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Typical Convexity (Concavity) of Dini-Hadamard Upper (Lower) Directional Derivatives of Functions on Separable Banach Spaces

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Dedicated to Hedy Attouch on the occasion of his 60th birthday.

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1. Introduction

By "typical" we mean "valid outside a small (or negligible) set". There are various concepts of "smallness" used in analysis: measure theoretic (null sets of different kind), topological (sets of the first Baire category), metric (σ-porous sets or directionally σ-porous sets), analytic (countable unions of sets that can be represented as (subsets of) graphs of certain classes of Lipschitz functions). We refer to [2] for detailed discussions relating to many specific types of these sets in Banach spaces and comparisons of their levels of smallness.

Here we basically deal with two types of small sets associated with the two last types of smallness concepts, namely directionally σ-porous sets and so called sparse sets. These two classes of sets are among the smallest: directionally σ-porous sets are sets of the first Baire category and at the same time Aronszajn null (hence Haar null, hence sets of Lebesgue measure zero if the space is finite dimensional). In turn, sparse sets is a proper subclass of the class of directionally σ-porous sets.

Typical convexity-relating properties of (Fréchet, Gâteaux, Hadamard) derivatives and directional derivatives of functions on separable Banach spaces have been extensively studied in the literature and a number of interesting results have been obtained – see e.g [2]. Just to mention a few: a convex continuous function is Gâteaux differentiable outside of a set which is a countable union of so called c-c-hypesurfaces: that is sets that are subsets of graphs of DC-functions[1] (Zajiček [15] – an extension of a classical theorem of Mazur which already cannot be further strengthened in particular because in $I\!R$ it reduces to the function being non-differentiable at most countable set), the upper Dini and the Clarke directional derivatives of a Lipschitz function coincide on a residual set (Giles-Sciffer [7]), moreover the upper Dini directional derivative of a Lipschitz function is convex (in direction) outside of a directionally σ-porous set (this

[1]Recall that a DC-function is a difference of two convex continuous functions bounded on bounded sets.

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was stated in Bessis-Clarke [3] for the case of a Hilbert space and proved for finite dimensional situation; the general result was proved by Preiss-Zajiček [12] – actually it was proved there that the upper Dini directional derivative coincides with Michel-Penot directional derivative up to a directionally σ-porous subset of the domain[2]), for an arbitrary function the set of points at which the Dini-Hadamard subdifferential contains more than one point is sparse (Benoist, Loewen-Wang [1, 9]).

In this note we strengthen some of these and other results mainly in one (or more) of the following three respects: extension to smaller classes of small sets (e.g. from genericity to σ-porosity), extension to a broader class of functions (mainly from Lipschitz to continuous or even arbitrary) and extension to a broader class of spaces (e.g. from separable Hilbert to general separable spaces). The point we wish to specially emphasize is that the proofs are basically very elementary and in certain cases much simpler than the proofs of the original results. They all exploit variants of the same techniques that should be largely attributed to Preiss and Zajiček [11, 12]. In the next section we give all necessary definitions (upper and lower directional derivatives of different kind, derivatives, smallness concepts). The third section contains the statements of all results obtained in the paper and some most elementary proofs. And the last section contains the main proofs.

2. Preliminaries

Throughout the paper X is a Banach space which is assumed to be separable if nothing else is specified; f a function on X with values in $[-\infty,\infty]$; $\operatorname{dom} f = \{x \in X : |f(x)| < \infty\}$; $t \to +0$ means $t > 0$ and $t \to 0$; $u \rightharpoonup x$ means "u converges weakly to x". All limiting relations are *sequential.*

2.1. Derivatives.

Definition 2.1. Let $x \in \operatorname{dom} f$, $h \in X$. Then

$$f_D^-(x;h) = \liminf_{t\to+0} \frac{f(x+th)-f(x)}{t}$$

is the *lower Dini directional derivative* of f at x at the direction h;

$$f_H^-(x;h) = \liminf_{\substack{t\to+0\\ \|u\|\to 0}} \frac{f(x+t(h+u))-f(x)}{t}$$

is the *lower Dini-Hadamard* (or just *Hadamard*) *directional derivative* of f at x at the direction h;

$$f_{WH}^-(x;h) = \liminf_{\substack{t\to+0\\ u\rightharpoonup 0}} \frac{f(x+t(h+u))-f(x)}{t}$$

is the *weak lower Hadamard directional derivative* of f at x at the direction h.

Replacing lim inf by lim sup in the three equalities, we get definitions of the corresponding *upper directional derivatives* $f_D^+(x;h)$, $f_H^+(x;h)$, $f_{WH}^+(x;h)$. If for a certain

[2] I am thankful to L. Zajiček and the reviewer for bringing my attention to this result.

x the corresponding lower and upper directional derivatives coincide (as functions of h), we say that f is *directionally differentiable* at x in the corresponding sense (Dini, Dini-Hadamard, weak Hadamard), call the functions *directional derivatives* and denote them in a natural way by: $f'_D(x;h)$, $f'_H(x;h)$, $f'_{WH}(x;h)$.

The following elementary properties of lower directional derivatives are of substantial importance:

(a) Each lower directional derivative is a homogeneous function of h, that is $f^-_\bullet(x;\lambda h) = \lambda f^-_\bullet(x;h)$ if $\lambda > 0$; hence at zero it may assume only two values: zero and minus infinity.

(b) $f^-_D(x;\cdot) = f^-_H(x,\cdot)$ if f is Lipschitz near x. In this case $f^-_H(x,\cdot)$ is (globally) Lipschitz with constant equal to the Lipschitz constant of f near x.

(c) $f^-_H(x,\cdot) = f^-_{WH}(x,\cdot)$ if $\dim X < \infty$.

(d) The lower Dini-Hadamard directional derivative is lower semicontinuous as a function of direction h.

(e) If X has a separable dual then the weak lower Dini-Hadamard directional derivative is sequentially weak lower semicontinuous in h.

Note also that in the literature the expression "subderivative" is often used (instead of "lower directional derivative") etc.

Definition 2.2. Let $x \in \operatorname{dom} f$, $x^* \in X^*$. It is said that x^* is the *Gâteaux* derivative of f at x: $x^* = f'_G(x)$, if

$$\lim_{t\to 0} \frac{f(x+th) - f(x)}{t} = \langle x^*, h\rangle$$

for every $h \in X$; x^* is the *Hadamard derivative* of f at x: $x^* = f'_H(x)$, if

$$\lim_{\substack{t\to 0\\ \|u\|\to 0}} \frac{f(x+t(h+u)) - f(x)}{t} = \langle x^*, h\rangle$$

for every $h \in X$; x^* is the *Fréchet* derivative of f at x: $x^* = f'_F(x)$, if

$$\lim_{\|h\|\to 0} \frac{f(x+th) - f(x) - \langle x^*, h\rangle}{\|h\|} = 0.$$

If one or another derivative exists, we say that f is *differentiable* at x in the corresponding sense.

It is clear that a Fréchet derivative is also a Hadamard derivative and the latter is a Gâteaux derivative. It is also a trivial matter to see that either derivative, if exists, is uniquely defined. Therefore it is often reasonable to drop subscripts and write $f'(x)$.

We next observe that the Dini and Dini-Hadamard directional derivatives are connected with the Gâteaux and Hadamard derivatives in an obvious way: the first (if exist) reduce to the latter when they are continuous linear functions of h. The same connection exists between the weak Hadamard directional derivative and Fréchet derivative in separable reflexive spaces.[3]

[3]In the paper we do not work with Fréchet derivatives – we just mention its connection with the weak lower Dini-Hadamard directional derivative.

Finally we shall need two well known kinds of convex directional derivatives.

Definition 2.3. Let f be a lower semicontinuous function and $x \in \mathrm{dom}\, f$. The quantity

$$f^{\uparrow}(x; h) = \sup_{\delta>0} \limsup_{\substack{t\to+0 \\ x'\to_f x}} \inf_{\|h'-h\|<\delta} \frac{f(x'+th') - f(x')}{t}$$

is called the *Clarke-Rockafellar directional derivative* of f at x along h. (As usual, $x' \to_f x$ means that $x' \to x$ and $f(x') \to f(x)$.)

Assume now that f is Lipschitz in a neighborhood of x. Then

$$f^{\diamond}(x; h) = \sup_{z} \limsup_{t\to+0} \frac{f(x + t(h + z)) - f(x + tz)}{t}$$

is the *Michel-Penot directional derivative* of f at x along h.

Both are usually considered function of direction h with x being a parameter. We need the following properties of the functions (under the made assumptions on the latter).

(f) Both $f^{\uparrow}(x; \cdot)$ and $f^{\diamond}(x; \cdot)$ are proper sublinear functions, the first lower semicontinuous and the second globally Lipschitz with the Lipschitz constant equal to the Lipschitz constant of f at x.

(g) If f is Lipschitz near x, then $f^{\uparrow}$ reduces to

$$f^{\circ}(x; h) = \limsup_{\substack{t\to+0 \\ x'\to x}} \frac{f(x'+th) - f(x')}{t}.$$

(h) A lower semicontinuous f is strictly Hadamard differentiable at x if and only if $f^{\uparrow}(x; \cdot)$ is a linear continuous function; a Lipschitz f is Gâteaux differentiable at x if and only if $f^{\diamond}(x; \cdot)$ is a linear function.

For more details concerning the two types of directional derivatives see [13, 10].

Strict Hadamard differentiability at x means that (f is Hadamard differentiable at x and) for any h

$$t^{-1}(f(x'+th') - f(x') - f'(x)h') \to 0, \quad \text{when } t \to +0,\ x' \to x,\ h' \to h.$$

2.2. Small sets.

Here we define the concepts of smallness that will be used below. For more information and details see [2, 17]. Let X be a Banach space and $S \subset X$. It is usually said that (the closure of) S *is nowhere dense* if for every $x \in S$ and every sufficiently small $\delta > 0$ there are $x' \in B(x, \delta)$ and a $\delta' > 0$ such that

$$B(x', \delta') \cap S = \emptyset. \tag{$*$}$$

Definition 2.4. If, moreover for any $x \in S$ it is possible to find a $r > 0$ and a sequence of pairs $(t_n, x_n) \to (+0, x)$ such that, for all n, $(*)$ holds with $x' = x_n$ and $\delta' = r\|x - x_n\|$, we say that S is *porous*. (The concept of porosity was first introduced in [5].)

If furthermore for any $x \in S$ there are a $u \in X$ with $\|u\| = 1$, an $r > 0$ and a sequence $t_n \to +0$, such that $(*)$ holds with $x' = x + t_n u$ and $\delta' = rt_n$, then S is said to be *directionally porous*.

Recall that a set of the *first Baire category* is a subset of a countable union of closed nowhere dense sets. A set is *σ-porous* if it is a union of a countable family of porous sets. The set is *directionally σ-porous* if it is a countable union of directionally porous sets.

It is immediate from definitions that

(j) Every σ-porous set is a set of the first Baire category and every directionally σ-porous set is σ-porous.

(k) A σ-porous set in $\mathbb{R}^n$ has n-dimensional Lebesgue measure zero.

The latter is immediate from the Lebesgue density theorem. However not every set which is both measure negligible and of the first category is σ-porous [14].

A σ-porous set may not be directionally σ-porous. The main difference between the two classes is that directionally σ-porous sets have much stronger measure-theoretic smallness property (see [2]):

(l) A directionally σ-porous set in a separable Banach space is Aronszajn null.

On the other hand, even in a Hilbert space there exist an Aronszajn null set whose complement is σ-porous ([2], Theorem 4.19 and Example 6.46).

Definition 2.5 ([4, 16]). A set $S \subset X$ is a *Lipschitz hypersurface* if there is a subspace $L \subset X$ of codimension one, vector $h \notin L$ and a Lipschtz function f defined on an open subset of L such that S coincides with the image of Graph f under the mapping $(x, \alpha) \mapsto x + \alpha h$ from $L \times \mathbb{R}$ into X. A set S is called *sparse* if it can be covered by a countable family of Lipschitz hypersurfaces.

It is clear that

(m) A hypersurface in $\mathbb{R}$ is a point, so a sparse set in $\mathbb{R}$ is at most countable.

It is also an easy matter to see that

(n) A Lipschitz hypersurface is directionally porous, so a sparse set is directionally σ-porous.

Indeed, let $L \subset X$ be a subspace of codimension one, let $h \notin L$, let f be a Lipschitz function on an open subset V of L and let $S = \{x = u + f(u)h : \ u \in V\}$. Now if K is strictly greater than the Lipschitz constant of f, then for an $x \in S$ the ball $B(x + tKh, t)$ cannot meet S for any $t > 0$.[4]

We shall see (in the proof of the second part of Theorem 3.1) that the converse is also true: S is a Lipschitz hypersurface if there are a u with $\|u\| = 1$ and an $r > 0$ such that for any $x \in S$ and any $t \neq 0$ $(*)$ holds with $x' = x + tu$ and $\delta' = rt$.

[4]Indeed, if $x' = u' + f(u')h$ belongs to the ball for some t, then $u' - u + (f(u') - f(u))h = tKh + \xi$ for some ξ with $\|\xi\| \le t$, then, as $h \notin L$ we have $(f(u') - f(u)) = tK$ and $u' - u = \xi$, that is $|f(u') - f(u)| = tK \ge K\|u' - u\| > |f(u') - f(u)|$.

3. Results

Theorem 3.1. *Let X be a separable Banach space. Then for any function f on X the following two statements hold true.*

(*a*) *The set of $x \in \operatorname{dom} f$ for which the inequality*

$$f_H^-(x; h_1) + f_H^-(x; h_2) > f_H^-(x; h_1 + h_2) \tag{1}$$

holds for some $h_1,\ h_2 \in X$ is directionally σ-porous. Thus for all x outside of a directionally σ-porous subset of $\operatorname{dom} f$ *either $f_H^-(x;\cdot)$ is an continuous superlinear function or $|f_H^-(x;h)| \equiv \infty$.*

(*b*) *The set of $x \in \operatorname{dom} f$ for which the inequality*

$$f_H^-(x; h) + f_H^-(x; -h) > 0 \tag{2}$$

holds for some $h \in X$ is sparse; in particular such is the set of $x \in \operatorname{dom} f$ for which the restriction of $f_H^-(x;\cdot)$ to a certain subspace of X is convex and not linear.

For locally Lipschitz functions on Hilbert spaces the first statement was proved in [3] (Theorems 3.1 and 4.1). The second statement for locally Lipschitz function was proved in [16] (with a reference to [8] as source of the idea for the proof).

Remark. The theorem can easily be reformulated for upper Dini-Hadamard directional derivatives. The only we need is to change the signs of all inequalities to the opposite.

Corollary 3.2 ([12]). *If f is locally Lipschitz function defined on an open set U, then it is intermediately Gâteaux differentiable at every point of its domain except at most for a directionally σ-porous set. This means that for all $x \in \operatorname{dom} f$ outside of a directionally σ-porous set there is an $x^* \in X^*$ such that*

$$f_D^+(x; h) \geq \langle x^*, h\rangle \geq f_D^-(x; h), \quad \forall\, h \in X.$$

Proof. As for a Lipschitz function the Dini and Hadamard directional derivatives coincide, the theorem says that $f_D^+(x;\cdot)$ is convex and $f_D^-(x;\cdot)$ is concave for all x outside of a directionally σ-porous subset of U. As f is Locally Lipschitz, both $f_D^-(x,\cdot)$ and $f_D^+(x,\cdot)$ are Lipschitz and $f_D^+(x;h) \geq f_D^-(x;h)$ for all h. So it remains to separate the graphs of the functions. □

The corollary is a strengthening of the separable version of a theorem by Fabian-Preiss [6] in which generic intermediate Gâteaux differentiability was proved for locally Lipschitz functions on Asplund generated spaces. [7] contains a simple proof of the Fabian-Preiss theorem for separable spaces. Later in [3] it was shown that for a locally Lipschitz functions on a Hilbert space the property holds outside of a directionally σ-porous, not just first category, subset of the domain of the function.

Corollary 3.3 ([9]). *For any function f on X the collection of x such that $\partial_H f(x)$ contains more than one element is a sparse set.*

Proof. If $x_1^* \neq x_2^*$ and both belong to $\partial_H f(x)$, then for any h for which, say $\langle x_1^* - x_2^*, h\rangle > 0$ we have

$$f_H^-(x;h) + f_H^-(x,-h) \geq \langle x_1^*, h\rangle + \langle x_2^*, -h\rangle > 0. \qquad \square$$

If f is convex continuous on a convex open set, then $\partial f(x) \neq \emptyset$ for all $x \in \operatorname{dom} f$. On the other hand, if $\partial f(x)$ is a singleton, the function is Gâteaux differentiable. Thus we arrive to the following well known result extending the classical Mazur theorem about generic Gâteaux differentiability of convex continuous functions on separable spaces.

Corollary 3.4. *Let f be a convex function defined and continuous on a convex open set U. Then f is Gâteaux differentiable at x for all $x \in \operatorname{dom} f$ outside of a sparse subset of U.*

Proof. For a convex continuous functions f we have $f'(x;h) = f_H^-(x;h)$ for all $x \in U$ and $h \in X$. The function $f'(x;\cdot)$ is convex continuous and the condition $f'(x;h) + f'(x;-h) \leq 0$ for all h implies that it is linear, hence the Gâteaux derivative of f at x. $\square$

In fact the set of points of Gâteaux non-differentiability of a convex continuous function on a separable Banach space admits a more precise description ([15], see also [2], Theorem 4.20): it is a countable union of so called c-c-hypersurfaces which are graphs of locally Lipschitz functions representable as differences of convex continuous functions bounded on bounded sets. We note that there are sparse sets which are not countable unions of c-c-hypersurfaces [16].

Theorem 3.5. *Let X be a Banach space with separable dual. Then for any function f on X the set of $x \in \operatorname{dom} f$ for which the inequality*

$$f_{WH}^-(x;h_1) + f_{WH}^-(x;h_2) > f_{WH}^-(x;h_1+h_2) \qquad (1_{WH})$$

holds for some $h_1,\ h_2 \in X$ is directionally σ-porous. Thus for all x outside of a directionally σ-porous subset of $\operatorname{dom} f$ either $f_H^-(x;\cdot)$ is a continuous superlinear function or $|f_{WH}^-(x;h)| \equiv \infty$.

The theorem is a generalization of the second main result of [3] (Theorem 3.2) in which locally Lipschitz functions on a separable Hilbert space were considered and the inequality was proved for h_2 taken from a certain finite dimensional subspace and h_1 from its complement. Observe also that the extension of the second part of Theorem 3.1 to weak lower Hadamard directional derivatives is trivial as $f_{WH}^-(x;h) \leq f_H^-(x;h)$ for all h.

Theorem 3.6. *Let $U \subset X$ be open, and let f be a continuous function defined on U. Then*

$$f_H^+(x;\cdot) = f^\uparrow(x;\cdot) \qquad (3)$$

for all $x \in U$ of a residual subset of U. If furthermore, f is locally Lipschitz on U, then

$$f_H^+(x;\cdot) = f^\diamond(x;\cdot) \qquad (4)$$

outside of a directionally σ-porous subset of U.

Recall that the main result of [7] is that (3) holds for a locally Lipschitz function on a residual subset of U. Thus the first part of the theorem extends this result by showing that the equality of the upper Dini-Hadamard and Clarke-Rockafellar super-derivatives is a generic phenomenon even if the function is only continuous, not necessarily Lipschitz. The second part of the theorem was established in [12]. We state and prove it mainly to emphasize the connection of the two results and to show that the proof of the second needs only a slight revision of the proof of the first. It is to be further noted that in the first statement the passage from residual to a complement of a directionally σ-porous set is impossible in principle: consider e.g. a Lipschitz function f on $[0,1]$ whose derivative is a.e. either 1 or -1 and the sets on which the derivative is equal to either of the numbers are dense. Then the Clarke directional derivative $f^{\uparrow}(x;h)$ (actually coinciding with better known Clarke's directional derivative f°) is equal to $|h|$ for all x.

The concluding result deals with Hadamard differentiability.

Theorem 3.7. *Let again f be a function on a separable Banach space X and assume that for any $x \in \operatorname{dom} f$ there is a set $S_x \subset X$ such that the Dini-Hadamard directional derivative*

$$f'_H(x;h) = \lim_{\substack{t \to +0 \\ h' \to h}} \frac{f(x+th') - f(x)}{t}$$

exists and finite for all $x \in \operatorname{dom} f$ and all $h \in S_x$.

(a) *If $S_x \equiv X$, then f is Hadamard differentiable at every $x \in \operatorname{dom} f$ outside of a sparse subset of* $\operatorname{dom} f$.

(b) *If the linear span of every S_x coincides with X, then f is Hadamard differentiable at every $x \in \operatorname{dom} f$ outside of a directionally σ-porous subset of* $\operatorname{dom} f$.

(c) *The conclusion of* (b) *is valid also if f is defined and locally Lipschitz on an open set and the linear spans of S_x are dense in X.*

The last statement was proved in [12]. We again note that under the assumptions of (c) we can speak about Gâteaux differentiability. Observe further that under the assumption of part (a) of Theorem 3.7, f is a continuous function on an open set U, then f is locally Lipschitz on a dense open subset of U. This follows from Proposition 6.45 of [2]. We note also that for a locally Lipschitz function the existence of directional derivative at every point of the domain implies generic Gâteaux differentiability for function on Asplund generated spaces [6], not just on separable Banach spaces.

Finally, combining Theorems 3.6 and 3.7, we get

Corollary 3.8. *If f is a continuous function on an open set U and the Dini-Hadamard directional derivative exists for all $x \in U$, then f is strictly Hadamard differentiable on a residual subset of U.*

4. Proofs.

In what follows we denote by Q the set of rational numbers, by $\mathcal{N}$ the set of natural numbers and by Q_+ the set of positive rationals.

Proof of Theorem 3.1. (a) Clearly (1) is tantamount to the existence of rational α_1 and α_2 such that

$$f_H^-(x;h_i) > \alpha_i, \quad i=1,2; \qquad f_H^-(x;h_1+h_2) < \alpha_1+\alpha_2. \tag{5}$$

The first inequality in (5) is equivalent to the existence of an $m \in \mathcal{N}$ such that

$$f(x+t(h_i+u)) - f(x) > t\alpha_i$$

for all t and u satisfying $0 \le t \le 1/m$ and $\|u\| \le 2/m$ respectively.

Let $\tilde{X}$ be a dense countable subset of X. Take $\tilde{h}_i \in \tilde{X}$ with $\|h_i - \tilde{h}_i\| < 1/m$ and $\|h_1+h_2-(\tilde{h}_1+\tilde{h}_2)\| \le 1/3m$. Then

$$f(x+t(\tilde{h}_i+u)) - f(x) > t\alpha_i, \quad \text{if } t \in [0,1/m], \ \|u\| \le 1/m. \tag{6}$$

On the other hand, the second inequality in (5) means that there are sequences of $t_n \to +0$ and $u_n \in X$ with $\|u_n\| \to 0$ as $n \to \infty$ such that

$$f(x+t_n(h_1+h_2+u_n)) - f(x) < t_n(\alpha_1+\alpha_2). \tag{7}$$

We may assume that $\|u_n\| \le 1/6m$. So setting $v_n = (h_1+h_2+u_n) - (\tilde{h}_1+\tilde{h}_2)$, we have $\|v_n\| \le (1/2m)$ and

$$f(x+t_n(\tilde{h}_1+\tilde{h}_2+v_n)) - f(x) < t_n(\alpha_1+\alpha_2). \tag{8}$$

Denote by S the collection of x for which (1) holds with some $h_1,\ h_2 \in X$, and let $S(\tilde{h}_1,\tilde{h}_2,\alpha_1,\alpha_2,m)$ stand for the collection of x for which (6) and (8) hold. The above discussion shows that S is contained in the union of $S(\tilde{h}_1,\tilde{h}_2,\alpha_1,\alpha_2,m)$ over all $(\tilde{h}_1,\tilde{h}_2,\alpha_1,\alpha_2,m) \in \tilde{X}^2 \times Q^2 \times \mathcal{N}$. We claim that $S(\tilde{h}_1,\tilde{h}_2,\alpha_1,\alpha_2,m)$ is a directionally porous set.

So fix a $(\tilde{h}_1,\tilde{h}_2,\alpha_1,\alpha_2,m) \in \tilde{X}^2 \times Q^2 \times \mathcal{N}$ and let $x \in S(\tilde{h}_1,\tilde{h}_2,\alpha_1,\alpha_2,m)$. Let $t_n \to +0$, and v_n with $\|v_n\| \le 1/2m$ be such that (6) holds for all n. Take a u with $\|u\| \le 1/2m$. We have using (6) and (8) (and taking into account that $\|v_n - u\| \le 1/m$)

$$\begin{aligned} & f(x+t_n(\tilde{h}_1+u)+t_n(\tilde{h}_2+v_n-u)) - f(x+t_n(\tilde{h}_1+u)) \\ &= f(x+t_n(\tilde{h}_1+\tilde{h}_2+v_n)) - f(x+t_n(\tilde{h}_1+u)) \\ &< t_n(\alpha_1+\alpha_2) + f(x) - f(x+t_n(\tilde{h}_1+u)) \le t_n\alpha_2. \end{aligned} \tag{9}$$

This means that $x+t_n(\tilde{h}_1+u) \notin S(\tilde{h}_1,\tilde{h}_2,\alpha_1,\alpha_2,m)$ if $\|u\| \le 1/2m$. In other words, $(x+t_n(\tilde{h}_1+(1/2m)B) \cap S(\tilde{h}_1,\tilde{h}_2,\alpha_1,\alpha_2,m) = \emptyset$. As $t_n \to 0$ as $n \to \infty$ this shows that $S(\tilde{h}_1,\tilde{h}_2,\alpha_1,\alpha_2,m)$ is a directionally porous set as claimed.

To complete the proof of the first statement we note that in case when $f_H^-(x;\cdot)$ is concave and not identical minus infinity, we necessarily have $f_H^-(x;0) = 0$ and as $f_H^-(x;\cdot)$ is lower semicontinuous, it is bounded below in a neighborhood of zero. Since it is concave, it is continuous at zero and since it is homogeneous it is continuous on X.

(b) Let P be the set of x for which (2) holds with some h. Here by the way, we only can consider vectors h with norm one, due to positive homogeneity of $f_H^-(x;\cdot)$. As $f_H^-(x;\cdot)$ is lower semicontinuous, (2) must hold also for some $h \in \tilde{S}$, where $\tilde{S}$ is a dense countable subset of the unit sphere. On the other hand, (2) is equivalent to the existence of $\alpha_1,\ \alpha_2 \in Q$ such that $f_H^-(x;h) > \alpha_1,\ f_H^-(x;-h) > \alpha_2$ and $\alpha_1 + \alpha_2 > 0$. We have $\alpha_1 = \alpha + \beta,\ \alpha_2 = -\alpha + \beta$, where $\beta = (\alpha_1 + \alpha_2)/2 > 0$. By definition of the Dini-Hadamard subderivative, it follows that there is an $m \in \mathcal{N}$ such that

$$f(x + t(h + u)) - f(x) > t\alpha + |t|\beta, \quad \text{if } 0 < |t| < 1/m,\ \|u\| < 1/m. \tag{10}$$

Thus, if we denote by $P(h,\alpha,\beta,m)$ the set of x for which (10) holds, then P is a subset of the union of $P(h,\alpha,\beta,m)$ when (h,α,β,m) runs through $\tilde{S} \times Q \times Q_+ \times \mathcal{N}$. As the latter is a countable set, the theorem will be proved if we show that every $P(h,\alpha,\beta,m)$ can be covered by a countable family of Lipschitz surfaces. This fact was actually proved in [9]. We give below a somewhat simpler and shorter proof, partly for the purpose of completeness.

Let $x \in P(h,\alpha,\beta,m)$. Take a $t \in [0, 1/m]$ and a u with $\|u\| \le 1/m$ and set $x' = x + t(h + u)$ We have by (10)

$$f(x' - t(h + u)) - f(x') = f(x) - f(x + t(h + u)) < -t\alpha - \beta|t|,$$

so that $x' \notin P(h,\alpha,\beta,m)$ if $t \ne 0$. In other words

$$\left(x + t\left(h + \frac{1}{m}B\right)\right) \cap P(h,\alpha,\beta,m) = \emptyset, \quad \text{if } 0 < |t| \le \frac{1}{m}. \tag{11}$$

Take a nonzero x^* such that $\langle x^*, h\rangle > 0$ and let $L = \operatorname{Ker} x^*$. Then $X = L \oplus I\!Rh$, so that every $u \in X$ has a unique representation $u = w + \lambda h$ with $w \in L,\ \lambda \in I\!R$. For any $n = 0, \pm1, \pm2, \ldots$ let $P_n(h,\alpha,\beta,m)$ be the collection of all $x = w + \lambda h \in P(h,\alpha,\beta,m)$ such that $(n/m) \le \lambda \le ((n+1)/m)$. Then $P(h,\alpha,\beta,m)$ is the union of all $P_n(h,\alpha,\beta,m)$. So it is enough to show that every $P_n(h,\alpha,\beta,m)$ can be covered by a Lipschitz hypersurface.

Take a $k > 0$ such that $\|x\| \le k \max\{\|w\|, |\lambda|\}$ whenever $x = w + \lambda h$. Denote by $B_L = L \cap B$ the unit ball in L. Then, with a slight abuse of notation, we can rewrite (11) as

$$\left((w,\lambda) + t\left((0,1) + \frac{1}{km}B_L \times [-1,1]\right)\right) \cap P(h,\alpha,\beta,m) = \emptyset, \quad \text{if } 0 < |t| \le \frac{1}{m}, \tag{12}$$

provided $x = w + \lambda h \in P(h,\alpha,\beta,m)$. It follows that for any w there maybe at most one λ such that $x = w + \lambda h \in P_n(h,\alpha,\beta,m)$. This means that $P_n(h,\alpha,\beta,m)$ can be viewed as the graph of a certain function $\varphi_n(w)$ satisfying $(n/m) \le \varphi_n(w) \le (n+1)/m$ for all points of its domain. We claim that this function is Lipschitz on its domain.

Indeed, if w and w' belong to $\operatorname{dom} \varphi$, then $|\varphi_n(w) - \varphi_n(w')| \le 1/m$. So if $\|w - w'\| \ge (1/km^2)$, then we have

$$|\varphi_n(w) - \varphi_n(w')| \le km\|w - w'\|. \tag{13}$$

Suppose now that $\|w - w'\| < (1/km^2)$. By (12)

$$(w', \varphi_n(w')) \notin (w, \varphi_n(w)) + t\left(\{0\} \times [0,1] + \frac{1}{km} B_L \times \{0\}\right) \quad \text{if } 0 < t \leq \frac{1}{m}.$$

But now $w' \in w + t(1/km)B_L$ if $t \geq \tau = km\|w - w'\|$. Therefore $\varphi_n(w') \notin \varphi_n(w) + [\tau, 1/m]$. On the other hand,

$$\varphi_n(w') \leq \frac{n+1}{m} \leq \varphi_n(w) + \frac{1}{m},$$

we must conclude that $\varphi_n(w') \leq \varphi_n(w) + \tau = \varphi_n(w) + km\|w - w'\|$. Applying the same argument with $t \leq 0$ and $\tau = -km\|w - w'\|$, we get $\varphi_n(w') \geq \varphi_n(w) - km\|w - w'\|$, that is (13) holds for all $w,\ w' \in \operatorname{dom} \varphi_n$. This proves the claim and the theorem.

Proof of Theorem 3.5. It is similar to the proof of the first part of Theorem 3.1. Indeed, let $x_1^*, x_2^*, \ldots$ be a dense countable subset of the unit sphere in X^*. Then $f_{WH}^-(x;h) > \alpha$ if and only if there is an $\alpha' > \alpha$ such that for any $N \in \mathcal{N}$ there is an $m \in \mathcal{N}$ such that $f(x + t(h+u)) - f(x) > t\alpha'$, whenever $0 \leq t \leq 1/m$, $\|u\| \leq N$ and $|\langle x_i^*, u\rangle| \leq 1/m,\ i = 1, \ldots, m$. Indeed, let $\mathcal{W}_N$ be the collection of sequences (t_n, v_n) such that $t_n \to +0$, $\|v_n\| \leq N$ and (v_n) converges weakly to zero. Then

$$f_{WH}^-(x;h) = \inf_{N \in \mathcal{N}} \inf_{(t_n, v_n) \in \mathcal{W}_N} \liminf_{n \to \infty} \frac{f(x + t_n(h + v_n)) - f(x)}{t_n}. \tag{14}$$

Suppose now that $f_{WH}^-(x;h) > \alpha' > \alpha$. Then there are N, $\delta > 0$ and a weak neighborhood V of zero such that $t^{-1}(f(x + t(h+u)) - f(x) > \alpha'$ for all $t \in [0, \delta]$ and $u \in V$ with $\|u\| \leq N$ and it remains to choose m to guarantee that $m\delta > 1$ and V contains $(1/m)B$ plus the annihilator of $\{x_1^*, \ldots, x_m^*\}$. The opposite implication is of course immediate from the definition.

On the other hand, if $f_{WH}^-(x;h) < \alpha$ then by (14) there is an $N \in \mathcal{N}$ and a sequence $(t_n, u_n) \in \mathcal{W}_N$ such that $t_n^{-1}(f(x + t_n(h + u_n)) - f(x)) < \alpha$ for all n.

Suppose now that there are h_i and $\alpha_i,\ i = 1,2$ such that $f_{WH}^-(x;h_i) > \alpha_i$ and $f_{WH}^-(x;h_1 + h_2) < \alpha_1 + \alpha_2$. The latter, as we have just seen, implies the existence of a sequence $(t_n, u_n) \in \mathcal{W}_N$ such that (7) holds for all n. Let $N \in \mathcal{N}$ satisfies $N > \sup\|u_n\|$. Choose $m \in \mathcal{N}$ so big that $f(x + t(h_i + u)) - f(x) > t\alpha_i$ whenever $0 \leq t \leq 1/m$, $\|u\| \leq N + 1/m$ and $|\langle x_i^*, u\rangle| \leq 2/m$.

Let further, as in the proof of Theorem 3.1, $\tilde{X}$ be a dense countable subset of X. Since that moment the proof can follow the proof of the first part of Theorem 3.1 word for word. We take $\tilde{h}_i \in \tilde{X}$ satisfying the same relations and shall see afterwords that for u with $\|u\| \leq 1/2m$ the calculation of (9) remains valid in our case which leads to the proof of the first statement.

Proof of Theorem 3.6. The Clarke-Rockafellar directional derivative cannot be smaller than the Dini-Hadamard superderivative. So to prove the first statement we have to show that the collection of $x \in U$ at which $f^{\uparrow}(x;h) > f_H^+(x;h)$ for some h is the set of first Baire category.

Let R be the set of x such that $f^{\uparrow}(x;h) > f^{+}_{H}(x;h))$ for some h. Let $x \in R$. Then there is an $\alpha \in I\!R$ and a $\gamma > 0$ such that $f^{\uparrow}(x;h) > \gamma + \alpha$ and $f^{+}_{H}(x;h) < \alpha$. The latter means that there is a $\delta > 0$ such that

$$f(x + t(h+u)) - f(x) < t\alpha, \quad \text{if } 0 < t < \delta,\ \|u\| < \delta, \tag{15}$$

while the first inequality means that there is a $\delta > 0$ and a sequence (t_n, x_n) such that $t_n \to +0$, $x_n \to x$, $f(x_n) \to f(x)$ and

$$f(x_n + t_n(h+u)) - f(x_n) > t_n(\gamma + \alpha), \quad \text{if } \|u\| < \delta,\ \forall\, n \in \mathcal{N}. \tag{16}$$

Let $R(h,\alpha,\gamma,\delta)$ be the set of all x for which (15), (16) holds (the latter for some sequences of $t_n \to +0$ and $x_n \to x$). Then R is the union of $R(h,\alpha,\gamma,\delta)$ over all $(h,\alpha,\gamma,\delta) \in \tilde{X} \times Q \times Q_+ \times Q_+$ and we have to verify that every $R(h,\alpha,\gamma,\delta)$ is a set of the first Baire category.

So let $x \in R(h,\alpha,\gamma,\delta)$. By continuity, we can find for any n a positive $\rho_n < \delta$ such that $|f(x_n + t_n u) - f(x_n)| < t_n\gamma$ if $\|u\| \le \rho_n$. Then (16) implies that for large n for which $t_n < \delta$

$$f((x_n + t_n u) + t_n h) - f(x_n + t_n u) > t_n\alpha$$

if $\|u\| \le \rho_n$, that is $x_n + t_n\rho_n B$ does not meet $R(h,\alpha,\gamma,\delta)$. This means that every neighborhood of x contains "bubbles" not belonging to $R(h,\alpha,\gamma,\delta)$ and therefore the closure of $R(h,\alpha,\gamma,\delta)$ is nowhere dense, hence a set of the first Baire category. This completes the proof of the first part of the theorem.

The proof of the second needs a minor modification. First, as we talk about the Michel-Penot directional derivative, we can choose x_n of the form $x_n = x + t_n z$ for some $z \in X$. On the other hand, as the function is Lipschitz near x, we can take $\rho_n \equiv \rho < \gamma/2K$, where K is the Lipschitz constant of f. Then the radius of the "bubble" not meeting $R(h,\alpha,\gamma,\delta)$ remains proportional to the radius of the neighborhood of x and we can conclude that $R(h,\alpha,\gamma,\delta)$ is a directionally porous set.

Proof of Theorem 3.7. (a) We have to verify that the set of $x \in \operatorname{dom} f$ such that for some $h_1,\ h_2$ either

$$f'_H(x;h_1) + f'_H(x;h_2) > f'_H(x;h_1+h_2) \tag{17}$$

or

$$f'_H(x;h_1) + f'_H(x;h_2) < f'_H(x;h_1+h_2) \tag{18}$$

is sparse. Clearly, it is enough to consider the set of x for which (17) holds with some $h_1,\ h_2$.

Arguing as in the proof of the first part of Theorem 3.1, we conclude that (17) is equivalent to the existence of $\tilde{h}_i \in \tilde{X}$, $\alpha_i \in Q$ $(i = 1,2)$ and an $m \in \mathcal{N}$ such that

$$f(x + t(\tilde{h}_i + u)) - f(x) > |t|\alpha_i; \qquad f(x + t(\tilde{h}_1 + \tilde{h}_2 + u)) - f(x) < |t|(\alpha_1 + \alpha_2) \tag{19}$$

if $0 < |t| \le 1/m$, $\|u\| \le 1/m$.

Denote by $R(\tilde{h}_1, \tilde{h}_2, \alpha_1, \alpha_2, m)$ the collection of x such that (19) holds. Then

$$f((x + t(\tilde{h}_1 + u)) + t(\tilde{h}_2 + v - u)) - f(x + t(\tilde{h}_1 + u)) < |t|\alpha_2$$

if $0 < |t| \leq 1/m$, $\|u\|, \|v\| \leq 1/2m$ which shows that $x + t(\tilde{h}_1 + u) \notin R(\tilde{h}_1, \tilde{h}_2, \alpha_1, \alpha_2, m)$ for such t and u, that is

$$x + t\left(\tilde{h}_1 + \frac{1}{m}B\right) \bigcap R(\tilde{h}_1, \tilde{h}_2, \alpha_1, \alpha_2, m) = \emptyset$$

if $0 < |t| \leq 1/m$. This is precisely the same as (11) with $P(h, \alpha, \beta, m)$ replaced by $R(\tilde{h}_1, \tilde{h}_2, \alpha_1, \alpha_2, m)$. From this point the proof repeats word for word the corresponding part of the proof of the second statement of Theorem 3.1.

(b) It follows from Theorem 3.1(a) that there is a directionally σ-porous set $D \subset \operatorname{dom} f$ such that for all $x \in \operatorname{dom} f \setminus D$ and all $h_1,\ h_2 \in S_x$ we have

$$f_H^+(x, h_1 + h_2) \leq f'_H(x; h_1) + f'_h(x; h_2) \leq f_H^-(x, h_1 + h_2).$$

This shows that $f'_H(x; h_1 + h_2)$ exists and is equal to $f'_H(x; h_1) + f'_H(x; h_2)$. Thus $f'_H(x; \cdot)$ is defined and linear on the span of S_x which is X.

(c) As f is Lipschitz near x,

$$|f_H^{\pm}(x; h) - f'_H(x, h')| \leq K\|h - h'|$$

for any $h \in X$ if f is Hadamard directionally differentiable along h'. Here K is the Lipschitz constant of f. On the other hand, the same argument as above shows that $f'_H(x; \cdot)$ is defined and linear on the span of S_x. Combining this two facts, we conclude that $H'(x; \cdot)$ is defined and linear on the closure of the span of S_x. □

References

[1] J. Benoist: The size of the Dini subdifferential, Proc. Amer. Math. Soc 129 (2001) 525–530.

[2] Y. Benyamini, J. Lindenstrauss: Geometric Nonlinear Functional Analysis. Volume 1, AMS, Providence (2000).

[3] D. N. Bessis, F. H. Clarke: Partial subdifferentials, derivatives and Rademacher's theorem, Trans. Amer. Math. Soc. 351 (1999) 2899–2926.

[4] H. Blumberg: Exceptional sets, Fundam. Math. 32 (1939) 3–32.

[5] E. P. Dolzhenko: Boundary properties of arbitrary functions, Izv. Akad. Nauk SSSR, Ser. Mat. 31 (1967) 3–14 (in Russian); Math. USSR, Izv. 1 (1967) 1–12 (in English).

[6] M. Fabian, D. Preiss: On intermediate differentiability of Lipschitz functions on certain Banach spaces, Proc. Amer. Math. Soc. 113 (1991) 733–740.

[7] J. R. Giles, S. Sciffer: Locally Lipschitz functions are generically pseudo-regular on separable Banach spaces, Bull. Aust. Math. Soc. 47 (1993) 205–212.

[8] S. V. Konyagin: Approximation properties of arbitrary sets in Banach spaces, Dokl. Akad. Nauk SSSR 239 (1978) 261–264 (in Russian); Sov. Math., Dokl. 19 (1978) 309–312 (in English).

[9] P. D. Loewen, X. Wang: On the multiplicity of Dini subgradients in separable spaces, Nonlinear Anal. 58 (2004) 1–10.

[10] P. Michel, J.-P. Penot: Calcul sous-différentiel pour des fonctions lipschitziennes et non-lipschitziennes, C. R. Acad. Sci., Paris, Sér. I 298 (1984) 684–687.

[11] D. Preiss, L. Zajíček: Sigma-porous sets in products of metric spaces and sigma-directionally porous sets in Banach spaces, Real Anal. Exch. 24 (1998) 295–314.

[12] D. Preiss, L. Zajíček: Directional derivatives of Lipschitz functions, Isr. J. Math. 125 (2001) 1–27.

[13] R. T. Rockafellar: The Theory of Subgradients and its Application to Optimization: Convex and Nonconvex Case, Heldermann, Berlin (1981).

[14] L. Zajíček: Sets of σ-porosity and sets of σ-porosity(q), Čas. Pešt. Mat. 101 (1976) 350–359.

[15] L. Zajíček: On the differentiation of convex functions in finite and infinite dimensional spaces, Czech. Math. J. 29 (1979) 340–348.

[16] L. Zajíček: Differentiability of the distance function and points of multi-valuedness of the metric projection in Banach space, Czech. Math. J. 33 (1983) 292–308.

[17] L. Zajíček: On σ-porous sets in Banach spaces, Abstr. Appl. Anal. 2005(5) (2005) 509–534.