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On Maximally q-Positive Sets

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Dedicated to Stephen Simons on the occasion of his 70th birthday.

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In his recent book *From Hahn-Banach to monotonicity* (Springer, Berlin, 2008), S. Simons has introduced the notion of SSD space to provide an abstract algebraic framework for the study of monotonicity. Graphs of (maximal) monotone operators appear to be (maximally) q-positive sets in suitably defined SSD spaces. The richer concept of SSDB space involves also a Banach space structure. In this paper we prove that the analog of the Fitzpatrick function of a maximally q-positive subset M in a SSD space $(B, \lfloor\cdot,\cdot\rfloor)$ is the smallest convex representation of M. As a consequence of this result it follows that, in the case of a SSDB space, the conjugate with respect to the pairing $\lfloor\cdot,\cdot\rfloor$ of any convex representation of M provides a convex representation of M, too. We also give a new proof of a characterization of maximally q-positive subsets of SSDB spaces in terms of such special representations.

1. Introduction and preliminaries

This work is in the setting of symmetrically self-dual spaces, a notion introduced recently by S. Simons [10, Def. 19.1]. A symetrically self-dual (SSD) space is a pair $(B, \lfloor\cdot,\cdot\rfloor)$ consisting of a nonzero real vector space B and a symmetric bilinear form $\lfloor\cdot,\cdot\rfloor : B \times B \longrightarrow \mathbb{R}$ which separates the points of B (that is, for every $b \in B \setminus \{0\}$ there exists $b' \in B$ such that $\lfloor b, b'\rfloor \neq 0$). The bilinear form $\lfloor\cdot,\cdot\rfloor$ induces the quadratic form on B defined by $q(b) := \frac{1}{2}\lfloor b, b\rfloor$. One says that a nonempty set $A \subseteq B$ is q-positive [10, Def. 19.5] if $b, c \in A \Longrightarrow q(b-c) \geq 0$. A set $M \subseteq B$ is called maximally q-positive [10, Def. 20.1] if it is q–positive and not properly contained in any other q–positive set. The theory of q-positive sets was introduced in [9] as a generalization of the theory of monotone operators. We next recall the fundamental notions and results of this theory, as developed in [10] (see also [9] for more details).

Given an arbitrary nonempty set $A \subseteq B$, the function $\Phi_A : B \longrightarrow \mathbb{R}\cup\{+\infty\}$ defined by $\Phi_A(b) := q(b) - \inf_{b\in A} q(b-a) = \sup_{b\in A}\{\lfloor b, a\rfloor - q(a)\}$ will be called the Fitzpatrick function of A. The latter expression shows that Φ_A is a proper convex function. If M is maximally q-positive then

$$\Phi_M(b) \geq q(b) \quad \forall\, b \in B, \tag{1}$$
$$\Phi_M(b) = q(b) \quad \Longleftrightarrow \quad b \in M. \tag{2}$$

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For any proper convex function $f : B \longrightarrow \mathbb{R} \cup \{+\infty\}$ satisfying $f \geq q$, one defines the set

$$\operatorname{pos} f := \{b \in B : f(b) = q(b)\}. \tag{3}$$

We will say that $f : B \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a convex representation of a nonempty set $A \subseteq B$ if $f \geq q$ and $\operatorname{pos} f = A$. By (1) and (2), one has:

Proposition 1.1 ([10, (20.2)]). *If $M \subseteq B$ is a maximally q-positive set then Φ_M is a convex representation of M.*

In view of the following proposition, which characterizes q-positivity in terms of convexity, $\operatorname{pos} f$ is q-positive provided that it is nonempty.

Proposition 1.2 (see [10, Lemma 19.8]). *Let $A \subseteq B$ be nonempty. Then A is q-positive if and only if there exists a convex function $f : B \to \mathbb{R} \cup \{+\infty\}$ such that $f \geq q$ and $A \subseteq \operatorname{pos} f$.*

Proof. If A is q-positive, using Zorn's Lemma we deduce the existence of a maximally q-positive set M that contains A. By Prop. 1.1, the conditions in the statement hold with $f = \Phi_M$. The converse result is Lemma 19.8 in [10]. □

Thus, by Prop. 1.2, if A admits a convex representation then it is q-positive. However, not every q-positive set admits a convex representation. A q-positive set having a convex representation is called S-q-positive [9, Def. 6.2]. By Prop. 1.1, the class of S-q-positive sets includes all maximally q-positive sets

Some of our main results will also require a Banach space structure. One says that $(B, \lfloor\cdot,\cdot\rfloor, \|.\|)$ is a symmetrically self-dual Banach (SSDB) space [10, Def. 21.1] if $(B, \lfloor\cdot,\cdot\rfloor)$ is a SSD space, $(B, \|\cdot\|)$ is a Banach space, the dual B^* is exactly $\{\lfloor\cdot, b\rfloor : b \in B\}$ and the isomorphism $i : B \longrightarrow B^*$ defined by $i(b) := \lfloor\cdot, b\rfloor$ is an isometry (in other terms, $\langle\cdot, b\rangle = \lfloor\cdot, i^{-1}(b)\rfloor$ and $\|b\| = \sup_{\|b'\|\leq 1} \lfloor b', b\rfloor$ for all $b \in B$). In this case, the quadratic form q is continuous and satisfies $|q(b)| \leq \frac{1}{2}\|b\|^2$ for all $b \in B$ (see Prop. 1.4 (c) below). We will denote by $\langle\cdot,\cdot\rangle$ the duality products between B and B^* and between B^* and the bidual space B^{**}, and the norm in B^* will be denoted by $\|\cdot\|$ as well.

The next proposition indicates that a SSDB space is reflexive as a Banach space.

Proposition 1.3. *If $(B, \lfloor\cdot,\cdot\rfloor, \|.\|)$ is a SSDB space then $(B, \|\cdot\|)$ is reflexive.*

Proof. Let $b^{**} \in B^{**}$. For every $b^* \in B^*$ we have

$$\left\langle i^{-1}(b^{**} \circ i), b^*\right\rangle = \left\langle i^{-1}(b^*), b^{**} \circ i\right\rangle = \langle b^*, b^{**}\rangle;$$

therefore b^{**} is noting but the evaluation functional at $i^{-1}(b^{**} \circ i)$. □

We will use some standard concepts and notations from convex analysis and monotone operator theory (see, e.g., the books [1, 10]). The superscript * will denote the standard Fenchel conjugate (with respect to the pairing $\langle\cdot,\cdot\rangle$), the symbol ∂ will mean subdifferential, and δ_A will represent the indicator function of a set $A \subseteq B$ (the function that is identically 0 on A and $+\infty$ on $B \setminus A$). We recall that the duality mapping $J : B \rightrightarrows B^*$

is $J := \partial\left(\frac{1}{2}\|\cdot\|^2\right)$; equivalently,

$$J(b) = \left\{ b^* \in B^* : \frac{1}{2}\|b\|^2 + \frac{1}{2}\|b^*\|^2 = \langle b, b^* \rangle \right\} \quad (b \in B).$$

For a proper convex function $f : B \longrightarrow \mathbb{R} \cup \{+\infty\}$, we will consider its Fenchel conjugate $f^@ : B \longrightarrow \mathbb{R} \cup \{+\infty\}$ with respect to the pairing $\lfloor \cdot, \cdot \rfloor$:

$$f^@(b) := \sup\{\lfloor c, b \rfloor - f(c) : c \in B\} \quad (b \in B).$$

Clearly, $f^@ = f^* \circ i$. We will also use the function $\theta_A : B \longrightarrow \mathbb{R} \cup \{+\infty\}$, associated to a q-positive set $A \subseteq B$, defined in [9, Lemma 6.1 (a)] as the largest l.s.c. convex function minorized by q that coincides with q on A.

The following proposition collects some basic results, which we will need.

Proposition 1.4. *For any SSDB space $(B, \lfloor \cdot, \cdot \rfloor, \|.\|)$, the following statements hold:*

(*a*) [9, (2)] *If $f : B \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a l.s.c. proper convex function then $f^{@@} = f$.*
(*b*) [10, (19.7)] *For every q-positive set $A \subseteq B$, $\Phi_A^@ \geq \Phi_A$.*
(*c*) [10, (21.1)] *For all $b \in B$, $|q(b)| \leq \frac{1}{2}\|b\|^2$.*
(*d*) [9, *Lemma* 6.1 (b)] *If $A \subseteq B$ is q–positive then $\theta_A = \Phi_A^@$.*

As mentioned above, the theory of q-positive sets was introduced as a generalization of the theory of monotone operators. This special case arises when the SSD space consists of $B = X \times X^*$, the product of a nonzero Banach space X with its dual X^*, and the bilinear mapping $\lfloor \cdot, \cdot \rfloor : (X \times X^*) \times (X \times X^*) \longrightarrow \mathbb{R}$ defined by

$$\lfloor (x, x^*), (y, y^*) \rfloor = \langle x, y^* \rangle + \langle y, x^* \rangle.$$

The associated quadratic form $q : X \times X^* \longrightarrow \mathbb{R}$ is then given by

$$q(x, x^*) = \langle x, x^* \rangle.$$

It turns out that a nonempty set $A \subseteq X \times X^*$ is q-positive if and only if $A = \operatorname{Graph}(T)$ for some monotone operator $T : X \rightrightarrows X^*$ [10, Examples 19.6]; we use $\operatorname{Graph}(T)$ to denote the graph of T :

$$\operatorname{Graph}(T) := \{(x, x^*) \in X \times X^* : x^* \in T(x)\}.$$

In this case the Fitzpatrick function of $A \subseteq X \times X^*$ reduces to the standard Fitzpatrick function of the operator whose graph is A, a very important tool in the study of monotonicity by convex analytic methods, which was introduced in [4, Def. 3.1]. Maximally q-positive sets are precisely the graphs of maximal monotone operators, and S-q-positive sets are the graphs of representable operators in the sense of [6, p. 27]. If X is reflexive and $X \times X^*$ is normed by $\|(x, x^*)\| := \sqrt{\|x\|^2 + \|x^*\|^2}$ then $(X \times X^*, \lfloor \cdot, \cdot \rfloor, \|.\|)$ is a SSDB space [10, Example 21.2 (a)].

We will use the following theorem in the proof of our main result.

Theorem 1.5 ([8, Thm. 10.6]). *Let X be reflexive and $T : X \rightrightarrows X^*$ be monotone. Then*

$$T \text{ is maximal monotone} \Longleftrightarrow \operatorname{Graph}(T) + \operatorname{Graph}(-J) = X \times X^*.$$

The next section contains the main results of this paper. We prove that the Fitzpatrick function of a maximally q-positive set M is the smallest convex representation of A. As a consequence of this result it follows that, in the case of SSDB spaces, for any convex representation f of A the function $f^* \circ i$ provides a convex representation of A as well. We also give a new proof, based on Thm. 1.5, of a result due to S. Simons [9, Thm. 4.3 (b)], which characterizes maximally q-positive sets in SSDB spaces as those sets that admit a convex representation f such that $f^@$ is a convex representation, too.

2. Main results

According to [2, (35) and Cor. 4.1], given a maximal monotone operator T from a Banach space X into its dual X^*, the largest l.s.c. convex function minorized by the duality product $\langle\cdot,\cdot\rangle$ on $X \times X^*$ that coincides with $\langle\cdot,\cdot\rangle$ on $\operatorname{Graph}(T)$ is the l.s.c. convex envelope of $\langle\cdot,\cdot\rangle + \delta_{\operatorname{Graph}(T)}$. Our first result extends this result to the context of SSDB spaces.

Proposition 2.1. *Let M be a maximally q-positive set in a SSDB space B. Then θ_M is the l.s.c. convex envelope of $q + \delta_M$.*

Proof. Let us denote by $\overline{co}\,(q + \delta_M)$ the l.s.c. convex envelope of $q+\delta_M$. Since, according to Prop. 1.1, Φ_M is a convex representation of M, we have $q \leq \Phi_M \leq q + \delta_M$; hence, as Φ_M is convex and l.s.c.,

$$q \leq \Phi_M \leq \overline{co}\,(q + \delta_M) \leq q + \delta_M. \tag{4}$$

If $x \in B$ satisfies $\overline{co}\,(q + \delta_M)(x) = q(x)$ then, by (4), $\Phi_M(x) = q(x)$, which, as Φ_M represents M, implies that $x \in M$. Conversely, if $x \in M$ then $q(x) = q(x) + \delta_M(x)$, which, in view of (4), yields $\overline{co}\,(q + \delta_M)(x) = q(x)$. We have thus proved that $\overline{co}\,(q + \delta_M) \geq q$ and that (3) holds with $f = \overline{co}\,(q + \delta_M)$; therefore $\overline{co}\,(q + \delta_M)$ represents M. On the other hand, as θ_M coincides with q on M, one has $\theta_M \leq q + \delta_M$, so that the inequality $\theta_M \leq \overline{co}\,(q + \delta_M)$ is an immediate consequence of the fact that θ_M is convex and l.s.c.. Since θ_M is the largest l.s.c. convex function minorized by q that coincides with q on M, we must also have the opposite inequality $\theta_M \geq \overline{co}\,(q + \delta_M)$. This finishes the proof. □

In the context of monotone operator theory it is well know that the Fitzpatrick function of a maximal monotone operator is its smallest convex representation [4, Thm. 3.10]. The next theorem has as an immediate corollary an extension of this result to the more general setting of maximally q-positive sets. Its direct proof is simpler, and of a different nature, than those provided in [4] and [2] for the particular case of maximal monotone operators.

Theorem 2.2. *Let A be a q-positive set in a SSD space B and f be a convex function such that $f \geq q$ and $f = q$ on A. Then $\Phi_A \leq f$.*

Proof. Let $x \in B$, $y \in A$ and $\lambda \in [0,1)$. Then

$$\begin{aligned}(1-\lambda)^2 q(x) + \lambda(1-\lambda)\lfloor x,y \rfloor + \lambda^2 q(y) &= q((1-\lambda)x + \lambda y) \\ &\leq f((1-\lambda)x + \lambda y) \\ &\leq (1-\lambda) f(x) + \lambda f(y) \\ &= (1-\lambda) f(x) + \lambda q(y);\end{aligned}$$

on subtracting $\lambda q(y)$ we obtain

$$(1-\lambda)^2 q(x) + \lambda(1-\lambda)\lfloor x,y \rfloor - \lambda(1-\lambda) q(y) \leq (1-\lambda) f(x),$$

and after dividing both sides of this inequality by $1-\lambda$ we get

$$(1-\lambda) q(x) + \lambda \lfloor x,y \rfloor - \lambda q(y) \leq f(x).$$

Setting $\lambda \longrightarrow 1^-$, we deduce that

$$\lfloor x,y \rfloor - q(y) \leq f(x),$$

which, by taking the supremum over $y \in A$ in the left hand side, yields

$$\Phi_A(x) \leq f(x).$$

□

Since, by Prop. 1.1, every maximally q-positive set is represented by its Fitzpatrick function, from the preceding theorem one obtains the announced generalization of [4, Thm. 3.10]:

Corollary 2.3. *Let M be a maximally q-positive set in a SSD space B. Then Φ_M is the smallest convex representation of M.*

Following [2, Corollaries 4.1 and 4.2], given a maximal monotone operator T on X, a l.s.c. convex function $h : X \times X^* \longrightarrow \mathbb{R} \cup \{+\infty\}$ represents T if and only if it lies between the Fitzpatrick function of T and the largest l.s.c. convex function minorized by the duality product $\langle \cdot, \cdot \rangle$ on $X \times X^*$ that coincides with $\langle \cdot, \cdot \rangle$ on $\operatorname{Graph}(T)$. An easy consequence of Cor. 2.3 is the following extension of this result to the SSDB framework.

Corollary 2.4. *Let M be a maximally q-positive set in a SSDB space B, and let $f : B \longrightarrow \mathbb{R} \cup \{+\infty\}$ be convex and l.s.c.. Then f represents M if and only if*

$$\Phi_M \leq f \leq \theta_M. \tag{5}$$

Proof. The only if statement follows immediately from Cor. 2.3 and the definition of θ_M. Conversely, since $\Phi_M \geq q$, from (5) we deduce that $f \geq q$ and that every $x \in \operatorname{pos} f$ satisfies $\Phi_M(x) = q(x)$, which implies $x \in M$; on the other hand, for every $x \in M$ one has $\theta_M(x) = q(x)$, hence by (5) and $\Phi_M \geq q$ we obtain $f(x) = q(x)$, that is, $x \in \operatorname{pos} f$. This shows that $\operatorname{pos} f = M$; therefore f is a convex representation of M. □

Our next result generalizes [6, (9)].

Proposition 2.5. *Let M be a maximally q-positive set in a SSDB space B. Then*

$$\theta_M^{@} = \Phi_M.$$

Proof. For every $x \in B$, by Prop. 2.1 we have

$$\begin{aligned}
\theta_M^{@}(x) &= (\theta_M^* \circ i)(x) = \theta_M^*(i(x)) = (q+\delta_M)^*(i(x)) = ((q+\delta_M)^* \circ i)(x) \\
&= (q+\delta_M)^{@}(x) = \sup_{y\in B}\{\lfloor y,x\rfloor - q(y) - \delta_M(y)\} \\
&= \sup_{y\in M}\{\lfloor y,x\rfloor - q(y)\} = \Phi_M(x).
\end{aligned}$$

□

In the context of monotone operator theory, it is well known that the transpose f^{*T} (defined by $f^{*T}(x,x^*) = f^*(x^*,x)$) of the conjugate f^* of a convex representation f of a maximal monotone operator also represents it [2, Thm. 5.3]. This result extends to the context of q-positive sets, as shown next.

Theorem 2.6. *Let M be a maximally q-positive set in a SSDB space B and $f : B \longrightarrow \mathbb{R}\cup\{+\infty\}$ be a l.s.c. proper convex function. Then f is a convex representation of M if and only if $f^{@}$ is a convex representation of M.*

Proof. Assume first that f is a convex representation of M. By Cor. 2.4, $\Phi_M \le f \le \theta_M$. Using Prop. 2.5, Cor. 2.4 and Prop. 1.4 (d) we get

$$\Phi_M = \theta_M^{@} \le f^{@} \le \Phi_M^{@} = \theta_M.$$

Hence, in view of Cor. 2.4, $f^{@}$ is a convex representation of M. The converse statement follows on combining the direct statement with Prop. 1.4 (a). □

The next theorem recalls a characterization, due to S. Simons [9, Thm. 4.3 (b)], of maximally q-positive sets in SSDB spaces. Implication 3) $\Longrightarrow$ 1) was proved in [9, p. 305] using Fenchel-Rockafellar duality theorem; in contrast, the direct proof we give here is based on Thm. 1.5, which is actually a special case of anolther result [10, Thm. 21.7] whose proof is based on Fenchel-Rockafellar duality theorem, too. Our proof also uses the maximal monotonicity of subdifferential operators, which, in the case of reflexive spaces, can also be easily derived from Fenchel-Rockafellar duality theorem (see [5, p. 347] and [10, Remark 18.9]).

Theorem 2.7 (see [9, Thm. 4.3 (b)]). *For every set A in a SSDB space B, the following statements are equivalent:*

1) *A is maximally q-positive.*
2) *A is S-q-positive, and every convex representation f of A satisfies $f^{@} \ge q$.*
3) *There exists a l.s.c. convex representation f of A such that $f^{@} \ge q$.*

Proof. *1)* $\Longrightarrow$ *2)*. Since every maximally q-positive set is represented by its Fitzpatrick function (Prop. 1.1), A is S-q-positive. The second part of assertion *2)* follows from Thm. 2.6.

2) $\Longrightarrow$ *3)*. This implication is obvious, given that the set of l.s.c. convex representations of A is nonempty as A is S-q-positive.

3) $\Longrightarrow$ *1)*. Let $x_0 \in B$ be such that $q(x_0 - a) \geq 0$ for every $a \in A$. We must show that $x_0 \in A$. Since, by Prop. 1.3, B is reflexive, and subdifferentials of l.s.c. proper convex functions are maximal monotone [7, Thm. A], in view of Thm. 1.5 we have $(x_0, i(x_0)) \in \text{Graph}(\partial f) + \text{Graph}(-J)$. Thus there exist $x \in B$ and $x^* \in -J(x)$ such that $i(x_0) - x^* \in \partial f(x_0 - x)$. We have

$$\begin{aligned}
0 &\leq f(x_0 - x) - q(x_0 - x) \\
&= \langle x_0 - x, i(x_0) - x^* \rangle - f^*(i(x_0) - x^*) - q(x_0 - x) \\
&= \langle x_0 - x, i(x_0) - x^* \rangle - f^@\left(x_0 - i^{-1}(x^*)\right) - q(x_0 - x) \\
&\leq \langle x_0 - x, i(x_0) - x^* \rangle - q\left(x_0 - i^{-1}(x^*)\right) - q(x_0 - x) \\
&= \langle x, x^* \rangle - q\left(i^{-1}(x^*)\right) - q(x) \leq \langle x, x^* \rangle + \frac{1}{2}\left\|i^{-1}(x^*)\right\|^2 + \frac{1}{2}\|x\|^2 \\
&= -\langle x, -x^* \rangle + \frac{1}{2}\|-x^*\|^2 + \frac{1}{2}\|x\|^2 = 0;
\end{aligned}$$

tle last inequality follows from Prop. 1.4 (d), whereas the last two equalities use the fact that i is an isometry and the definition of J. We thus deduce that $f(x_0 - x) = q(x_0 - x)$ and, since $-q(i^{-1}(x^*)) \leq \frac{1}{2}\|i^{-1}(x^*)\|^2$ and $-q(x) \leq \frac{1}{2}\|x\|^2$, also $-q(x) = \frac{1}{2}\|x\|^2$. This first equality means that $x_0 - x \in A$; hence $q(x) = q(x_0 - (x_0 - x)) \geq 0$, which, as $-q(x) = \frac{1}{2}\|x\|^2$, yields $x = 0$. Therefore $x_0 \in A$. □

It is worth noticing that implication *3)* $\Longrightarrow$ *1)* holds true even without the lower semicontinuity assumption. In fact, if A has a convex representation f then it also has a l.s.c. convex representation, namely, the l.s.c. hull of f. This is an easy consequence of Cor. 2.4 and the fact that the Fitzpatrick function is l.s.c..

In the special situation when q-positivity corresponds to monotonicity (see Section 1), Thm. 2.7 yields the following result of R. S. Burachik and B.-F. Svaiter.

Corollary 2.8 (see [3, Thm. 3.1]). *Suppose that X is a reflexive Banach space. For every operator $A : X \rightrightarrows X^*$, the following statements are equivalent:*

1) *A is maximal monotone.*

2) *A is representable, and every convex representation f of A satisfies*

$$f^*(x^*, x) \geq \langle x, x^* \rangle \quad \forall\ (x, x^*) \in X \times X^*. \tag{6}$$

3) *There exists a convex representation f of A such that* (6) *holds.*

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