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Fixed Point Theorems for Mappings Satisfying a Condition of Integral Type in Partially Ordered Sets*

J. Harjani
Departamento de Matemáticas, Universidad de Las Palmas de Gran Canaria, Campus de Tafira Baja, 35017 Las Palmas de Gran Canaria, Spain

K. Sadarangani
Departamento de Matemáticas, Universidad de Las Palmas de Gran Canaria, Campus de Tafira Baja, 35017 Las Palmas de Gran Canaria, Spain
ksadaran@dma.ulpgc.es

Dedicated to Professor José Rodríguez Expósito on the occasion of his 60th birthday.

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The purpose of this paper is to present some fixed point theorems for monotone operators in a metric space endowed with a partial order using a general contractive condition of integral type.

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1. Preliminaries

Recently the Banach contraction principle [8] was discussed in a metric space endowed with a partial order where some applications to matrix equations [16] and to ordinary differential equations [11, 13] are presented. The usual contraction condition is weakened but at the expense that the operator is monotone. The main idea in [11, 16] involves combining the ideas in the contraction principle with those in the monotone iterative technique [2, 3].

This article presents new results for contractions satisfying a condition of integral type in ordered metric spaces and these results are slight extensions of those in [11, 16].

Existence of fixed point in partially ordered sets starts with Tarki's theorem [18]. Recently, a lot of papers have treated this equation (see, for example [5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 16, 19]).

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2. Fixed point theorems

Suppose $(X, \leq)$ is a partially ordered set and $f : X \longrightarrow X$. We say f is non-decreasing if $x, y \in X$, $x \leq y$ implies $f(x) \leq f(y)$.

In a recent paper [1], R. Agarwal, M. El-Gebeily and D. O'Regan established the following theorem.

Theorem 2.1. *Let $(X, \leq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Assume there is a non-decreasing function $\psi : [0, \infty) \longrightarrow [0, \infty)$ with $\lim_{n\to\infty} \psi^n(t) = 0$ for each $t > 0$ and also suppose $F : X \longrightarrow X$ is a nondecreasing mapping with*

$$d(F(x), F(y)) \leq \psi \left(\max\{d(x, y), d(x, F(x)), d(y, F(y)), \frac{1}{2} [d(x, F(y)) + d(y, F(x))]\} \right),$$

for all $x \geq y$. Also suppose either F is continuous or if $(x_n) \subset X$ is a nondecreasing sequence with $x_n \to x$ in X then $x_n \leq x$ for all $n \in \mathbb{N}$.

If there exists $x_0 \in X$ with $x_0 \leq F(x_0)$, then F has a fixed point.

Now, we present our main result in this paper.

Previously, we define for $F : X \longrightarrow X$

$$m(x, y) = \max \left\{ d(x, y), d(x, F(x)), d(y, F(y)), \frac{1}{2} [d(x, F(y)) + d(y, F(x))] \right\}.$$

Theorem 2.2. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $F : X \to X$ be a continuous and nondecreasing mapping such that there exists $k \in [0, 1)$ with*

$$\int_0^{d(F(x),F(y))} \varphi(t)dt \leq k \int_0^{m(x,y)} \varphi(t)dt \quad \text{for } x \geq y, \tag{1}$$

where $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a Lebesgue-integrable mapping such that $\int_0^\varepsilon \varphi(t) > 0$ for $\varepsilon > 0$. If there exists $x_0 \in X$ with $x_0 \leq F(x_0)$ then F has a fixed point.

Proof. If $F(x_0) = x_0$ then the proof is finished. Suppose that $x_0 < F(x_0)$. Since $x_0 < F(x_0)$ and F is nondecreasing, we obtain by induction that

$$x_0 \leq F(x_0) \leq F^2(x_0) \leq \cdots \leq F^n(x_0) \leq F^{n+1}(x_0) \leq \ldots.$$

Put $x_{n+1} = F^n(x_0)$. Then for each integer $n \geq 1$, from (1) and, as the elements x_n and x_{n+1} are comparable, we get

$$\int_0^{d(x_n,x_{n+1})} \varphi(t)dt = \int_0^{d(F(x_{n-1}),F(x_n))} \varphi(t)dt \leq k \int_0^{m(x_{n-1},x_n)} \varphi(t)dt. \tag{2}$$

Taking into account that

$$\begin{aligned}
&m(x_{n-1}, x_n)\\
&= \max\Big\{d(x_{n-1}, x_n), d(x_{n-1}, F(x_{n-1})),\\
&\qquad d(x_n, F(x_n)), \frac{1}{2}[d(x_{n-1}, F(x_n)) + d(x_n, F(x_{n-1}))]\Big\}\\
&= \max\Big\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{1}{2}[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)]\Big\}\\
&= \max\Big\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{1}{2}[d(x_{n-1}, x_{n+1})]\Big\},
\end{aligned}$$

and, as

$$\frac{d(x_{n-1}, x_{n+1})}{2} \leq \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2} \leq \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}$$

we obtain

$$m(x_{n-1}, x_n) = \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}.$$

Substituting into (2) we obtain

$$\begin{aligned}
\int_0^{d(x_n,x_{n+1})} \varphi(t)dt &\leq k \int_0^{\max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\}} \varphi(t)dt\\
&= k \max\left\{\int_0^{d(x_{n-1},x_n)} \varphi(t)dt, \int_0^{d(x_n,x_{n+1})} \varphi(t)dt\right\}. \qquad (3)
\end{aligned}$$

If $\max\left\{\int_0^{d(x_{n-1},x_n)} \varphi(t)dt, \int_0^{d(x_n,x_{n+1})} \varphi(t)dt\right\} = \int_0^{d(x_n,x_{n+1})} \varphi(t)$, then, by (3),

$$\int_0^{d(x_n,x_{n+1})} \varphi(t)dt \leq k \int_0^{d(x_n,x_{n+1})} \varphi(t)dt.$$

and, as $k \in [0, 1)$, we have that $\int_0^{d(x_n,x_{n+1})} \varphi(t)dt = 0$. By our hypothesis about φ, we get $d(x_n, x_{n+1}) = 0$, or, equivalently, $x_n = x_{n+1} = F(x_n)$ and x_n is a fixed point of F.

If $\max\left\{\int_0^{d(x_{n-1},x_n)} \varphi(t)dt, \int_0^{d(x_n,x_{n+1})} \varphi(t)dt\right\} = \int_0^{d(x_{n-1},x_n)} \varphi(t)$ then, from (3), we get

$$\int_0^{d(x_n,x_{n+1})} \varphi(t)dt \leq k \int_0^{d(x_{n-1},x_n)} \varphi(t)dt. \qquad (4)$$

Using induction we have

$$\int_0^{d(x_n,x_{n+1})} \varphi(t)dt \leq k \int_0^{d(x_{n-1},x_n)} \varphi(t)dt \leq \cdots \leq k^n \int_0^{d(x_0,x_1)} \varphi(t)dt.$$

Taking limit as $n \to \infty$

$$\lim_{n\to\infty} \int_0^{d(x_n,x_{n+1})} \varphi(t)dt = 0. \qquad (5)$$

On the other hand, by (4), as $k \in [0,1)$,

$$\int_0^{d(x_n,x_{n+1})} \varphi(t)dt \le k \int_0^{d(x_{n-1},x_n)} \varphi(t)dt < \int_0^{d(x_{n-1},x_n)} \varphi(t)dt$$

and, as φ is a non-negative function, we obtain that $\{d(x_n,x_{n+1})\}$ is a non-negative and non-increasing sequence. We put $\lim_{n\to\infty} d(x_n,x_{n+1}) = a$.

In what follows, we will prove that $a = 0$.

Suppose that $a > 0$. As $0 < a \le d(x_n,x_{n+1})$ for all n, and, taking into account our assumption about φ,

$$0 < \int_0^{a} \varphi(t)dt \le \int_0^{d(x_n,x_{n+1})} \varphi(t)dt.$$

Taking limit as $n \to \infty$ and, from (5),

$$0 < \int_0^{a} \varphi(t)dt \le \lim_{n\to\infty} \int_0^{d(x_n,x_{n+1})} \varphi(t)dt = 0,$$

which is a contradiction. Therefore,

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \tag{6}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence.

Suppose that $\{x_n\}$ is not a Cauchy sequence there exists an $\varepsilon > 0$ and subsequences $\{m(p)\}$ and $\{n(p)\}$ such that $m(p) < n(p) < m(p+1)$ with

$$d(x_{m(p)},x_{n(p)}) \ge \varepsilon \quad \text{and} \quad d(x_{m(p)},x_{n(p)-1}) < \varepsilon. \tag{7}$$

Then

$$m(x_{m(p)-1},x_{n(p)-1}) = \max\Big\{ d(x_{m(p)-1},x_{n(p)-1}), d(x_{m(p)-1},x_{m(p)}), d(x_{n(p)-1},x_{n(p)}), \frac{1}{2}[d(x_{m(p)-1},x_{n(p)}) + d(x_{m(p)},x_{n(p)-1})]\Big\}.$$

By (5), we have

$$\lim_{p\to\infty} \int_0^{d(x_{m(p)-1},x_{m(p)})} \varphi(t)dt = \lim_{p\to\infty} \int_0^{d(x_{n(p)-1},x_{n(p)})} \varphi(t)dt = 0. \tag{8}$$

By the triangular inequality and (7)

$$d(x_{m(p)-1},x_{n(p)-1}) \le d(x_{m(p)-1},x_{m(p)}) + d(x_{m(p)},x_{n(p-1)}) < d(x_{m(p)-1},x_{m(p)}) + \varepsilon$$

and, by (5), this implies

$$\lim_{p\to\infty} \int_0^{d(x_{m(p)-1},x_{n(p)-1})} \varphi(t)dt \le \int_0^{\varepsilon} \varphi(t)dt. \tag{9}$$

Again, using the triangular inequality and (7), we get

$$\begin{aligned}
&\frac{1}{2}[d(x_{m(p)-1}, x_{n(p)}) + d(x_{m(p)}, x_{n(p)-1})] \\
&\leq \frac{1}{2}[d(x_{m(p)-1}, x_{m(p)}) + d(x_{m(p)}, x_{n(p)-1}) + d(x_{n(p)-1}, x_{n(p)}) + d(x_{m(p)}, x_{n(p)-1})] \\
&= \frac{1}{2}[d(x_{m(p)-1}, x_{m(p)}) + 2d(x_{m(p)}, x_{n(p)-1}) + d(x_{n(p)-1}, x_{n(p)})] \\
&= \frac{1}{2}[d(x_{m(p)-1}, x_{m(p)}) + d(x_{n(p)-1}, x_{n(p)})] + d(x_{m(p)}, x_{n(p)-1}) \\
&< \frac{1}{2}[d(x_{m(p)-1}, x_{m(p)}) + d(x_{n(p)-1}, x_{n(p)})] + \varepsilon.
\end{aligned}$$

Taking into account (6), we obtain

$$\lim_{p\to\infty} \int_0^{\frac{1}{2}[d(x_{m(p)-1},x_{n(p)})+d(x_{m(p)},x_{n(p)-1})]} \varphi(t)dt \leq \int_0^{\varepsilon} \varphi(t)dt. \tag{10}$$

From (1) and (7), we can get

$$\begin{aligned}
&\int_0^{\varepsilon} \varphi(t)dt \leq \int_0^{d(x_{m(p)},x_{n(p)})} \varphi(t)dt \\
&= \int_0^{d(F(x_{m(p)-1}),F(x_{n(p)-1}))} \varphi(t)dt \leq k\int_0^{m(x_{m(p)-1},x_{n(p)-1})} \varphi(t)dt \\
&= k \max\left(\int_0^{d(x_{m(p)-1},x_{n(p)-1})} \varphi(t)dt, \int_0^{d(x_{m(p)-1},x_{m(p)})} \varphi(t)dt, \right. \\
&\qquad \left. \int_0^{d(x_{n(p)-1},x_{n(p)})} \varphi(t)dt, \int_0^{\frac{1}{2}[d(x_{m(p)-1},x_{n(p)})+d(x_{m(p)},x_{n(p)-1})]} \varphi(t)dt \right),
\end{aligned}$$

and, taking limit as $p \to \infty$, and taking into account (8), (9) and (10), we obtain

$$\int_0^{\varepsilon} \varphi(t)dt \leq k\int_0^{\varepsilon} \varphi(t)dt.$$

As $k \in [0, 1)$, this implies $\int_0^{\varepsilon} \varphi(t)dt = 0$ which is a contradiction.

Therefore, $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$.

Finally, we prove that $z \in X$ is a fixed point of F.

As F is a continuous mapping and $\lim_{n\to\infty} x_n = z$, then

$$z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} F(x_n) = F(z)$$

and the proof is complete. □

In what follows, we prove that Theorem 2.2 is still valid for F not necessarily continuous, assuming the following hypothesis in X (which appears in Theorem 1 of [1]):

$$\text{if } (x_n) \subset X \text{ is a nondecreasing sequence with } x_n \to x \text{ then } x_n \leq x \text{ for all } n \in \mathbb{N}. \tag{11}$$

Theorem 2.3. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $F : X \longrightarrow X$ be a nondecreasing mapping such that there exists $k \in [0, 1)$ with*

$$\int_0^{d(F(x),F(y))} \varphi(t)dt \leq k \int_0^{m(x,y)} \varphi(t)dt, \quad \text{for } x \geq y,$$

where $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a Lebesgue-integrable mapping such that $\int_0^{\varepsilon} \varphi(t)dt > 0$ for $\varepsilon > 0$. Assume that X satisfies (11) and there exists $x_0 \in X$ with $x_0 \leq F(x_0)$, then F has a fixed point.

Proof. Following the proof of Theorem 2.2, we only have to check that $F(z) = z$.

From (2) and (11), we have

$$\begin{aligned}\int_0^{d(F(z),x_{n+1})} \varphi(t)dt &\leq k \int_0^{m(z,x_n)} \varphi(t)dt \\ &= k \max\left\{ \int_0^{d(z,x_n)} \varphi(t)dt, \int_0^{d(z,F(z))} \varphi(t)dt, \right. \\ &\qquad \left. \int_0^{d(x_{n+1},x_n)} \varphi(t)dt, \int_0^{\frac{1}{2}[d(z,x_{n+1})+d(x_n,F(z))]} \varphi(t)dt \right\},\end{aligned}$$

and, taking limit as $n \to \infty$, and, by (5), we get

$$\int_0^{d(F(z),z)} \varphi(t)dt \leq k \int_0^{d(F(z),z)} \varphi(t)dt,$$

which implies that $\int_0^{d(F(z),z)} \varphi(t)dt = 0$. By our assumption about φ, this gives us

$$d(F(z), z) = 0$$

and this proves that z is a fixed point of F. □

Remark 2.4. If we assume that φ is a nonincreasing function in Theorem 2.2 its proof is less complicated.

In fact, perhaps the more difficult part in Theorem 2.2 is to prove that $\{x_n\}$ is a Cauchy sequence. Under assumption that φ is a nonincreasing function, for $m > n$ we can get

$$\begin{aligned}\int_0^{d(x_m,x_n)} \varphi(t)dt &\leq \int_0^{d(x_m,x_{m-1})+d(x_{m-1},x_{m-2})+\cdots+d(x_{n+1},x_n)} \varphi(t)dt \\ &= \int_0^{d(x_{n+1},x_n)} \varphi(t)dt + \int_{d(x_{n+1},x_n)}^{d(x_{n+2},x_{n+1})+d(x_{n+1},x_n)} \varphi(t)dt \\ &\qquad + \cdots + \int_{d(x_{n+1},x_n)+\cdots+d(x_{m-1},x_{m-2})}^{d(x_{n+1},x_n)+\cdots+d(x_{m-1},x_{m-2})+d(x_m,x_{m-1})} \varphi(t)dt.\end{aligned}$$

Applying a simple change of variables, our integrals can be transformed in

$$\int_0^{d(x_m,x_n)} \varphi(t)dt \leq \sum_{i=n+1}^{m} \int_0^{d(x_i,x_{i-1})} \varphi\left(s + \sum_{j=n+1}^{i-1} d(x_j, x_{j-1})\right) ds$$

and, as φ is a nonincreasing function, we can get

$$\int_0^{d(x_m,x_n)} \varphi(t)dt$$
$$\leq \sum_{i=n+1}^{m} \int_0^{d(x_i,x_{i-1})} \varphi\left(s + \sum_{j=n+1}^{i-1} d(x_j, x_{j-1})\right) ds \leq \sum_{i=n+1}^{m} \int_0^{d(x_i,x_{i-1})} \varphi(s)ds.$$

Taking into account (4) in the proof of Theorem 2.2, we obtain

$$\int_0^{d(x_m,x_n)} \varphi(t)dt \leq \sum_{i=n+1}^{m} \int_0^{d(x_i,x_{i-1})} \varphi(t)dt$$
$$\leq \sum_{i=n+1}^{m} k^{i-1} \int_0^{d(x_0,x_1)} \varphi(t)dt = \left(\int_0^{d(x_0,x_1)} \varphi(t)dt\right)(k^n + \cdots + k^{m-1})$$
$$\leq \left(\int_0^{d(x_0,x_1)} \varphi(t)dt\right)\left(\frac{k^n}{1-k}\right).$$

Taking limit as $n \to \infty$ we have

$$\lim_{m,n\to\infty} \int_0^{d(x_m.x_n)} \varphi(t)dt = 0. \tag{12}$$

Now, suppose that $\{x_n\}$ is not a Cauchy sequence. This means that there exists an $\varepsilon > 0$ such that for any $p \in \mathbb{N}$ we can find $m(p), n(p) \in \mathbb{N}$ with $m(p), n(p) > p$ satisfying $d(x_{m(p)}, x_{n(p)}) \geq \varepsilon$. Consequently,

$$\int_0^{d(x_{m(p)},x_{n(p)})} \varphi(t)dt \geq \int_0^{\varepsilon} \varphi(t)dt > 0,$$

and, taking limit as $p \to \infty$, we get

$$\lim_{p\to\infty} \int_0^{d(x_{m(p)},x_{n(p)})} \varphi(t)dt \geq \int_0^{\varepsilon} \varphi(t)dt > 0$$

and this contradicts to (12).

Remark 2.5. If we put $\varphi(t) = 1$ in (1) of Theorem 2.2, we have

$$d(F(x), F(y)) \leq k\ m(x,y) \quad \text{for } x \geq y$$

and our Theorem 2.2 is a particular case of Theorem 2.2 of [1] for the function $\psi(t) = kt$ with $k \in [0,1)$.

Remark 2.6. If we put $\varphi(t) = 1$ in (1) of Theorem 2.2 then the condition $d(F(x), F(y)) \leq kd(x,y)$ for $x \geq y$ implies $d(F(x), F(y)) \leq k\ m(x,y)$ and Theorem 2.1 in [11] and Theorem 2.1 in [16] are particular cases of our Theorem 2.2.

Remark 2.7. We present an example where it can be appreciated that hypotheses in Theorem 2.2 do not guarantee uniqueness of the fixed point. This example appears in [11].

Let $X = \{(1,0),(0,1)\} \subset \mathbb{R}^2$ and consider the usual order

$$(x,y) \leq (z,t) \Leftrightarrow x \leq z \text{ and } y \leq t.$$

Thus, $(X, \leq)$ is a partially ordered set, whose different elements are not comparable. Besides, (X, d_2) is a complete metric space considering d_2 the euclidean distance. The identity map $f(x,y) = (x,y)$ is trivially continuous and nondecreasing and condition (1) of Theorem 2.2 is satisfied for any $k \in [0,1)$ and φ nonnegative Lebesgue-integrable function since elements in X are only comparable to themselves. Moreover, $(1,0) \leq f(1,0) = (1,0)$ and f has two fixed points in X.

Remark 2.8. In Theorem 2.1 of [11] and Theorem 2.3 of [16] it is proved the uniqueness of the fixed point adding the following conditon

$$\text{every pair of elements of } X \text{ has a lower bound or an upper bound.} \tag{13}$$

We have not been able to prove this fact for our Theorem 2.2. We think that condition (13) is not sufficient for the uniqueness of the fixed point in Theorem 2.2.

Remark 2.9. If X is a totally ordered set then we can obtain the uniqueness of the fixed point in Theorem 2.2.

In fact, if y is other fixed point of F then, as $F^n(z) = z$ and $F^n(y) = y$ for $n \in \mathbb{N}$, and, as y and z are comparable, the condition (1) of Theorem 2.2 give us

$$\int_0^{d(y,z)} \varphi(t)dt = \int_0^{d(F^n(y),F^n(z))} \varphi(t)dt \leq k \int_0^{m(F^{n-1}(y),F^{n-1}(z))} \varphi(t)dt.$$

But

$$\begin{aligned} m(y,z) &= m(F^{n-1}(y), F^{n-1}(z)) \\ &= \max\left\{d(y,z), d(y,F(y)), d(z,F(z)), \frac{1}{2}[d(y,F(z)) + d(F(y),z)]\right\} \\ &= \max\{d(y,z), 0, 0, d(y,z)\} = d(y,z) \end{aligned}$$

and, consequently,

$$\int_0^{d(y,z)} \varphi(t)dt \leq k \int_0^{d(y,z)} \varphi(t)dt$$

and, as $k \in [0,1)$, this implies that $\int_0^{d(y,z)} \varphi(t)dt = 0$. By our assumption about φ, we obtain $d(y,z) = 0$, or, equivalently, $y = z$.

Remark 2.10. Theorem 2.2 is false if we admit zero value near zero for the mapping φ. The following example proves this fact. Let $(\mathbb{N}, d)$ be with the trivial metric ($d(x,y) = 0$ iff $x = y$ and $d(x,y) = 1$ if $x \neq y$). Then $(\mathbb{N}, d)$ is a complete metric space.

We consider in $\mathbb{N}$ the usual order and let $f : \mathbb{N} \longrightarrow \mathbb{N}$ be defined by $f(n) = n + 1$. Obviously, f is continuous (the topology generated by d is the discret topology) and nondecreasing.

Let $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be defined by $\varphi(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq 1 \\ e^{-t} & \text{if } t > 1. \end{cases}$

Now, since for each $n, m \in \mathbb{N}$, $d(f(n), f(m)) \leq 1$, we have, for every $k \in [0, 1)$

$$\int_0^{d(f(n),f(m))} \varphi(t)dt \leq \int_0^1 \varphi(t)dt = 0 \leq k \int_0^{d(n,m)} \varphi(t) = 0$$

and, consequently, the condition (1) of Theorem 2.2 is satisfied. Moreover, $0 \leq f(0) = 1$ and we can see that f has no fixed points.

Remark 2.11. In [17] it is proved the following theorem.

Let (X, d) be a complete metric space, $k \in [0, 1)$, $F : X \longrightarrow X$ a mapping such that, for each $x, y \in X$,

$$\int_0^{d(F(x),F(y))} \varphi(t)dt \leq k \int_0^{m(x,y)} \varphi(t)dt,$$

where $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a Lebesgue-integrable mapping such that $\int_0^{\varepsilon} \varphi(t)dt > 0$ for $\varepsilon > 0$. Then F has a unique fixed point $z \in X$.

By using Zermelo's well ordering theorem the set X can be well-ordered and the condition (1) of our Theorem 2.2 is valid for each $x, y \in X$. Moreover, $x_0 = \min X$ satisfies $x_0 \leq F(x_0)$ and our Theorem 2.2 give us the above mentioned result for the particular case that F is a continuous and nondecreasing function. The uniqueness of fixed point is obtained in virtue a well ordering in a set X implies that X is a totally ordered set and Remark 2.9 applies.

In connection with the condition (11) it is proved in [11] the following lemma.

Lemma 2.12. *If X is a totally ordered set and*

$$d(a, c) \geq d(b, c) \quad \text{for } a \leq b \leq c \tag{14}$$

then the condition (11) *holds.*

Consequently, our Theorem 2.2 also gives us the result mentioned in Remark 2.11 for the particular case that F is a nondecreasing function and the distance satisfies condition (14).

Remark 2.13. Theorem 2.1 uses nondecreasing functions $\psi : [0, \infty) \longrightarrow [0, \infty)$ with $\lim_{n\to\infty} \psi^n(t) = 0$ for each $t > 0$. In the sequel, we present a function ψ which can be expressed by an integral.

Put $\psi(t) = \int_0^t \varphi(s)ds$, where $\varphi(s) = \frac{1}{(1+s)^2}$.

Then, a simple calculus, give us $\psi(t) = \frac{t}{1+t}$ and, it is easily proved that $\psi^n(t) = \frac{t}{1+nt}$ and that ψ is a non-decreasing function. Obviously, $\lim_{n\to\infty} \psi^n(t) = 0$.

One would like to be able to replace (1) in Theorem 2.2 with the integral form of Ciric's condition [4], that is

$$\int_0^{d(F(x),F(y))} \varphi(t)dt \leq k \int_0^{M(x,y)} \varphi(t)dt, \tag{15}$$

where $M(x,y) = \max\{d(x,y), d(x,F(x)), d(y,F(y)), d(x,F(y)), d(y,F(x))\}$.

The following example proves that this is not possible (this example appears in [17]).

Let $f : \mathbb{N} \longrightarrow \mathbb{N}$ defined by $f(n) = n + 1$.

In $\mathbb{N}$ we consider the usual order and the euclidean distance d. Obviously, $(\mathbb{N}, d)$ is a complete metric space and f is continuous and nondecreasing function. Moreover, $0 \leq f(0) = 1$.

On the other hand, we consider $\phi, \varphi : [0,\infty) \longrightarrow [0,\infty)$, where $\phi(t) = (t+1)^{t+1} - 1$ and $\varphi(t) = \phi'(t)$.

Then, for $n > m$

$$M(n,m) = \max\{n-m, 1, n-m-1, n-m+1\} = n-m+1.$$

Note that, for any $t \in \mathbb{N}$ with $t \geq 1$, we have

$$\begin{aligned}(t+2)^{t+2} - 1 &= (t+1+1)^{t+2} - 1 \geq (t+1)^{t+2} + 1^{t+2} - 1 \\ &= (t+1)^{t+1}(t+1) \geq 2(t+1)^{t+1} \\ &\geq 2(t+1)^{t+1} - 2 = 2[(t+1)^{t+1} - 1].\end{aligned}$$

Consequently, $\phi(t+1) \geq 2\phi(t)$.

Since $\varphi(t) = \phi'(t)$, we can get

$$\int_0^{d(f(n),f(m))} \varphi(t)dt = \int_0^{n-m} \varphi(t)dt = \phi(n-m) \leq \frac{1}{2}\phi(n-m+1) = \frac{1}{2}\int_0^{M(n,m)} \varphi(t)dt$$

and the condition (15) is satisfied. However, f has no fixed point.

It is possible to prove a weaker theorem involving condition (15).

Let $O(x,n) = \{x, f(x), f^2(x), \ldots, f^n(x)\}$ and $O(x) = \{x, f(x), f^2(x), \ldots, f^n(x), \ldots\}$. Then $O(x,n)$ is called the nth orbit of x and $O(x)$ the orbit of x. $\delta(A)$ will denote the diameter of A.

Theorem 2.14. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X,d) is a complete metric space. Let $F : X \longrightarrow X$ a nondecreasing mapping such that there exists $k \in [0,1)$ with*

$$\int_0^{d(F(x),F(y))} \varphi(t)dt \leq k \int_0^{M(x,y)} \varphi(t)dt, \quad \textit{for } x \geq y,$$

where $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a Lebesgue-integrable mapping such that $\int_0^\varepsilon \varphi(t)dt > 0$ for $\varepsilon > 0$. Assume that F is continuous or that X satisfies (11) and suppose that there exists $x_0 \in X$ with $x_0 \leq F(x_0)$ and $O(x_0)$ is bounded. Then F has a fixed point.

Proof. As in the proof of Theorem 2.2, we consider the nondecreasing sequence

$$x_0 \leq F(x_0) \leq F^2(x_0) \leq \cdots \leq F^n(x_0) \leq \cdots$$

Put $x_n = F^n(x_0)$. As $O(x_0)$ is bounded and for each $m \in \mathbb{N}$, $O(x_m, n) \subset O(x_0)$ for every $n \in \mathbb{N}$ and as $O(x_m, n)$ is finite for every $n \in \mathbb{N}$, there exist integers i, j satisfying $0 \leq i < j \leq n$ such that $\delta(O(x_m, n)) = d(x_{m+i}, x_{m+j})$.

We claim that for m fixed and for every $n \in \mathbb{N}$ and $n > 0$, there exists k with $0 < k \leq n$ such that

$$\delta(O(x_m, n)) = d(x_m, x_{m+k}). \tag{16}$$

In fact, we may assume that $\delta(O(x_m, n)) > 0$ for $m, n \in \mathbb{N}$ (with $n > 0$) fixed since if $\delta(O(x_m, n)) = 0$, then F has a fixed point and the proof is finished.

Suppose that $\delta(O(x_m, n)) = d(x_{m+i}, x_{m+j})$ with $0 < i < j \leq n$. Then, by our assumption, we can get

$$\begin{aligned}\int_0^{\delta(O(x_m,n))} \varphi(t)dt &= \int_0^{d(x_{m+i},x_{m+j})} \varphi(t)dt = \int_0^{d(F^{m+i}(x_0),F^{m+j}(x_0))} \varphi(t)dt \\ &\leq k \int_0^{M(F^{m+i-1}(x_0),F^{m+j-1}(x_0))} \varphi(t)dt \leq k \int_0^{\delta(O(x_m,n))} \varphi(t)dt\end{aligned}$$

and, as $k \in [0, 1)$, this gives us $\delta(O(x_m, n)) = 0$ and this contradicts the fact that $\delta(O(x_m, n)) > 0$. Therefore $i = 0$.

Now, let m and n be integers with $m > n$. By our assumption

$$\int_0^{d(x_n,x_m)} \varphi(t)dt \leq k \int_0^{M(x_{n-1},x_{m-1})} \varphi(t)dt \leq k \int_0^{\delta(O(x_{n-1},m-n))} \varphi(t)dt.$$

By (16), $\delta(O(x_{n-1}, m-n)) = d(x_{n-1}, x_{k_1+n-1})$ for some $0 < k_1 \leq m-n$ and, consequently,

$$\int_0^{d(x_n,x_m)} \varphi(t)dt \leq k \int_0^{\delta(O(x_{n-1},m-n))} \varphi(t)dt = k \int_0^{d(x_{n-1},x_{k_1+n-1})} \varphi(t)dt.$$

Repeating the same process we get

$$\begin{aligned}\int_0^{d(x_n,x_m)} \varphi(t)dt &\leq k \int_0^{d(x_{n-1},x_{k_1+n-1})} \varphi(t)dt \\ &\leq k^2 \int_0^{M(x_{n-2},x_{k_1+n-2})} \varphi(t)dt \leq k^2 \int_0^{\delta(O(x_{n-2},k_1))} \varphi(t)dt \\ &= k^2 \int_0^{d(x_{n-2},x_{k_2+n-2})} \varphi(t)dt, \quad \text{for some } 0 < k_2 \leq k_1 \leq m-n \leq m \\ &\vdots \\ &\leq k^n \int_0^{d(x_0,x_h)} \varphi(t)dt \quad \text{for some } 0 < h \leq m-n \leq m \\ &\leq k^n \int_0^{\delta(O(x_0,m))} \varphi(t)dt.\end{aligned}$$

As the orbit of x_0 is bounded, $\int_0^{\delta(O(x_0,m))} \varphi(t)dt < \infty$ and, taking limit as $m, n \to \infty$, we get

$$\lim_{m,n\to\infty} \int_0^{d(x_m,x_n)} \varphi(t)dt = 0.$$

And using the same reasoning that Theorem 2.2 we can prove that $\{x_n\}$ is a Cauchy sequence and hence convergent.

Put $\lim_{n\to\infty} x_n = z$.

Finally, if F is a continuous mapping, applying the same argument that Theorem 2.2, we can prove that z is a fixed point.

If F satisfies the condition (11) we get

$$\begin{aligned}\int_0^{d(x_{n+1},F(z))} \varphi(t)dt &\le k \int_0^{M(x_n,z)} \varphi(t)dt \\ &= k \max\left\{ \int_0^{d(x_n,z)} \varphi(t)dt, \int_0^{d(x_n,x_{n+1})} \varphi(t)dt, \int_0^{d(z,F(z))} \varphi(t)dt \right. \\ &\qquad \left. \int_0^{d(x_n,F(z))} \varphi(t)dt, \int_0^{d(z,x_{n+1})} \varphi(t)dt \right\},\end{aligned}$$

and, taking limit as $n \to \infty$, we obtain

$$\int_0^{d(z,F(z))} \varphi(t)dt \le k \int_0^{d(z,F(z))} \varphi(t)dt.$$

This implies that $d(z, F(z)) = 0$ and this says us that $z = F(z)$. □

Remark 2.15. If $(X, \le)$ is a totally ordered set we can obtain the uniqueness of the fixed point in Theorem 2.14.

In fact, suppose that z and w are fixed points of F. Then by our assumption, $z \le w$ or $w \le z$ and, consequently,

$$\begin{aligned}\int_0^{d(z,w)} \varphi(t)dt &= \int_0^{d(F(z),F(w))} \varphi(t)dt \\ &\le k \int_0^{M(z,w)} \varphi(t)dt = k \int_0^{d(z,w)} \varphi(t)dt,\end{aligned}$$

which implies that $\int_0^{d(z,w)} \varphi(t)dt = 0$ and, this gives us that $d(z, w) = 0$. Therefore, $z = w$.

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