small density character for the weak* topology:

Theorem 1.4. *Let X be a Banach space and let $F \subset X^*$ be a norming and* weak* *separable subspace. Let us assume there are $\sigma(X, F)$- slicely isolated families $\mathcal{N}_n$ for $n = 1, 2, ...$ such that for every x in X and every $\varepsilon > 0$ there are $n \in \mathbb{N}$, $N \in \mathcal{N}_n$ and a weakly compact set $C \subset X$ such that $x \in N \subseteq C + \varepsilon B_X$. Then X admits an equivalent $\sigma(X, F)$-lower semicontinuous and locally uniformly rotund norm.*

We shall see in this paper that our condition of being slicely isolated corresponds with the so called rigidity condition inside Deville's lemma. Our free-coordinate approach to **LUR** renormings is explained here with the construction of convex functions describing slicely isolated families of sets in a normed space as biorthogonal systems:

Theorem 1.5. *Let X be a normed space and F be a norming subspace in X^*. Let $\mathcal{B} := \{B_i : i \in I\}$ be a uniformly bounded family of subsets of X. Then the following are equivalent:*

(*1*) *The family $\mathcal{B}$ is $\sigma(X, F)$-slicely isolated*
(*2*) *There is a family $\mathcal{L} := \{\varphi_i : X \to \mathbb{R}^+, i \in I\}$ of convex $\sigma(X, F)$-lower semicontinuous functions such that*
  (*a*) *$B_i \subset \{x \in X : \varphi_i(x) > 0\}$ for every $i \in I$,*
  (*b*) *$\varphi_i(B_j) = \{0\}$ whenever $i \neq j$.*

Theorems 1.2, 1.3 and 1.5 have the following straightforward consequence:

Theorem 1.6. *A Banach spaces X, with a norming subspace $F \subset X^*$, admits an equivalent $\sigma(X, F)$-lower semicontinuous and locally uniformly rotund norm if, and only if, there are countably many families of convex and $\sigma(X, F)$-lower semicontinuous functions $\{\varphi_i^n : X \to \mathbb{R}^+, i \in I_n\}_{n=1}^{\infty}$ such that there are open subsets*

$$G_i^n \subset \{\varphi_i^n > 0\} \cap \{\varphi_j^n = 0 : j \neq i, j \in I_n\}$$

such that $\{G_i^n : i \in I_n, n \in \mathbb{N}\}$ is a basis for the norm topology of X.

2. Lower semicontinuous convex functions and LUR renormings.

Let $(X, \|\cdot\|)$ be a normed space with a norming subspace $F \subset X^*$. Let us denote by $\|\cdot\|_F$ the equivalent norm associated with it:

$$\|\cdot\|_F := \sup\{|\langle\cdot, f\rangle| : f \in B_{X^*} \cap F\}.$$

The former expression is plenty of sense for elements in the bidual X^{**}, nevertheless it is equal zero on $F^\perp \subsetneq X^{**}$ and it is only seminorm on X^{**}. We are going to make extensive use of it in what follows.

We shall begin with the construction of convex and lower semicontinuous functions related to the norm-distance function to a fixed convex set. Such a function is convex. Moreover in order to control the lower semicontinuity too, we need a small modification given in the next result:

Proposition 2.1. *Let X be a normed space and F a norming subspace in the dual space X^*. If D is a weak* compact and convex subset of X^{**} and we define*

$$\varphi(x) := \inf\{\|x - d\|_F : d \in D\}$$

Then φ is a convex, $\sigma(X, F)$-lower semicontinuous, and 1-Lipschitz function from $(X, \|\cdot\|_F)$ to $\mathbb{R}^+$.

Proof. Convexity and Lipschitz conditions are an easy exercise. Let us check the lower semicontinuity. So let us fix $r \geq 0$ and take a net $(x_\alpha)_{\alpha\in A}$ in X with $\varphi(x_\alpha) \leq r$ for every $\alpha \in A$ and let $x \in X$ be the $\sigma(X, F)$-limit of the net $(x_\alpha)_{\alpha\in A}$. We will see that $\varphi(x) \leq r$ too. Let us fix an $\varepsilon > 0$ and choose $c_\alpha^{**} \in D$ such that

$$\sup\{|\langle x_\alpha - c_\alpha^{**}, f\rangle| : f \in B_{X^*} \cap F\} \leq r + \varepsilon$$

for every $\alpha \in A$. Since D is weak* compact we can find a cluster point (x^{**}, c^{**}) of the net $\{(x_\alpha, c_\alpha^{**}) : \alpha \in A\}$ in $X^{**} \times X^{**}$ for the topology $\sigma(X^{**}, X^*)$. Then we have that x^{**} does coincide with x when both linear functionals are restricted to F and thus

$$\langle x - c^{**}, f\rangle = \langle x^{**} - c^{**}, f\rangle \leq r + \varepsilon \ \text{ for all } f \in B_{X^*} \cap F$$

and so $\varphi(x) \leq r + \varepsilon$. Since the reasoning is valid for every $\varepsilon > 0$ we have $\varphi(x) \leq r$ as required. □

Definition 2.2. The function φ defined in Proposition 2.1 shall be called the $\|\cdot\|_F$ -distance to the set D.

We now arrive to the following interplay result that contains Theorem 1.5 in the introduction:

Theorem 2.3. *Let $(X, \|\cdot\|)$ be a normed space and F be a norming subspace in X^*. Let $\mathcal{B} := \{B_i : i \in I\}$ be a uniformly bounded family of subsets of X. The following are equivelent:*

(*1*) *The family $\mathcal{B}$ is $\sigma(X, F)$-slicely isolated.*

(*2*) *There is a family* $\mathcal{L} := \{\varphi_i : X \to \mathbb{R}^+, i \in I\}$ *of convex* $\sigma(X, F)$*-lower semicontinuous functions such that*

$$\{x \in X : \varphi_i(x) > 0\} \cap \bigcup\{B_j : j \in I\} = B_i$$

for every $i \in I$*.*

(*3*) *There is a family* $\mathcal{L} := \{\psi_i : X \to \mathbb{R}^+, i \in I\}$ *of convex* $\sigma(X, F)$*-lower semicontinuous functions and numbers* $0 \le \alpha \le \beta$ *such that*

$$\psi_i(b_i) > \beta \ge \alpha \ge \psi_i(b_j)$$

for every $b_i \in B_i, b_j \in B_j$ *and* $i, j \in I$*.*

Proof. Assume that (*1*) holds. Applying Proposition 2.1 we may consider φ_i to be the F- distance to the convex bounded set:

$$\overline{\operatorname{co}\{B_j : j \ne i, j \in I\}}^{\sigma(X^{**},X^*)}$$

for every $i \in I$. Fix any $i_0 \in I$. The inclusion

$$\{x \in X : \varphi_{i_0}(x) > 0\} \cap \bigcup\{B_j : j \in I\} \subset B_{i_0}$$

in the assertion (*2*) follows immediately from the very definition of φ_{i_0}. Let us prove the reverse inclusion. Our hypothesis on the slicely isolated character of the family $\mathcal{B}$ tells us that for the element $x \in B_{i_0}$ there is a $\sigma(X, F)$-open half space H in X with $x \in H$ and $H \cap B_i = \emptyset$ for all $i \in I$ with $i \ne i_0$. Let us write $H = \{y \in X : f(y) > \mu\}$ where $f \in B_{X^*} \cap F$. Then we have $\varphi_{i_0}(x) \ge f(x) - \mu > 0$ and the inclusion is proved.

The condition (*2*) clearly implies (*3*) with $\alpha = \beta = 0$. Finally, assume (*3*), we then have that $\psi_i(y) \le \alpha$ for every $y \in \operatorname{co}\{B_j : j \ne i, j \in I\}$ by the convexity of the function ψ_i, and also for every $y \in \overline{\operatorname{co}\{B_j : j \ne i, j \in I\}}^{\sigma(X,F)}$ by the lower semicontinuity of ψ_i. Consequently we have

$$x \notin \overline{\operatorname{co}\{B_j : j \ne i, j \in I\}}^{\sigma(X,F)}$$

for every $x \in B_i$ and every $i \in I$. A straightforward application of Hahn-Banach separation theorem then yields the $\sigma(X, F)$-slicely isolated property for the family $\mathcal{B}$. □

A normed space X with a locally uniformly rotund norm decomposes a σ-discrete basis of the norm topology into a σ-slicely isolated network, [14, 12]. It is possible to recover the basis from the network and to have the σ-slicely property as presented in Theorem 1.3. In order to prove it we need the following

Proposition 2.4. *Let* X *be a normed space with a norming subspace* $F \subset X^*$*. Given a uniformly bounded and* $\sigma(X, F)$*-slicely isolated family*

$$\mathcal{A} := \{A_i : i \in I\}$$

of subsets in X*, there exist decompositions* $A_i = \bigcup_{n=1}^{\infty} A_i^n$ *with*

$$A_i^1 \subset A_i^2 \subset \cdots \subset A_i^n \subset A_i^{n+1} \subset \cdots \subset A_i$$

for every $i \in I$*, and such that for every* $n \in \mathbb{N}$ *the family*

$$\{A_i^n + B_{\|\cdot\|_F}(0, 1/4n) : i \in I\}$$

is $\sigma(X, F)$*-slicely isolated and norm discrete.*

Proof. For $i \in I$ let us denote by φ_i the $\|\cdot\|_F$-distance to $\overline{\mathrm{co}(A_j : j \neq i)}^{\sigma(X^{**},X^*)}$. Theorem 2.3 gives us the scalpel to split up the sets of the family using these convex functions. Indeed, let us define $A_i^n := \{x \in A_i : \varphi_i(x) > 1/n\}$; then we have $A_i = \bigcup_{n=1}^{\infty} A_i^n$. Moreover, if $x \in A_i^n + B_{\|\cdot\|_F}(0, 1/4n)$ then we have

$$\varphi_i(x) > 3/(4n).$$

Indeed, let us write $x = y + z$, where $y \in A_i^n, \|z\|_F < 1/4n$. Since $\varphi_i(y) > 1/n$ we can select a number ρ with $\varphi_i(y) > \rho > 1/n$ and we will have for every fixed $c^{**} \in \overline{\mathrm{co}(A_j : j \neq i)}^{\sigma(X^{**},X^*)}$ that $\|y - c^{**}\|_F > \rho$. So we can find some $f \in B_{X^*} \cap F$ with $f(y - c^{**}) > \rho$. Now we see that $f((y+z) - c^{**}) > \rho - 1/4n$ and so $\|x - c^{**}\|_F > \rho - 1/4n$ for every $c^{**} \in \overline{\mathrm{co}(A_j : j \neq i)}^{\sigma(X^{**}.X^*)}$. Consequently we see that $\varphi_i(x) \geq \rho - 1/4n > 3/4n$. On the other hand for $y \in A_j$ with $j \neq i$, we know that $\varphi_i(y) = 0$; then for $x \in A_j^n + B_{\|\cdot\|_F}(0, 1/4n)$ if we write $x = y + z$, with $y \in A_j^n$ and $\|z\|_F < 1/4n$ we have, for every $c^{**} \in \overline{\mathrm{co}(A_j : j \neq i)}^{\sigma(X^{**},X^*)}$

$$\|x - c^{**}\|_F < \|y - c^{**}\|_F + 1/(4n).$$

From where it follows that

$$\varphi_i(x) = \inf\left\{\|x - c^{**}\|_F : c^{**} \in \overline{\mathrm{co}(A_j : j \neq i)}^{\sigma(X^{**}.X^*)}\right\} \leq 1/(4n)$$

since $\varphi_i(y) = 0$. All together means that the family

$$\{A_i^n + B_{\|\cdot\|_F}(0, 1/(4n)) : i \in I\}$$

verifies the conditions in *(3)* of the Theorem 2.3 with the functions $\varphi_i = \psi_i, i \in I$, and constants $\alpha = 1/(4n), \beta = 3/(4n)$. Thus it is $\sigma(X, F)$-slicely isolated as we wanted to prove.

Moreover the former family is discrete for the norm topology. Indeed for any $z \in X$, if we fix $\delta > 0$ such that

$$1/4n + \delta < 3/4n - \delta$$

we have that

$$B_{\|\cdot\|_F}(z, \delta) \cap \bigcup\{A_i^n + B_{\|\cdot\|_F}(0, 1/4n) : i \in I\}$$

has non empty intersection with at most one member of the family because every time the intersection is non empty we can see that $\varphi_i(z) > 3/4n - \delta$ if

$$B_{\|\cdot\|_F}(z, \delta) \cap \{A_i^n + B_{\|\cdot\|_F}(0, 1/4n)\} \neq \emptyset$$

but $\varphi_i(z) < 1/4n + \delta$ when

$$B_{\|\cdot\|_F}(z,\delta) \cap \{A_j^n + B_{\|\cdot\|_F}(0, 1/4n)\} \neq \emptyset$$

for any $j \neq i$ and $j \in I$. This fact can be seen as above writing now $z = x + y$ with $x \in B_{\|\cdot\|_F}(z,\delta) \cap \{A_i^n + B_{\|\cdot\|_F}(0, 1/4n)\}$ and $\|y\|_F < \delta$ in the first case and $x \in B_{\|\cdot\|_F}(z,\delta) \cap \{A_j^n + B_{\|\cdot\|_F}(0, 1/4n)\}$ with $\|y\|_F < \delta$ for the second one. □

Our way to prove Theorem 1.3 pass through the next result strengthening the network condition of Theorem 1.2:

Theorem 2.5. *Let X be a normed space and F a norming subspace in the dual space X^*. Let us assume the space X admits an equivalent $\sigma(X,F)$-lower semicontinuous and* LUR *norm. Then the norm topology of X admits a network*

$$\mathcal{N} = \bigcup_{n=1}^{\infty} \mathcal{N}_n$$

where each family $\mathcal{N}_n$ is $\sigma(X,F)$-slicely isolated and consists of sets which are differences of two $\sigma(X,F)$-closed and convex subsets of X. Moreover, for every $n \in \mathbb{N}$ there is $\delta_n > 0$ such that $\mathcal{N}_n + B(0,\delta_n)$ is norm discrete.

Proof. Let us take the network $\mathcal{M} = \bigcup_{n=1}^{\infty} \mathcal{M}_n$ of the norm topology such that every family $\mathcal{M}_r := \{M_{r,i} : i \in I_r\}$ is $\sigma(X,F)$-slicely isolated, see Theorem 1.2 and Chapter 3 in [14], and let us perform the decomposition from Proposition 2.4 for it, i.e. denoting by $\varphi_{r,i}$ the $\|\cdot\|_F$-distance to $\overline{\mathrm{co}\{M_{r,j} : j \neq i\}}^{\sigma(X^{**},X^*)}$, we define

$$N_{r,i}^n = \left\{ x \in \overline{\mathrm{co}(M_{r,i})}^{\sigma(X,F)} : \varphi_{r,i}(x) > 3/(4n) \right\}.$$

The fact that the family $\mathcal{N}_r^n := \{N_{r,i}^n : i \in I_r\}$ is $\sigma(X,F)$-slicely isolated follows from Theorem 2.3 since the lower semicontinuity and convexity of the functions $\varphi_{r,i}$ tell us that $\varphi_{r,j}(y) = 0$ for every $y \in \overline{\mathrm{co}(M_{r,i})}^{\sigma(X,F)}$ and $j \neq i, j \in I_r$. Moreover, we easily get that for every $0 < \mu$ we have $\varphi_{r,i}(z) > 3/(4n) - \mu$ whenever $z \in N_{r,i}^n + B_{\|\cdot\|_F}(z,\mu)$, and $\varphi_{r,i}(z) < \mu$ whenever $z \in N_{r,j}^n + B_{\|\cdot\|_F}(z,\mu)$. Let us choose δ_n such that $0 < 2\delta_n < 3/4n - \delta_n$; then we have that the norm open sets $\{N_{r,i}^n + B_{\|\cdot\|_F}(0,\delta_n) : i \in I_r\}$ are pairwise disjoint and they form a norm discrete and $\sigma(X,F)$-slicely isolated family by Theorem 2.3. Moreover, each set $N_{r,i}^n$ is the difference of convex and $\sigma(X,F)$-closed subsets of X: $\overline{\mathrm{co}(M_{r,i})}^{\sigma(X,F)}$ and $\{x \in X : \varphi_{r,i}(x) \leq 3/(4n)\}$. The union of all these families:

$$\bigcup \{\mathcal{N}_r^n : r, n = 1, 2, ...\}$$

is clearly the network we are looking for. Indeed, given $x \in X$ there is $r \in \mathbb{N}$ and $i \in I_r$ such that $x \in M_{r,i} \subset x + B_{\|\cdot\|_F}(0, \frac{\varepsilon}{3})$. Then for $n \in \mathbb{N}$ big enough we have $x \in N_{r,i}^n$, $x \in \overline{\mathrm{co}(M_{r,i})}^{\sigma(X,F)} \subset x + B_{\|\cdot\|_F}(0, \frac{2\varepsilon}{3})$ and we have $x \in \overline{\mathrm{co}(N_{r,i}^n)}^{\sigma(X,F)} + B_{\|\cdot\|_F}(0,\delta_n) \subset x + B_{\|\cdot\|_F}(0,\varepsilon)$ if we take the integer n big enough again. □

We now arrive to:

Proof of Theorem 1.3. Necessity. From the proof of Theorem 2.5 we continue with the notation and observe that when we add open balls of sufficiently small radii to elements of the network provided above we will have a basis of the norm topology we are looking for. Indeed

$$\bigcup_{n,r}^{\infty}\{N_{r,i}^n + B_{\|\cdot\|_F}(0,\delta_n) : i \in I_r\}$$

is such a basis of the norm topology. Indeed, for a given $x \in X$ and $\varepsilon > 0$ we find some $p \in \mathbb{N}$ and $i \in I_p$ with $x \in N_{p,i} \subset B(x,\varepsilon/2)$. There is $m_0 \in \mathbb{N}$ such that $x \in N_{p,i}^m$ whenever $m \geq m_0$. It now follows that for m big enough we have $N_{p,i}^m + B_{\|\cdot\|_F}(0,\delta_m) \subset B(x,\varepsilon)$ since $x \in N_{p,i} \subset B(x,\varepsilon/2)$ and δ_m goes to zero when m goes to infinity. The sufficiency follows from Theorem 1.2. Moreover, the needed implication will be proved in the next section, see Corollary 3.3. □

3. The connection lemma

Now we are in position to present our main result here. For a slicely isolated family of sets it is always possible to construct an equivalent norm, such that, the premise for the LUR condition on the new norm for a sequence and a point x implies that the sequence is eventually in the same set of the family to which the limit point x belongs. The construction is done applying Deville's master lemma, p. 279 in [2]:

Lemma 3.1 (Deville's master lemma). *Let $(X, \|\cdot\|)$ be a normed space, let I be a set and let $(\varphi_i)_{i\in I}$ and $(\psi_i)_{i\in I}$ be families of non-negative convex functions on X which are uniformly bounded on bounded subsets of X. For every $x \in X$, $m \in \mathbb{N}$ and $i \in I$ define*

$$\theta_m(x) = \sup\left\{\varphi_i(x)^2 + 2^{-m}\psi_i(x)^2 : i \in I\right\}, \tag{1}$$

$$\theta(x) = \|x\|^2 + \sum_{m=1}^{\infty} 2^{-m}(\theta_m(x) + \theta_m(-x)). \tag{2}$$

Then the Minkowski functional of the set $\{x \in X : \theta(x) \leq 1\}$ is an equivalent norm $|||\cdot|||$ on X such that if a sequence (x_n) and a point $x \in X$ satisfy the LUR *condition:*

$$\lim_n (2|||x_n|||^2 + 2|||x|||^2 - |||x_n + x|||^2) = 0,$$

then there is a sequence (i_n) in I such that:

$$\lim_n \varphi_{i_n}(x) = \lim_n \varphi_{i_n}(x_n) = \lim_n \varphi_{i_n}((x+x_n)/2) = \sup\{\varphi_i(x) : i \in I\} \tag{3}$$

$$\lim_n \left[\frac{1}{2}(\psi_{i_n}^2(x_n) + \psi_{i_n}^2(x)) - \psi_{i_n}^2\left(\frac{1}{2}(x_n + x)\right)\right] = 0. \tag{4}$$

Our main lemma here reads as follows:

Lemma 3.2 (Connection lemma). *Let $(X, \|\cdot\|)$ be a normed space and F be a norming subspace in X^*. Let $\mathcal{B} := \{B_i : i \in I\}$ be a uniformly bounded and slicely isolated family of subsets of X for the $\sigma(X, F)$-topology. Then there is an equivalent and $\sigma(X, F)$-lower semicontinuous norm $\|\cdot\|_{\mathcal{B}}$ on X such that for every $i_0 \in I$, every $x \in B_{i_0}$, and every sequence $(x_n)_{n\in\mathbb{N}}$ in X the condition*

$$\lim_n \left(2\|x_n\|_{\mathcal{B}}^2 + 2\|x\|_{\mathcal{B}}^2 - \|x_n + x\|_{\mathcal{B}}^2\right) = 0$$

implies that:

(*1*) *There is $n_0 \in \mathbb{N}$ such that*

$$x_n, \frac{1}{2}(x_n + x) \notin \overline{\mathrm{co}\bigcup\{B_i : i \neq i_0, i \in I\}}^{\sigma(X,F)}$$

for all $n \geq n_0$

(*2*) *For every positive δ there is $n_\delta \in \mathbb{N}$ such that*

$$x_n \in \overline{\mathrm{co}(B_{i_0}) + \delta B_X}^{\sigma(X,F)}$$

whenever $n \geq n_\delta$.

Proof. Let us fix an index $i \in I$ and define the nonnegative, convex and $\sigma(X, F)$-lower semicontinuous function φ_i as the $\|\cdot\|_F$- distance to

$$\overline{\mathrm{co} \cup \{B_j : j \neq i, j \in I\}}^{\sigma(X^{**},X^*)}.$$

Let us choose a point $a_i \in B_i$ and set $D_i = \mathrm{co}B_i$ and $D_i^\delta = D_i + B(0, \delta)$, where we denote by $B(0, \delta)$ the open ball $\{x \in X : \|x\|_F < \delta\}$, for every $\delta > 0$ and $i \in I$. We denote by $p_{i,\delta}$ the Minkowski functional of the convex body $\overline{D_i^\delta}^{\sigma(X,F)} - a_i$. Then we define the norm p_i by the formula

$$p_i^2(x) = \sum_{q=1}^{\infty} \frac{1}{q^2 2^q} p_{i,1/q}(x)^2$$

for every $x \in X$. It is well defined and $\sigma(X, F)$-lower semicontinuous. Indeed, since $B(0, \delta) + a_i \subset \overline{D_i^\delta}^{\sigma(X,F)}$ we have for every $x \in X$, and $\delta > 0$, that $p_{i,\delta}(\delta x/\|x\|_F) \leq 1$, thus $\delta p_{i,\delta}(x) \leq \|x\|_F$ and hence the above series converges. Finally we define the nonnegative, convex and $\sigma(X, F)$-lower semicontinuous function

$$\psi_i(x) := p_i(x - a_i)$$

for every $x \in X$. We are now in position to apply Deville's master Lemma for our φ_i's and ψ_i's to get an equivalent and $\sigma(X, F)$-lower semicontinuous norm $\|\cdot\|_{\mathcal{B}}$ on X. Take $i_0 \in I$, $x \in B_{i_0}$ and a sequence (x_n) in X satisfying

$$\lim_n \left(2\|x_n\|_{\mathcal{B}}^2 + 2\|x\|_{\mathcal{B}}^2 - \|x_n + x\|_{\mathcal{B}}^2\right) = 0$$

implies the existence of a sequence of indexes (i_n) in I such that (*3*) and (*4*) in Lemma 3.1 hold. Our hypothesis on the slicely isolated character of the family $\mathcal{B}$ tell us after

Theorem 2.3 that since the point x belongs to the set B_{i_0} of the family $\mathcal{B}$, we have $\varphi_{i_0}(x) > 0$, but $\varphi_i(x) = 0$ for all $i \in I$ with $i \neq i_0$. From the assertion (*3*) from Lemma 3.1 it now follows that there exists a positive integer n_0 such that $i_n = i_0$, $\varphi_{i_o}(x_n) > 0$ and $\varphi_{i_0}(\frac{1}{2}(x + x_n)) > 0$ for all $n \geq n_0$, from where the conclusion (*1*) of our lemma follows. Moreover, the equation (*4*) in Lemma 3.1 is now in form

$$\lim_n [2^{-1}(\psi_{i_0}^2(x_n) + \psi_{i_0}^2(x)) - \psi_{i_0}^2(2^{-1}(x_n + x))] = 0,$$

and so by the usual convexity argument, and for every $q \in \mathbb{N}$, we have that

$$\lim_n [2^{-1}((p_{i_0,1/q}(x_n - a_{i_0}))^2 + (p_{i_0,1/q}(x - a_{i_0}))^2) - (p_{i_0,1/q}(2^{-1}(x_n + x) - a_{i_0}))^2] = 0,$$

and consequently

$$\lim_n p_{i_0,1/q}(x_n - a_{i_0}) = p_{i_0,1/q}(x - a_{i_0}).$$

Fix a positive number δ and then $q \in \mathbb{N}$ such that $1/q < \delta$. Since $x - a_{i_0} \in D_{i_0}^{1/q} - a_{i_0}$ we have that $p_{i_0,1/q}(x - a_{i_0}) < 1$ because $D_{i_0}^{1/q} - a_{i_0}$ is norm open. Therefore, there is $n_\delta \in \mathbb{N}$ such that for $n \geq n_\delta$ we have that $p_{i_0,1/q}(x_n - a_{i_0}) < 1$ and thus $x_n - a_{i_0} \in \overline{D_{i_0}^{\delta}}^{\sigma(X,F)} - a_{i_0}$, that is $x_n \in \overline{(co(B_{i_0}) + B(0,\delta))}^{\sigma(X,F)}$, which is (*2*) for $\|\cdot\|_F$. Since the proof is valid for all $\delta > 0$ and $\|\cdot\|_F$ is an equivalent norm the proof is over. □

A direct consequence of the connection lemma is a straightforward proof of the renorming implication in Theorem 1.2

Corollary 3.3. *In a normed space $(X, \|\cdot\|)$, with a norming subspace F in X^*, we have an equivalent $\sigma(X,F)$-lower semicontinuous and locally uniformly rotund norm whenever X admits a network for the norm topology which is σ-slicely isolated family for $\sigma(X,F)$, i.e. there are $\sigma(X,F)$-slicely isolated families*

$$\mathcal{N}_n : n = 1, 2, ...$$

such that for every x in X and every $\varepsilon > 0$ there are $n \in \mathbb{N}$ and $N \in \mathcal{N}_n$ with the property that $x \in N \in \mathcal{N}_n$ and $\|\cdot\| - \operatorname{diam}(N) < \varepsilon$.

Proof. It is not a restriction to assume that every family $\mathcal{N}_n$ is uniformly bounded since we can make intersections with countably many balls centered in the origin and covering X without losing the character of slicely isolatedness and the network condition of the whole family. So we can consider the norms say $\|\cdot\|_{\mathcal{N}_n}$ constructed using the connection lemma for each family $\mathcal{N}_n$ and to define the new norm by the formula:

$$\|x\|_{\mathcal{N}}^2 := \sum_{n=1}^{\infty} c_n \|x\|_{\mathcal{N}_n}^2$$

for every $x \in X$, where the sequence (c_n) is chosen accordingly for the convergence of the series. This is possible because all the norms $\|\cdot\|_{\mathcal{N}_n}$ are equivalent to the original one and hence there are numbers d_n such that

$$\|\cdot\|_{\mathcal{N}_n} \leq d_n \|\cdot\|,$$

so it is enough to take $c_n := \frac{1}{d_n^2 2^n}$. Consider x and a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that

$$\lim_n \left(2\|x_n\|_{\mathcal{N}}^2 + 2\|x\|_{\mathcal{N}}^2 - \|x_n + x\|_{\mathcal{N}}^2\right) = 0.$$

Fix an $\varepsilon > 0$. We know that there is $q \in \mathbb{N}$ and $N_0 \in \mathcal{N}_q$ with $x \in N_0 \subset x + \varepsilon B_X$. The condition

$$\lim_n \left(2\|x_n\|_{\mathcal{N}}^2 + 2\|x\|_{\mathcal{N}}^2 - \|x_n + x\|_{\mathcal{N}}^2\right) = 0$$

implies that

$$\lim_n \left(2\|x_n\|_{\mathcal{N}_q}^2 + 2\|x\|_{\mathcal{N}_q}^2 - \|x_n + x\|_{\mathcal{N}_q}^2\right) = 0$$

by convexity arguments. The connection lemma now says that for every positive δ there is $n_\delta \in \mathbb{N}$ such that

$$x_n \in \overline{\operatorname{co}(N_0) + \delta B_X}^{\sigma(X,F)}$$

whenever $n \geq n_\delta$. Thus $\|x_n - x\| \leq \varepsilon + \delta$ for $n \geq n_\delta$ and $\lim_n x_n = x$ in $(X, \|\cdot\|)$ as we wanted to prove. □

Let us present now the renorming Theorem 1.2 as developed in Chapter 3 of [14]:

Theorem 3.4. *For a normed space $(X, \|\cdot\|)$, and a norming subspace $F \subset X^*$, the following conditions are equivalent each to other:*

(*1*) *The space X admits an equivalent $\sigma(X,F)$-lower semicontinuous* LUR *norm.*
(*2*) *There are $\sigma(X,F)$-slicely isolated families $\mathcal{N}_n, n = 1, 2, ...$ such that $\bigcup_{n\in\mathbb{N}} \mathcal{N}_n$ is a network for the norm topology.*
(*3*) *For every $\varepsilon > 0$ we can write $X = \bigcup_{n\in\mathbb{N}} X_n^\varepsilon$ where for every $n \in \mathbb{N}$ and every $x \in X_n^\varepsilon$ there is a $\sigma(X,F)$-open half space $H \subset X$ such that $x \in H$ and* $\operatorname{diam}(X_n^\varepsilon \cap H) < \varepsilon$

Proof. (*1*) ⇒ (*3*) can be found in Theorem 2 of [16], in the Main Theorem of [11], see Lemma 14 and implication *c*) ⇒ *b*) in page 629, as well as in Theorem 3.1 of [14].

(3) ⇒ (2) can be found in Proposition 2.24 of [14]. For completeness we prove the latter. Take any $p \in \mathbb{N}$. Using Stone's theorem, Theorem 4.4.1 in [3], when starting from a cover of X by open sets of diameter less than $1/p$, we get norm-discrete families $\mathcal{B}_i^p, i \in \mathbb{N}$ of open sets such that the family $\mathcal{B}_1^p \cup \mathcal{B}_2^p \cup \cdots$ covers X, that each element of this family has diameter less than $1/p$, and moreover that $\inf\{\|b - b'\| : b \in B, b' \in B'\} > 1/2^i$ whenever $i \in \mathbb{N}$ and B, B' are distinct elements of $\mathcal{B}_i^p$. Define now

$$\mathcal{N}_{i,p,n} = \{X_n^{1/2^i} \cap B : B \in \mathcal{B}_i^p\}, \quad i, p, n \in \mathbb{N}.$$

Fix any $i, p, n \in \mathbb{N}$. We shall show that $\mathcal{N}_{i,p,n}$ is a $\sigma(X,F)$-slicely isolated family. So, take any $N \in \mathcal{N}_{i,p,n}$ and then any $x \in N$. Find $B \in \mathcal{B}_i^p$ so that $N = X_n^{1/2^i} \cap B$. Find a $\sigma(X,F)$-open half space $H \subset X$ such that $x \in H$ and $\operatorname{diam}(X_n^{1/2^i} \cap H) < 1/2^i$. Consider any $B' \in \mathcal{B}_i^p$ distinct from B. We want to prove that $(X_n^{1/2^i} \cap B') \cap H = \emptyset$. So, assume that there is $y \in (X_n^{1/2^i} \cap B') \cap H$. Then $y \in X_n^{1/2^i} \cap H$, and hence $\|y - x\| < 1/2^i$. However we also have that $\|y - x\| > 1/2^i$, a contradiction. We thus proved that the family $\mathcal{N}_{i,p,n}$

is $\sigma(X,F)$-slicely isolated. It remains to prove that the union $\bigcup\{\mathcal{N}_{i,p,n} : i,p,n \in \mathbb{N}\}$ is a network for the norm topology in X. So fix any $x \in X$ and $\varepsilon > 0$. Take $p \in \mathbb{N}$ so big that $\varepsilon > 1/p$. Find $i \in \mathbb{N}$ and $B \in \mathcal{B}_i^p$ so that $x \in B$. Find then $n \in \mathbb{N}$ so that $x \in X_n^{1/2^i}$. Thus $x \in X_n^{1/2^i} \cap B \in \mathcal{N}_{i,p,n}$ and $\operatorname{diam}(X_n^{1/2^i} \cap B) < \frac{1}{p} < \varepsilon$.

Finally we observe that *(2)* $\Rightarrow$ *(1)* is the content of Corollary 3.3. □

Remark 3.5. Our approach in the proof of *(3)* $\Rightarrow$ *(2)* in Theorem 3.4 does not need any convexification argument as the ones based on Bourgain Namioka supper-lemma, [16, 4], or those developped in [14]. The convex structure here is inside the proof of the connection lemma; it is in the fact that the functions used there are already convex. Since the functions φ_i and ψ_i are convex, as they have been defined, we get free of charge the construction of the equivalent norm using now Deville's master Lemma. With our approach here the convexification can be described on the elements of the σ-slicely (for $\sigma(X,F)$) isolated network fo the norm topology as differences of $\sigma(X,F)$-closed convex sets. In all previous approaches this was done on the sets X_n^ε from the above decomposition $X = \bigcup_{n\in\mathbb{N}} X_n^\varepsilon$.

The main results in the work [4] provides extensions of Corollary 3.3 when the Kuratowski index of non-compactness is used instead of the diameter. We are going to go further when the dual unit ball is weak* separable as an application of our connection lemma above, but using a more general measure of non-compactness. Actually we are going to present the proof of a theorem in the introduction:

Proof of Theorem 1.4. Without any lost of generality we can, and we do assume, that every family $\mathcal{N}_n$ is uniformly bounded since the intersection of a slicely isolated family of sets with a fixed ball is slicely isolated too. Let us construct, with the use of the connection lemma, equivalent $\sigma(X,F)$-lower semicontinuous norms $\|\cdot\|_{\mathcal{N}_n}$, for every $n \in \mathbb{N}$, which verify the conclusion of Lemma 3.2 for the families $\mathcal{N}_n$. We pick a countable weak* dense set $T =: \{t_n : n = 1,2,...\}$ in F. Define then a new, equivalent norm $|||\cdot|||$ on X by the formula

$$|||x|||^2 := \sum_{m=1}^{\infty} c_m(\|x\|^2_{\mathcal{N}_m} + \langle x, t_m\rangle^2)$$

for every $x \in X$, where the sequence (c_n) is chosen so that the above series converges. Consider any $x \in X$ and any sequence (x_n) in X satisfying the premise

$$\lim_n \left(2\,|||x_n|||^2 + 2\,|||x|||^2 - |||x_n + x|||^2\right) = 0.$$

We shall show first that the set $\{x_1, x_2, ...\}$ is weakly relatively compact. To do so, fix any $\varepsilon > 0$. From the assumptions, find $m \in \mathbb{N}, N \in \mathcal{N}_m$, and a weakly compact set $C \subset X$ so that $x \in N \subset C + \varepsilon B_X$. The connection lemma yields $n \in \mathbb{N}$ such that

$$\{x_n, x_{n+1}, ...\} \subset \overline{\operatorname{co} N + \varepsilon B_X}^{\sigma(X,F)} \subset$$

and we may continue, using Krein-Shmulyan theorem, in the following chain of inclusions:

$$\subset \overline{\operatorname{co}(C + \varepsilon B_X) + \varepsilon B_X}^{\sigma(X,F)} \subset \overline{\operatorname{co} C} + 2\varepsilon B_X.$$

Hence

$$\overline{\{x_1, x_2, ...\}}^{w*} \subset \{x_1, x_2, ..., x_n\} \cup \overline{\overline{\mathrm{co}C} + 2\varepsilon B_X}^{w*}$$
$$\subset \{x_1, x_2, ..., x_n\} \cup \overline{\mathrm{co}C} + 2\varepsilon B_{X^{**}} \subset X + 2\varepsilon B_{X^{**}}.$$

Here $\varepsilon > 0$ was arbitrary, so $\{x_1, x_2, ..., x_n, ...\}$ is a relatively weakly compact set. Moreover, by the convexity arguments again we have that $\lim_n \langle x_n, t\rangle = \langle x, t\rangle$ for every $t \in T$. Thus, every $\sigma(X, X^*)$ cluster point y of the sequence $\{x_n : n = 1, 2, ...\}$ does coincide with x on the $\sigma(X^*, X)$-dense subset T of X^*, so they coincide in all X^* and $x = y$. The sequence (x_n) itself is $\sigma(X, X^*)$ convergent to x by its weak relative compactness. We conclude that the new norm $|||\cdot|||$ is weakly locally uniformly rotund and $\sigma(X, F)$-lower semicontinuous. From results in [12] we will have a LUR renorming on X. The fact that it can be obtained $\sigma(X, F)$-lower semicontinuous also follows from analysis in [14]. Indeed, the main lemma in Section 3, Chapter 3 of [14], tell us that the weak topology has a $\sigma(X, F)$-slicely isolated network, so a metric finer than the weak topology with a $\sigma(X, F)$-slicely isolated network can be cosntructed on X, see Theorem 3.21 in [14], from where it follows that the same is valid for the norm topology, see Corollary 3.23 in [14]. Thus the space X admits an equivalent $\sigma(X, F)$-lower semicontinuous LUR norm. □

Let us finish with an open question:

Question. Given a scattered compact space K, is there any characterization of the LUR renormability of $C(K)$ by means of any σ-discreteness property for the family of all clopen subsets of K? Indeed, we know that if $\mathcal{A}$ is the familly of all clopen subsets of the scattered compact spaces K and $C(K)$ admits an equivalent pointwise lower semicontinuous and LUR norm, then the family of clopen sets is a countable union $\mathcal{A} = \bigcup_{n\in\mathbb{N}} \mathcal{A}_n$ of families such that every $\mathcal{A}_n$ provides a set of characteristic functions $\{\mathbf{1}_A : A \in \mathcal{A}_n\}$ which is pointwise slicely discrete, but it is unknown what else is needed to have a reverse implication true.

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References

[1] C. Bessaga, A. Pelczynski: Selected Topics in Infinite-dimensional Topology, Monografie Matematyczne 58, PWN - Polish Scientific Publishers, Warszawa (1975).

[2] R. Deville, G. Godefroy, V. Zizler: Smoothness and renormings in Banach spaces, Pitman Monographs and Surveys in Pure and Applied Mathematics 64, Longman, Harlow (1993).

[3] R. Engelking: General Topology, Monografie Matematyczne 60, PWN - Polish Scientific Publishers, Warszawa (1977).

[4] F. García, L. Oncina, J. Orihuela, S. Troyanski: Kuratowski index of non compactness and renormings in Banach spaces, J. Convex Analysis 11(2) (2004) 477–494.

[5] R. Haydon: Trees in renorming theory, Proc. London Math. Soc. 78(3) (1999) 541–584.

[6] R. Haydon: Locally uniformly rotund norms in Banach spaces and their duals, J. Funct. Anal. 254 (2008) 2023–2039.

[7] R. Haydon, A. Moltó, J. Orihuela: Spaces of functions with countably many discontinuities, Israel J. Math. 158 (2007) 19–39.

[8] G. Köthe: Topological Vector Spaces I, Springer, Berlin (1969).

[9] J. F. Martínez Romero: Renormings in $C(K)$ Spaces, PhD Thesis, Valencia University, Burjassot (2007).

[10] J. F. Martínez Romero, A. Moltó, J. Orihuela, S. Troyanski: On locally uniformly rotund renormings on $C(K)$ spaces, Can. J. Math, to appear.

[11] A. Moltó, J. Orihuela, S. Troyanski: Locally uniformly rotund renorming and fragmentability, Proc. Lond. Math. Soc., III. Ser. 75(3) (1997) 619–640.

[12] A. Moltó, J. Orihuela, S. Troyanski, M. Valdivia: On weakly locally uniformly rotund Banach spaces, J. Funct. Anal. 163(2) (1999) 252–271.

[13] A. Moltó, J. Orihuela, S. Troyanski, M. Valdivia: Kadec and Krein-Milman properties, C. R. Acad. Sci., Paris, Sér. I, Math. 331(6) (2000) 459–464.

[14] A. Moltó, J. Orihuela, S. Troyanski, M. Valdivia: A nonlinear transfer technique for renorming, Lecture Notes in Mathematics 1951, Springer, Berlin (2009).

[15] A. Moltó, J. Orihuela, S. Troyanski, M. Valdivia: Continuity properties up to a counatble partition, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. 100 (2006) 279–294.

[16] M. Raja: Locally uniformly rotund norms, Mathematika 46 (1999) 343–358.

[17] S. Troyanski: On locally uniformly convex and differentiable norms in certain non-separable Banach sapces, Stud. Math. 37 (1971) 173–180.

[18] V. Zizler: Non separable Banach spaces, in: Handbook of the Geometry of Banach Spaces, Vol. 2, W. B. Johnson, J. Lindenstrauss (eds.), North-Holland, Amsterdam (2003) 1743–1816.