Journal of Convex Analysis
Volume 17 (2010), No. 1, 203–210

A Calculus of Prox-Reguarity

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Received: September 12, 2008
Revised manuscript received: February 11, 2009

In this paper we show that the operations of composition and addition, under appropriate conditions, preserve prox-regularity. The class of prox-regular functions covers all l.s.c., proper, convex functions, lower-$\mathcal{C}^2$ functions, strongly amenable functions (i.e. convexly composite functions), and pln functions, hence a large core of functions of interest in variational analysis and optimization. These functions, despite being in general nonconvex, possess many of the properties that one would expect only to find in convex or near convex (lower-$\mathcal{C}^2$) functions e.g. the Moreau-envelopes are $\mathcal{C}^{1+}$, a localization of the subgradient mapping is hypomonotone, etc... In this paper, we add to this list of convex-like properties by showing, under suitable conditions, that locally the subdifferential of the sum of prox-regular functions is equal to the sum of subdifferentials.

Keywords: Prox-regular, proximal mapping, Moreau envelope, convex, convexly composite, pln, strongly amenable, subgradients, amenable functions

2000 Mathematics Subject Classification: Primary 49A52, 58C06, 58C20; Secondary 90C30

1. Introduction

Let f be a proper, l.s.c. function on $I\!R^n$, we denote by $\partial f(x)$ the set of limiting proximal subgradients of f at any point $x \in \operatorname{dom} f$ (i.e. $f(x)$ is finite).

Definition 1.1. The function f is prox-regular at $\bar{x}$ relative to $\bar{v}$ if $\bar{x} \in \operatorname{dom} f$, $\bar{v} \in \partial f(\bar{x})$, and there exist $\epsilon > 0$ and $r > 0$ such that

$$f(x') > f(x) + \langle v, x' - x \rangle - \frac{r}{2} |x' - x|^2$$

whenever $|x' - \bar{x}| < \epsilon$ and $|x - \bar{x}| < \epsilon$, with $x' \neq x$ and $|f(x) - f(\bar{x})| < \epsilon$, while $|v - \bar{v}| < \epsilon$ with $v \in \partial f(x)$.

First note that it is obvious that if f is convex, it is prox-regular at $\bar{x}$ for every subgradient $\bar{v} \in \partial f(\bar{x})$. The same is true for lower-$\mathcal{C}^2$ functions, pln (primal-lower-

*Research by this author was supported in part by the Natural Sciences and Engineering Research Council of Canada under grant OGP41983.

ISSN 0944-6532 / $ 2.50

nice) functions ([15]), and *strongly amenable* functions, cf. [21]. A function f is strongly amenable at $\bar{x}$ if it can be written as $f = g \circ F$ in a neighborhood of $\bar{x}$ for a mapping $F : I\!R^n \to I\!R^m$ of class C^2 and a proper, l.s.c., convex function $g : I\!R^m \to \overline{I\!R}$ satisfying at $\bar{x}$ with respect to the convex set $D = \operatorname{dom} g$ the basic constraint qualification that

$$\text{there is no vector } y \neq 0 \text{ in } N_D\left(F(\bar{x})\right) \text{ with } \nabla F(\bar{x})^* y = 0.$$

(Here $N_D\left(F(\bar{x})\right)$ refers to the normal cone to D at $F(\bar{x})$.) Amenability itself merely requires F to be of class $\mathcal{C}^1$, whereas *full* amenability is the subcase of strong amenability where g is also piecewise linear-quadratic. Full amenability already covers most applications that arise in the framework of nonlinear programming and its extensions. For more on amenability, see [21].

Prox-regular functions were first introduced in [16]. Additional results were provided in [17] and [21]. For a comprehensive survey, see [21]. Extensions to non-finite dimensional spaces were given in [1]–[5]. Other applications and results can be found in [7]–[14], [18]–[20], and [23].

The initial goal in the development of prox-regularity was to demonstrate that many of the important properties of convex functions can be found in a large class of nonconvex functions e.g. differentiable Moreau-envelopes, single-valued proximal mappings, etc... In Section 2 of this paper we give a brief survey of some of the important properties of prox-regular functions and sets (a set is prox-regular if its indicator function is prox-regular).

In Section 3 of this paper, we show that under suitable conditions prox-regularity is preserved under addition and the composition of a prox-regular function with a $\mathcal{C}^2$ mapping. In addition, we show in Section 3 that under suitable conditions the subdifferential of the sum of prox-regular functions is equal to the sum of subdifferentials. This is somewhat of a surprise since prox-regular functions are in general not "regular" and this regularity is one of the few conditions that guarantees (see for example [21, Cor. 10.9]) that the subdifferential of a sum is the sum of subdifferentials. In Section 3 we also obtain rules for generating new prox-regular sets from known ones.

For more on operations that preserve prox-regularity, see [8] where the authors show that on the space of $n \times n$ symmetric matrices the composition of a symmetric l.s.c. function f with the eigenvalue mapping is prox-regular if and only if f is prox-regular.

2. Prox-Regular functions and sets

The fact that the subgradient mapping of a convex function is monotone plays a vital role in many applications. Something similar is true for prox-regular functions in that a selection or localization of the subgradient mapping is r-monotone i.e. a subset of the subgradient mapping plus rI is monotone, where $r > 0$ and I is the identity mapping. In the next theorem, the *f-attentive ϵ-localization* of ∂f at $(\bar{x}, \bar{v})$ (where $\epsilon > 0$) is the mapping $T : I\!R^n \rightrightarrows I\!R^n$ defined by

$$T(x) = \begin{cases} \left\{ v \in \partial f(x) \,\middle|\, |v - \bar{v}| < \epsilon \right\} & \text{when } |x - \bar{x}| < \epsilon \text{ and } |f(x) - f(\bar{x})| < \epsilon, \\ \emptyset & \text{otherwise.} \end{cases}$$

The graph of T is the intersection of gph ∂f with the product of an f-attentive neighborhood of $\bar{x}$ and an ordinary neighborhood of $\bar{v}$. (The f-attentive topology on $\mathbb{R}^n$ is the weakest topology in which f is continuous.)

Theorem 2.1 ([16, Thm. 3.2]). *The function f is prox-regular at a point $\bar{x}$ relative to $\bar{v}$ if and only if the vector $\bar{v}$ is a proximal subgradient of f at $\bar{x}$ and there exist $\epsilon > 0$ and $r > 0$ such that, for the f-attentive ϵ-localization T of ∂f, the mapping $T + rI$ is monotone, i.e., one has $\langle v_1 - v_0, x_1 - x_0 \rangle \geq -r\,|x_1 - x_0|^2$ whenever $|x_i - \bar{x}| < \epsilon$, $|f(x_i) - f(\bar{x})| < \epsilon$, and $|v_i - \bar{v}| < \epsilon$ with $v_i \in \partial f(x_i)$, $i = 0, 1$.*

To illustrate further that prox-regular functions have many convex-like properties, we recall some of the results of [16] dealing with the properties of Moreau-envelopes e_λ and proximal mappings P_λ. Recall that

$$e_\lambda(x) := \min_{x'} \left\{ f(x') + \frac{1}{2\lambda} |x' - x|^2 \right\},$$

$$P_\lambda(x) := \operatorname*{argmin}_{x'} \left\{ f(x') + \frac{1}{2\lambda} |x' - x|^2 \right\}.$$

Theorem 2.2 ([16, Thms. 4.4, 4.6, 5.2]). *Suppose that f is prox-regular at $\bar{x} \in$ argmin f for $\bar{v} = 0$ with respect to ϵ and r, and let T be the f-attentive ϵ-localization of ∂f at $(\bar{x}, 0)$. Then for any $\lambda \in (0, 1/r)$ there is a convex neighborhood X_λ of $\bar{x}$ such that*

(*a*) *the mapping P_λ is single-valued and Lipschitz continuous on X_λ with*

$$P_\lambda = (I + \lambda T)^{-1}, \qquad |P_\lambda(x') - P_\lambda(x)| \leq \frac{1}{1 - \lambda r} |x' - x|, \qquad P_\lambda(\bar{x}) = \bar{x},$$

(*b*) *the function e_λ is $\mathcal{C}^{1+}$ and lower-$\mathcal{C}^2$ on X_λ with*

$$e_\lambda + \frac{r}{2(1 - \lambda r)} |\cdot|^2 \text{convex}, \qquad \nabla e_\lambda = \lambda^{-1} [I - P_\lambda] = \left[\lambda I + T^{-1}\right]^{-1}.$$

Proximal regularity can also be defined for sets. A set $C \subset \mathbb{R}^n$ is prox-regular at a point $\bar{x} \in C$ for a vector $\bar{v}$ if and only if its indicator function δ_C is prox-regular at $\bar{x}$ for $\bar{v}$. With this definition of prox-regular sets you can then characterize the prox-regularity of a function in terms of the prox-regularity of its epigraph: a function is prox-regular at $\bar{x}$ for the subgradient $\bar{v}$ if and only if its epigraph is prox-regular at $(\bar{x}, f(\bar{x})$ for the normal vector $(\bar{v}, -1)$; see [2] and [16].

Convex sets, "weakly convex" sets ([24]), "proximally smooth" sets ([6]), and sets with the "Shapiro" property ([22]) are all examples of prox-regular sets. Strong amenability provides further examples: a set $C \subset \mathbb{R}^n$ is *strongly amenable* at one of its points $\bar{x}$ if its indicator function δ_C is strongly amenable at $\bar{x}$, or equivalently, there is an open neighborhood U of $\bar{x}$ and a $\mathcal{C}^2$ mapping $F : U \to \mathbb{R}^m$ along with a closed, convex set $D \subset \mathbb{R}^m$ such that

$$C \cap U = \left\{ x \in U \,\middle|\, F(x) \in D \right\},$$

and the constraint qualification is satisfied that no nonzero vector $y \in N_D(F(\bar{x}))$ has $\nabla F(\bar{x})^* y = 0$. The following theorem summarizes some of the results of [19] concerning the local differentiability of the distance functions to prox-regular sets. In particular a set is prox-regular if and only if the projection mapping is locally single-valued. For additional results on the projection onto prox-regular sets see [20].

Theorem 2.3 ([19, Thm. 1.3]). *For a closed subset C of a Hilbert space H, and any point $\bar{x} \in C$, the following properties are equivalent:*

(*a*) C *is prox-regular at* $\bar{x}$*;*

(*b*) d_C *is continuously differentiable on* $O \backslash C$ *for some open neighborhood* O *of* $\bar{x}$*;*

(*c*) d_C *is Fréchet differentiable on* $O \backslash C$ *for some open neighborhood* O *of* $\bar{x}$*;*

(*d*) d_C *is Gâteaux differentiable on* $O \setminus C$ *for some open neighborhood* O *of* $\bar{x}$*, and* P_C *is nonempty-valued on* O*;*

(*e*) d_C^2 *is* C^{1+} *on an open neighborhood* O *of* $\bar{x}$*; i.e., Fréchet differentiable on* O *with the derivative mapping* $D(d_C^2)(x) : H \rightrightarrows H$ *depending Lipschitz continuously on* x*;*

(*f*) P_C *is single-valued and strongly-weakly continuous (i.e., from the strong topology in the domain to the weak topology in the range) on a neighborhood of* $\bar{x}$*;*

(*g*) C *has the Shapiro property at* $\bar{x}$ *i.e. there is a constant* $k > 0$ *along with a neighborhood* O *of* $\bar{x}$ *such that*

$$d_{T_C(x)}(x' - x) \le k|x' - x|^2 \quad \text{for all } x, x' \in C \cap O,$$

where $T_C(x)$ *denotes the general tangent cone (contingent cone) to* C *at* x*.*

Then there is a neighborhood O *of* $\bar{x}$ *on which* P_C *is single-valued, monotone and Lipschitz continuous with* $P_C = (I + N_C^r)^{-1}$ *on* O *for some* $r > 0$*, whereas* $D(d_C) = [I - P_C]/d_C$ *on* $O \setminus C$*.*

If the set C *is weakly closed relative to a (strong) neighborhood of* $\bar{x}$ *(which is always the case when the space* H *is finite-dimensional), then one can add the following to the set of equivalent properties:*

(*h*) P_C *is single-valued around* $\bar{x}$*.*

3. Calculus of Prox-Regularity

Consider a mapping $F : I\!R^n \to I\!R^m$ and a proper function $g : I\!R^m \to \overline{I\!R}$. We say that the constraint qualification for F and g is satisfied at $\bar{x}$ if $F(\bar{x}) \in \operatorname{dom} g$ and

$$\text{there is no vector } y \neq 0 \text{ in } \partial^\infty g\,(F(\bar{x})) \text{ with } \nabla F(\bar{x})^* y = 0. \tag{1}$$

This constraint qualification guarantees that $\partial f(x) \subset \nabla F(x)^* \partial g\,(F(x))$ in an f-attentive neighborhood of $\bar{x}$, cf. [21, Thm. 10.6]. In (1), $\partial^\infty g\,(F(\bar{x}))$ refers to the set of horizon subgradients. A vector v is a horizon subgradient for f at $\bar{x}$ (cf. [21, Def. 8.3]) if there exist a sequence v_n of regular subgradients to f at x_n and $\lambda_n \searrow 0$ with $\lambda_n v_n \to v$, $x_n \to x$ and $f(x_n) \to f(x)$. Note that since every regular subgradient is a limiting proximal subgradient (cf. [21, Cor. 8.47]), we may assume that $v_n \in \partial f(x_n)$ in the previous definition of horizon subgradient.

Theorem 3.1. *Let* $f(x) = g\,(F(x))$*, where* $F : I\!R^n \to I\!R^m$ *is of class* $\mathcal{C}^2$*,* $g : I\!R^m \to \overline{I\!R}$ *is l.s.c. and proper, and suppose that the constraint qualification* (1) *is satisfied at* $\bar{x}$ *(which lies in* $\operatorname{dom} f$*). Assume further that* $\bar{v} \in \partial f(\bar{x})$ *is a vector such that the* *function g is prox-regular at $F(\bar{x})$ for every $y \in \partial g(F(\bar{x}))$ with $\nabla F(\bar{x})^* y = \bar{v}$. Then f is prox-regular at $\bar{x}$ for $\bar{v}$.*

Proof. As already mentioned, (1) guarantees the existence of $\epsilon_0 > 0$ such that

$$\partial f(x) \subset \nabla F(x)^* \partial g(F(x)) \quad \text{when } |x - \bar{x}| < \epsilon_0, \ |f(x) - f(\bar{x})| < \epsilon_0. \tag{2}$$

The constraint qualification further ensures that the mapping

$$S : (x, v) \to \left\{ y \in \partial g(F(x)) \;\middle|\; \nabla F(x)^* y = v \right\}$$

has closed graph and is locally bounded at $(\bar{x}, \bar{v})$ with respect to the f-attentive topology. In particular, $S(\bar{x}, \bar{v})$ is a compact set. By our assumptions, for each $y \in S(\bar{x}, \bar{v})$ the function g is prox-regular at $F(\bar{x})$ for y with constants (say) r_y and ϵ_y. The compactness of $S(\bar{x}, \bar{v})$ enables us to make this uniform: there exist $\epsilon_1 > 0$ and $\bar{r} > 0$ such that

$$\left.\begin{array}{l} \langle y_1 - y_0, u_1 - u_0 \rangle \geq -\bar{r}\, |u_1 - u_0|^2 \\ \text{whenever } y_i \in \partial g(u_i), |u_i - F(\bar{x})| < \epsilon_1, |g(u_i) - g(F(\bar{x}))| < \epsilon_1, \\ \text{dist}\,(y_i, S(\bar{x}, \bar{v})) < \epsilon_1. \end{array}\right\} \tag{3}$$

Thus in combining (2) and (3) with the established properties of S we obtain that there exist $\epsilon > 0$, $\bar{r} > 0$, and Y a compact set such that

$$\left.\begin{array}{l} |x_i - \bar{x}| < \epsilon, \\ |f(x_i) - f(\bar{x})| < \epsilon \\ v_i \in \partial f(x_i), \\ |v_i - \bar{v}| < \epsilon \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} \exists y_i \in \partial g(F(x_i)) \cap Y \quad \text{with} \\ \nabla F(x_i)^* y_i = v_i \quad \text{and} \\ \langle y_1 - y_0, F(x_1) - F(x_0) \rangle \geq -\bar{r}\, |F(x_1) - F(x_0)|^2 \end{array}\right. \tag{4}$$

Note that when $v_i = \nabla F(x_i)^* y_i$ then

$$\begin{aligned} &\langle v_1 - v_0, x_1 - x_0 \rangle \\ &= \langle \nabla F(x_1)^* y_1 - \nabla F(x_0)^* y_1 + \nabla F(x_0)^* y_1 - \nabla F(x_0)^* y_0, x_1 - x_0 \rangle \\ &= \langle [\nabla F(x_1)^* - \nabla F(x_0)^*]\, y_1, x_1 - x_0 \rangle + \langle \nabla F(x_0)^* (y_1 - y_0), x_1 - x_0 \rangle. \end{aligned} \tag{5}$$

Because Y is compact and F is of class $\mathcal{C}^2$ we know there exists $r_1 > 0$ such that for all $y \in Y$

$$|[\nabla F(x_1)^* - \nabla F(x_0)^*]\, y| \leq r_1\, |x_1 - x_0|\,.$$

Thus for all $y \in Y$

$$\langle [\nabla F(x_1)^* - \nabla F(x_0)^*]\, y, x_1 - x_0 \rangle \geq -r_1\, |x_1 - x_0|^2\,. \tag{6}$$

On the other hand, we can write the final term in (5) as $\langle \nabla\phi(x_0), x_1 - x_0 \rangle$ for the $\mathcal{C}^2$ function $\phi(x) = \langle y_1 - y_0, F(x) \rangle$. Let r_2 be an upper bound for the eigenvalues of the Hessian of the mapping $x \to \langle \eta, F(x) \rangle$ where x ranges over $\bar{x} + \epsilon\bar{B}$ and η ranges over the compact set $Y - Y$. Then in particular $\phi(x) \leq \phi(x_0) + \langle \nabla\phi(x_0), x - x_0 \rangle + r_2\, |x - x_0|^2$ when $|x - \bar{x}| \leq \epsilon$. It follows that

$$\begin{aligned} \langle \nabla\phi(x_0), x_1 - x_0 \rangle &\geq\; - r_2\, |x_1 - x_0|^2 + \phi(x_1) - \phi(x_0) \\ &=\; - r_2\, |x_1 - x_0|^2 + \langle y_1 - y_0, F(x_1) - F(x_0) \rangle \\ &\geq\; - r_2\, |x_1 - x_0|^2 - \bar{r}\, |F(x_1) - F(x_0)|^2\,. \end{aligned} \tag{7}$$

But there is also a constant λ such that $|F(x_1) - F(x_0)| \le \lambda |x_1 - x_0|$ when $|x_i - \bar{x}| < \epsilon$. Using this fact with the estimates (6) and (7) for the two terms at the end of (5) we obtain

$$\langle v_1 - v_0, x_1 - x_0 \rangle \ge -r_1 |x_1 - x_0|^2 - r_2 |x_1 - x_0|^2 - \bar{r}\lambda^2 |x_1 - x_0|^2 .$$

Thus for $r = r_1 + r_2 + \bar{r}\lambda^2$ and the same ϵ as in (4) we have

$$\left.\begin{array}{l} |x_i - \bar{x}| < \epsilon, |f(x_i) - f(\bar{x})| < \epsilon \\ v_i \in \partial f(x_i), |v_i - \bar{v}| < \epsilon \end{array}\right\} \Longrightarrow \langle v_1 - v_0, x_1 - x_0 \rangle \ge -r |x_1 - x_0|^2$$

This means that f is prox-regular at $\bar{x}$ for $\bar{v}$. □

Theorem 3.2. *Let f_i, $i = 1, 2$ be extended real-valued functions on $I\!R^n$. Consider $\bar{x} \in [\operatorname{dom} f_1 \cap \operatorname{dom} f_2]$, and assume that*

$$\textit{the only choice of } v_i \in \partial f_i^\infty(\bar{x}) \textit{ with } v_1 + v_2 = 0 \textit{ is } v_1 = v_2 = 0. \tag{8}$$

Let $\bar{v} \in \partial f(\bar{x})$, where $f(x) = f_1(x) + f_2(x)$. Assume further that for each $v_i \in \partial f_i(\bar{x})$ with $v_1 + v_2 = \bar{v}$, the function f_i is prox-regular at $\bar{x}$ for v_i. Then f is prox-regular at $\bar{x}$ for $\bar{v}$ and there exist $\epsilon > 0$ such that

$$[\partial f_1(x) + \partial f_2(x)] \cap B(\bar{v}, \epsilon) = \partial f(x) \cap B(\bar{v}, \epsilon) \tag{9}$$

whenever $|x - \bar{x}| < \epsilon$ with $|f(x) - f(\bar{x})| < \epsilon$.

Proof. Let $g(u_1, u_2) := f_1(u_1) + f_2(u_2)$ and $F(x) := (x, x)$. It is an easy exercise to verify that all conditions in the previous theorem are satisfied. This gives that f is prox-regular at $\bar{x}$ for $\bar{v}$. The fact that $\partial f(x) \cap B(\bar{v}, \epsilon) \subset [\partial f_1(x) + \partial f_2(x)] \cap B(\bar{v}, \epsilon)$ follows from the constraint qualification (8) and [21, Cor. 10.9].

Now suppose that the other inclusion in (9) is not verified. If so, there exist x_n converging to $\bar{x}$ with $f(x_n) \to f(\bar{x})$ and $v_n^i \in \partial f_i(x_n)$ with $v_n^1 + v_n^2$ converging to $\bar{v}$ and such that $v_n^1 + v_n^2$ is not a subgradient to f at x_n.

First note that we must have $f_i(x_n) \to f_i(\bar{x})$ for each i. This follows from the fact that each f_i is l.s.c. i.e. $\liminf_{x \to \bar{x}} f_i(x) \ge f_i(\bar{x})$ and that $f_1(x_n) + f_2(x_n) \to f_1(\bar{x}) + f_2(\bar{x})$.

It follows from (8) that for $i = 1, 2$, v_n^i can not be unbounded. To see this, assume that v_n^1 is unbounded and that (without loss of generality) $\|v_n^1\| \ge \|v_n^2\|$. By passing to a subsequence if necessary, we may assume that $v_n^i / \|v_n^1\| \to v_i$ for each i. Note that $\|v_1\| = 1$ and that $v_i \in \partial f_i^\infty(\bar{x})$ (recall that $f_i(x_n) \to f_i(\bar{x})$ and the comments on horizon subgradients made before Theorem 3.1). We then have $v_1 + v_2 = 0$ with $v_i \in \partial f_i^\infty(\bar{x})$ and $v_1 \ne 0$; this contradicts (8).

We may therefore assume that $v_n^i \to v_i$ with $v_i \in \partial f_i(\bar{x})$ and $v_1 + v_2 = \bar{v}$. From our assumptions f_i is prox-regular at $\bar{x}$ for v_i. Therefore for $i = 1, 2$, there exist $r_i > 0$ and $\epsilon_i > 0$ such that

$$f_i(y) \ge f_i(x) + \langle v, y - x \rangle - (r_i/2) |y - x|^2 \tag{10}$$

whenever $|y - \bar{x}| < \epsilon_i$, $|x - \bar{x}| < \epsilon_i$, $|f_i(x) - f_i(\bar{x})| < \epsilon_i$ while $|v - v_i| < \epsilon_i$ with $v \in \partial f_i(x)$. It therefore follows that eventually v_n^i is a proximal subgradient to f_i at

x_n. It is trivial to verify that the sum of proximal subgradients is a proximal subgradient of the sum. Hence $v_n^1 + v_n^2$ is a proximal subgradient to f at x_n. In particular, $v_n^1 + v_n^2$ is a subgradient to f at x_n. This contradicts our earlier assumption and completes the proof. □

The previous results can be applied to indicator functions to obtain rules for generating new prox-regular sets from known ones. The following corollaries are straightforward applications of the previous theorems.

Corollary 3.3. *Let $C = \{x \mid F(x) \in D\}$, where $F : I\!R^n \to I\!R^m$ is of class $\mathcal{C}^2$ and $D \subset I\!R^m$ is closed. Let $\bar{x} \in C$, and suppose that the following constraint qualification is satisfied:*

$$\text{there is no vector } y \neq 0 \text{ in } N_D(F(\bar{x})) \text{ with } \nabla F(\bar{x})^* y = 0.$$

Assume further that $\bar{v} \in N_C(\bar{x})$ is a vector such that the set D is prox-regular at $F(\bar{x})$ for every $y \in N_D(F(\bar{x}))$ with $\nabla F(\bar{x})^* y = \bar{v}$. Then C is prox-regular at $\bar{x}$ for $\bar{v}$.

Corollary 3.4. *Let C_i, $i = 1, 2$, be closed subsets of $I\!R^n$, and let $\bar{x} \in C = C_1 \cap C_2$. Suppose that*

$$\text{the only choice of } v_i \in N_{C_i}(\bar{x}) \text{ with } v_1 + v_2 = 0 \text{ is } v_1 = v_2 = 0.$$

Let $\bar{v} \in N_C(\bar{x})$, and assume that for each choice of $v_i \in N_{C_i}(\bar{x})$ with $v_1 + v_2 = \bar{v}$, the set C_i is prox-regular at $\bar{x}$ for v_i. Then C is prox-regular at $\bar{x}$ for $\bar{v}$ and there exist $\epsilon > 0$ such that

$$[N_{C_1}(\bar{x}) + N_{C_2}(\bar{x})] \cap B(\bar{v}, \epsilon) = N_C(\bar{x}) \cap B(\bar{v}, \epsilon)$$

whenever $|x - \bar{x}| < \epsilon$.

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