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Abstract Results on the Finite Extinction Time Property: Application to a Singular Parabolic Equation

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We start by studying the finite extinction time for solutions of the abstract Cauchy problem $u_t + Au + Bu = 0$ where A is a maximal monotone operator and B is a positive operator on a Hilbert space H. We use a suitable spectral energy method to get some sufficient conditions which guarantee this property. As application we consider a singular semilinear parabolic equation: $Au = -\Delta u$, $Bu = a(x)u^q$, $a(x) \geq 0$ bounded and $-1 < q < 1$, on a regular bounded domain Ω and Dirichlet boundary conditions.

Keywords: Finite extinction time for abstract Cauchy problems, singular semilinear parabolic equations, semi-classical analysis

1. Introduction

Let H be a real Hilbert space endowed with its inner product $(.,.)$ and its related norm $\|.\|$. Our aim is to investigate the extinction time phenomenon for the solutions of the abstract Cauchy problem

$$\begin{cases} u_t + Au \ni 0,\ t > 0 \text{ in } H, \\ u(0) = u_0, \end{cases} \tag{1}$$

where A is a maximal monotone operator with $0 \in A(0)$.

It is well-known ([17]) that if $u_0 \in \overline{D(A)}$ then there exists a unique $u \in \mathcal{C}([0, +\infty) : H)$ solution of (1). Then, we can define the *extinction time* associated to u_0 by

$$T(u_0) = \sup\{t \geq 0 \text{ such that } \forall \tau \in [0, t[, u(\tau) \neq 0\}, \tag{2}$$

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that is, $u(t) \neq 0$ for all $t < T(u_0)$ and $u(t) = 0$ for any $t \geq T(u_0)$ if $T(u_0)$ is finite. We shall always assume that $u_0 \neq 0$ (which yields that $T(u_0) > 0$). Clearly, we have

$$T(u_0) = t + T(u(t)), \quad \text{for any } t \in [0, T(u_0)]. \tag{3}$$

The finite extinction time phenomenon has been intensively studied in the literature by many authors (see, e.g. the surveys [2], [24], [23] and their references). In spite of the many equations for which we know that this property holds, the only abstract results in the literature is the 1974 Brezis' result ([18]) for multivalued maximal monotone operators (see some generalizations in the above mentioned surveys). The main goal of this paper is to obtain some sufficient conditions for the appearance of this phenomenon in the above abstract framework, getting also some estimates, from above and from below, on the extinction time $T(u_0)$. In a next paper ([10]) we shall present the extension of the abstract results of this paper to the framework of m-accretive operators in general Banach spaces.

Our general method is inspired in the spectral energy method, called "semi-classical method", introduced by V. A. Kondratiev and L. Véron in [32] for the special case of $Au = -\Delta u + a(x)u^q$, with $0 < q < 1$, $a \in L^\infty$ and non-negative (see also the improvement made in [11]). The key-stone in such a method consists in estimating the solutions of the parabolic equation by means of the solutions of a family of Schrödinger equations, thanks to the regularizing effect with respect to the L^∞-norm. Some extensions to more general operators was carried out in [8]. See also the improvements made in [13] by using an integral method which gives a Dini-like condition for the extinction of the associated solution.

More precisely, the main goal of this paper is to introduce a new variant of the "semi-classical method" by replacing the L^∞-estimates of solutions of the parabolic equations by $L^\infty-$estimates of solutions of an auxiliary elliptic equation. The key-stone of this approach is the study of the behavior of the "Rayleigh-like quotients" family

$$\lambda_1(h) = \inf_{u \in H,\ \|u\|^2 \geq h} (A^\circ u, u), \tag{4}$$

for any $h > 0$. Here A° denotes the minimal section of the operator (see [17]). It is clear that $\lambda_1(0) = 0$. We shall assume that

$$\lambda_1(h) > 0, \quad \text{for any } h > 0. \tag{5}$$

Our sufficient condition to have extinction in a finite time can be stated in terms of the behavior of $\lambda_1(h)$ for very small h. More precisely we need the integrability of the improper integral

$$\int_0^1 \frac{dh}{\lambda_1(h)} < +\infty. \tag{6}$$

(see Theorem 2.1). Among the many applications of the above result we mention the special case of the homogeneous operators satisfying $A(au) = a^k A(u)$ *for any* $a \in R^+$ *and* $u \in D(A)$: in that case the mere assumption $k < 1$ implies that the solution of the abstract Cauchy problem (1) corresponding to $u_0 \in \overline{D(A)}$ satisfies the finite extinction time property. We point out that the assumption $k < 1$ in the above statement is optimal in the class of homogeneous operators since it is well known ([1]) that if $k > 1$ then there is not extinction in finite time of solutions of (1).

It is a curious fact that an estimate from below for $T(u_0)$ can be obtained independently of the assumption (10). Indeed, in Theorem 2.6 we prove that if $u_0 \in D(A)\backslash\{0\}$ then

$$T(u_0) \geq \frac{\|u_0\|}{\|A^\circ u_0\|},$$

Moreover, we prove that the equality

$$T(u_0) = \frac{\|u_0\|}{\|A^\circ u_0\|},$$

holds if and only if $A^\circ u(t) = A^\circ u_0$ for all $t \in [0, T(u_0))$.

We shall obtain some other estimates for $T(u_0)$ by assuming certain additional conditions on operator A, in particular when $A = \partial\varphi$ for some convex, l.s.c. and proper function $\varphi : D(\varphi) \subset H \to (-\infty, +\infty]$.

As we shall see later, Theorem 2.1 can be adapted to the perturbed problem

$$\begin{cases} u_t + Au + Bu \ni 0,\ t > 0 \text{ in } H, \\ u(0) = u_0, \end{cases} \tag{7}$$

where B is a perturbation operator defined on a subset $D(B)$ of H, with $\overline{D(A)} \subset D(B)$, and which we assume to be an "absorption" operator in the sense that

$$(Bu, u) \geq 0 \quad \text{for any } u \in \overline{D(A)}. \tag{8}$$

As mentioned before, the main idea in our approach is the study of the dependence on h of the "Rayleigh-like quotient" $\lambda_1(h)$. We shall end Section 2 (which contains the proofs of the above results and their improvements) by studying such dependence for several concrete maximal operators associated to some quasilinear parabolic problems such as the ones associated to the p-Laplacian operator, the nonlinear diffusion operator and an higher order version of the p-Laplacian operator.

Section 3 is devoted to present a careful study of the behavior of function $\lambda_1(h)$, on the Hilbert space $H = L^2(\Omega)$, for the case of the semilinear (possibly singular) parabolic equation

$$\begin{cases} u_t - \Delta u + a(x)u^q\chi_{u>0} = 0 & \text{in } \Omega \times (0, +\infty), \\ u = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ u(x, 0) = u_0(x) & \text{on } \Omega, \end{cases} \tag{9}$$

when we assume $-1 < q < 1$, and $u_0(x) \geq 0$ a.e. $x \in \Omega$. Here $\chi_{u>0}$ denotes the characteristic function of the set $\{(x, t) \in \Omega \times (0, +\infty) : u(x, t) > 0\}$. We shall show that $\lambda_1(h)$ has a minimizer which satisfies an elliptic equation: the associated Euler-Lagrange equation. By estimating the L^∞-norm of the solution of this elliptic equation, we shall be able to give an explicit upper bound for $T(u_0)$ assumed that a is measurable, positive a.e. and bounded on Ω such that

$$\left|\ln \frac{1}{a}\right|^s \in L^1(\Omega),$$

for some $s > N/2$ (see Theorem 3.16).

We end this Introduction by pointing out that a variant of the "semi-classical method" introduced in this paper can be applied to the case of other high order operators different to the higher order version of the p-Laplacian operator presented in Section 2 (see ([13])).

2. Abstract results on the extinction time property

2.1. A sufficient condition for the existence of a finite extinction time

We start by proving the following result:

Theorem 2.1. *Assume that the maximal monotone operator A satisfies* (5) *and that the following improper integral converges near zero*

$$\int_0^1 \frac{dh}{\lambda_1(h)} < +\infty. \tag{10}$$

Then, for any $u_0 \in \overline{D(A)}$ the corresponding solution of (1) *vanishes in a finite time $T(u_0)$. Moreover*

$$T(u_0) \leq \frac{1}{2} \int_0^{\|u_0\|^2} \frac{dh}{\lambda_1(h)}. \tag{11}$$

Proof. Suppose that (10) holds and assume, for the moment, that $u_0 \in D(A)$. By multiplying (1) by u, we have that $\frac{d^+}{dt} \|u\|^2 + 2(A^\circ u, u) = 0$ for any $t \geq 0$. So, from the definition of function $\lambda_1(h)$ we get that $-\frac{d^+}{dt} \|u\|^2 \geq 2\lambda_1(\|u\|^2)$. From (5) and the mean value theorem, function $t \mapsto \|u(t)\|^2$ is a decreasing function on $[0, T(u_0))$. Therefore, $t \mapsto \lambda_1(\|u(t)\|^2)$ is continuous except at most on a countable set. Moreover, $1 \leq -\frac{\frac{d^+}{dt}\|u\|^2}{2\lambda_1(\|u\|^2)}$ for any $t \in [0,\, T(u_0))$. Then, if we set $F(t) = \int_{\|u(t)\|^2}^{\|u_0\|^2} \frac{dh}{\lambda_1(h)}$ we get that F has a right derivative except at most on a countable set and it satisfies $2 \leq \frac{d^+F}{dt}(t)$. Thus, by the mean value theorem, $2t \leq F(t) \leq \int_0^{\|u_0\|^2} \frac{dh}{\lambda_1(h)}$ for any $t \in [0,\, T(u_0))$. Passing to the limit, as $t \nearrow T(u_0)$ we get the result. For the general case $u_0 \in \overline{D(A)}$, let $u_{0,n} \in D(A)$ such that $u_{0,n} \to u_0$ in H. Since the corresponding mild solutions of (1) verify that $u_n \to u$ in $\mathcal{C}([0,T] : H)$ for any $T < +\infty$ we get that $T(u_{0,n}) \to T(u_0)$. From the previous step we know that

$$T(u_{0,n}) \leq \frac{1}{2} \int_0^{\|u_{0,n}\|^2} \frac{dh}{\lambda_1(h)}$$

for any n and thus estimate (11) holds by passing to the limit when $n \to +\infty$. $\square$

Corollary 2.2. *Assume that A is a maximal monotone operator such that it is $k-$homogeneous (i.e. $A(au) = a^k A(u)$ for any $a \in R^+$ and $u \in D(A)$). Assume that $\lambda(1) = \inf\{(Av, v)$ with $\|v\| \geq 1\}$ is finite. Then if $-1 < k < 1$ the solution of the abstract Cauchy problem* (1) *corresponding to $u_0 \in \overline{D(A)}$ satisfies the finite extinction time property.*

We notice that the proof of Corollary 2.2 is obvious since the assumptions imply the condition (10).

It is possible to get an extension to the case of perturbed problem, i.e. for $B \neq 0$ but, previously, we have to introduce a new version of the "Rayleigh-like quotients" family which takes into account this new term. Let us define

$$\widetilde{\lambda_1}(h) = \inf_{\|v\|^2 \geq h} (A^\circ v, v) + (Bv, v), \tag{12}$$

for any $0 \leq h$. This leads to

Theorem 2.3. *Let $u_0 \in \overline{D(A)}$. Assume that B satisfies* (8) *and is such that problem* (7) *admits a mild solution $u(t)$. Assume also that*

$$\widetilde{\lambda_1}(h) > 0 \quad \textit{for any } 0 < h \leq \|u_0\|^2 .$$

Then

$$T(u_0) \leq \frac{1}{2} \int_0^{\|u_0\|^2} \frac{1}{\widetilde{\lambda_1}(h)} dh.$$

The proof is similar to the one of Theorem 2.1 and does not need any additional explanation.

Remark 2.4. Sufficient conditions on operators A and B ensuring the existence of a strong solution $u(t)$ of problem (1) can be found in [17], [4] and [5]. The regularity question about when the above mild solutions satisfy the problem in a strong sense and its application to different quasilinear operators was studied in [6].

2.2. Abstract estimates from below for the extinction time

In this subsection, we use some properties of maximal monotone operators. The following lemma is the key-stone for our estimates from below on the extinction time.

Lemma 2.5. *Let $u_0 \in D(A)\backslash\{0\}$. Let $r \neq 0$ be any given exponent. Then, if $u(t)$ is the corresponding solution of* (1)*, we have*

$$\frac{1}{r}(\|u_0\|^r - \|u(t)\|^r) = \int_0^t \|u(\tau)\|^{r-2} (A^\circ u(\tau), u(\tau)) d\tau, \tag{13}$$

and

$$\frac{1}{r}(\|u_0\|^r - \|u(t)\|^r) \leq \int_0^t \|u(\tau)\|^{r-1} \|A^\circ u(\tau)\| d\tau, \tag{14}$$

for any $0 \leq t < T(u_0)$.

Proof. By an approximating argument we know that for a.e. t in $[0, T(u_0))$, since $u_0 \in D(A)\backslash\{0\}$, the function $t \mapsto \|u(t)\|^r$ is differentiable and we have

$$\frac{d}{dt}\|u(t)\|^r = \|u\|^{r-2}(u_t, u),$$

and

$$\|u\|^{r-2}(u_t, u) + \|u\|^{r-2}(A^\circ u, u) = 0. \tag{15}$$

Since all the above quantities are bounded, it suffices to integrate between 0 and t to get (13) The second estimate (14) holds by applying the Cauchy-Schwarz inequality. □

As mentioned at the Introduction, an estimate from below for $T(u_0)$ can be obtained independently of the assumption (10):

Theorem 2.6. *Let $u_0 \in D(A)\backslash\{0\}$. Then*

$$T(u_0) \geq \frac{\|u_0\|}{\|A^\circ u_0\|}, \tag{16}$$

for problem (1) *and with $T(u_0) \in (0, +\infty]$. Moreover, the equality*

$$T(u_0) = \frac{\|u_0\|}{\|A^\circ u_0\|}, \tag{17}$$

holds if and only if $A^\circ u(t) = A^\circ u_0$ for all $t \in [0, T(u_0))$, where $u(t)$ is the corresponding solution of (1)*. Finally, given any $u_0 \neq 0$, in an arbitrary Hilbert space H, there exists a maximal monotone operator A such that the equality* (17) *holds for a finite $T(u_0)$.*

Proof. Without lost of generality we can assume that $A^\circ u_0 \neq 0$ and $T(u_0) < +\infty$ since if $A^\circ u_0 = 0$ (i.e. if $0 \in Au_0$) then for any $t \geq 0$, $u(t) = u_0$ and thus $T(u_0) = +\infty = \frac{\|u_0\|}{\|A^\circ u_0\|}$, moreover, if $A^\circ u_0 \neq 0$ and $T(u_0) = +\infty$ then (13) holds trivially.

Now, when $A^\circ u_0 \neq 0$ and $T(u_0) < +\infty$ we apply Lemma 2.5, with $r = 1$. Making $t \to T(u_0)$ we get

$$\|u_0\| \leq \int_0^{T(u_0)} \|A^\circ u(t)\| \, dt \leq T(u_0) \|A^\circ u_0\|,$$

since $t \mapsto \|Au(t)\|$ is a non-increasing function (see Theorem 3.1 of [17]). This proves (16).

In order to prove (17) we assume that $T(u_0)$ is finite (otherwise the property trivially holds). Notice that if $A^\circ u(t) = A^\circ u_0$ for any $t \in [0, T(u_0))$ then we get (17). The converse is also true. Suppose that we have (17), then

$$u_0 = \int_0^{T(u_0)} A^\circ u(\tau) \, d\tau,$$

which yields

$$\|u_0\| \leq \int_0^{T(u_0)} \|A^\circ u(\tau)\| \, d\tau \leq T(u_0) \|A^\circ u_0\| = \|u_0\|.$$

So,

$$\int_0^{T(u_0)} \|A^\circ u(\tau)\| \, d\tau = T(u_0) \, \|A^\circ u_0\| \,. \tag{18}$$

Since $\tau \mapsto \|A^\circ u(\tau)\|$ is a non increasing function, let us prove that if we suppose that there exists a time $t_0 \in (0, T(u_0))$ such that $\|A^\circ u(t_0)\| < \|A^\circ u_0\|$ then (17) becomes false (i.e. necessarily, $\|A^\circ u(t)\| = \|A^\circ u_0\|$, for any $t \in [0, T(u_0))$, assumed (17)). Indeed, by contradiction, we first prove that for all $t \in [0, T(u_0))$,

$$\left\| \int_0^t A^\circ u(\tau) d\tau \right\| = t \, \|A^\circ u_0\| \,.$$

This is clear since otherwise there exists a time $t_0 \in (0, T(u_0))$ such that

$$\left\| \int_0^{t_0} A^\circ u(\tau) d\tau \right\| < t_0 \, \|A^\circ u_0\| \,,$$

and then

$$\begin{aligned} \left\| \int_0^{T(u_0)} A^\circ u(\tau) d\tau \right\| &\le \left\| \int_0^{t_0} A^\circ u(\tau) d\tau \right\| + \left\| \int_{t_0}^{T(u_0)} A^\circ u(\tau) d\tau \right\| \\ &< t_0 \, \|A^\circ u_0\| + (T(u_0) - t_0) \, \|A^\circ u_0\| = T(u_0) \, \|A^\circ u_0\| \,, \end{aligned}$$

which is impossible by (18).
Let us prove, in a second step, that $\forall t \in [0, T(u_0))$,

$$\left\| \int_0^t A^\circ u(\tau) d\tau \right\|^2 = t^2 \, \|A^\circ u_0\|^2 \,.$$

Indeed, since the functions are Lipschitz continuous, we can differentiate in t

$$\left(\int_0^t A^\circ u(\tau) \, d\tau, A^\circ u(t) \right) = t \, \|A^\circ u_0\|^2 \,, \tag{19}$$

and by the Cauchy-Schwarz inequality,

$$t \, \|A^\circ u_0\|^2 \le \left\| \int_0^t A^\circ u(\tau) \, d\tau \right\| \, \|A^\circ u(t)\| = t \, \|A^\circ u(t)\|^2 \le t \, \|A^\circ u_0\|^2 \,.$$

Then the above inequality becomes an equality. So, there exists a real number $\gamma(t)$ such that

$$\int_0^t A^\circ u(\tau) \, d\tau = \gamma(t) \, A^\circ u(t).$$

By (19), $\gamma(t) = t$. Hence, for all $t \in [0, T(u_0))$,

$$\int_0^t A^\circ u(\tau) \, d\tau = t \, A^\circ u(t).$$

By induction, $t \mapsto A^\circ u(t)$ is $\mathcal{C}^\infty$ on $(0, T(u_0))$. By differentiating it, $\frac{d}{dt} A^\circ u(t) = 0$, which means that $t \mapsto A^\circ u(t)$ is constant on $(0, T(u_0))$. But $t \mapsto A^\circ u(t)$ is right-continuous at $t = 0$, so $A^\circ u(t) = A^\circ u_0$.
To end the proof of Theorem 2.6, let H be any Hilbert space and let $u_0 \in H$ with $u_0 \neq 0$. We define, on the whole space H, the operator $Au = u_0$. A is clearly monotone and maximal (since $R(I + A) = H$, see [17]). Then $u_t = -u_0$ and as a consequence $u(t) = u_0(1 - t)$ which implies that $T(u_0) = 1 = \frac{\|u_0\|}{\|Au_0\|}$. □

Remark 2.7. As a more sophisticated example of a maximal monotone operator leading to the conclusion in the above argument we can offer the following one: let $N \geq 1$ and Ω be a Lipschitz bounded domain of R^N. To simplify the exposition, let assume that $|\Omega| = 1$. We set $u_0 \equiv 1$, the constant function on Ω. Let $q \in (0, 1)$. We consider the operator $Au = -\Delta u + |u|^{q-1} u$ on $H = L^2(\Omega)$, with domain $D(A) = H^2(\Omega)$ (i.e. we assume Neumann boundary conditions). Clearly, the associated PDE is now reduced to an ODE and the (unique) solution is $u(x, t) = (1 - (1 - q)t)^{\frac{1}{1-q}}$. So on one hand $T(u_0) = \frac{1}{1-q}$. On the other hand, $\frac{\|u_0\|}{\|A^\circ u_0\|} = 1$. So, we are concluding that for any $\varepsilon > 0$, it is possible to find A such that

$$\frac{\|u_0\|}{\|A^\circ u_0\|} \leq T(u_0) \leq \frac{\|u_0\|}{\|A^\circ u_0\|}(1 + \varepsilon),$$

and always with the same initial data u_0 and with the same $A^\circ u_0$.

Corollary 2.8. *Under the assumptions of Theorem* 2.6

$$\|A^\circ u_0\| \, (T(u_0) - t) \geq \|u(t)\| \geq \|u_0\| - t \, \|A^\circ u_0\| \,, \tag{20}$$

for all $T(u_0) \geq t \geq 0$, *whether* $T(u_0)$ *is finite or not.*

Proof. By applying Theorem 2.6 for $t \in [0, T(u_0)]$ we get

$$\|A^\circ u_0\| \, (T(u_0) - t) \geq \|A^\circ u(t)\| \, (T(u_0) - t) \geq \|u(t)\| \,,$$

since $T(u_0) - t = T(u(t))$. For the right-hand side, we copy the proof of Theorem 2.6 with the estimate in Lemma 2.5 for $r = 1$. □

Corollary 2.9. *Under the assumptions of Theorem* 2.6, *if* $T(u_0)$ *is finite then*

$$\lim_{t \to T(u_0)} \frac{\|u(t)\|}{\|A^\circ u(t)\|} = 0. \tag{21}$$

As consequence, if there exists $\varepsilon > 0$ *such that* $\|u(t)\| \geq \varepsilon \, \|A^\circ u(t)\|$ *for all* $t > 0$ *then* $T(u_0) = +\infty$.

Proof. It is enough to use that $T(u_0)$ finite implies that $T(u_0) - t \geq \frac{\|u(t)\|}{\|A^\circ u(t)\|} \geq 0$, which ends the proof. □

A trivial case for which $T(u_0) = +\infty$ is given in the following result for not necessary linear operators A (its proof is straighforward).

Proposition 2.10. *Suppose that A is a maximal monotone operator on an Hilbert space H. Assume that there exist $(u_0, \lambda) \in D(A) \times R^+$ with $u_0 \neq 0$ such that $A^\circ u_0 = \lambda u_0$ and that the restriction of A° to the cone Ru_0 is linear. Then the solution of* (1) *is $u(t) = u_0 e^{-\lambda t}$, so that $T(u_0) = +\infty$.*

Let us show now some improvements of Theorem 2.6.

Theorem 2.11. *Under the assumptions of Theorem* 2.6*, let us assume that there exists $\xi \in D(A)$ such that $u(t) \neq \xi$ for any $t \in (0, T(u_0))$. Then we have*

$$T(u_0) \geq \sup_{\xi \in D(A) \setminus \{u(t),\, t \geq 0\}} \frac{\|\xi\| - \|u_0 - \xi\|}{\|A^\circ \xi\|}. \tag{22}$$

Proof. Let $\xi \in D(A)$. For all $t \in (0, T(u_0))$, we have

$$\frac{1}{2}\frac{d^+}{dt} \|u(t) - \xi\|^2 + (A^\circ u, u(t) - \xi) = 0,$$

which implies that

$$\frac{1}{2}\frac{d^+}{dt} \|u(t) - \xi\|^2 + (A^\circ u - A^\circ \xi, u(t) - \xi) + (A^\circ \xi, u(t) - \xi) = 0.$$

Since A is monotone,

$$\frac{1}{2}\frac{d^+}{dt} \|u(t) - \xi\|^2 + (A^\circ \xi, u(t) - \xi) \leq 0,$$

and by Cauchy-Schwarz,

$$\frac{1}{2}\frac{d^+}{dt} \|u(t) - \xi\|^2 \leq \|A^\circ \xi\| \; \|u(t) - \xi\|.$$

Therefore,

$$\frac{1}{2}\frac{\frac{d^+}{dt} \|u(t) - \xi\|^2}{\|u(t) - \xi\|} \leq \|A^\circ \xi\|.$$

We end the proof by applying the mean value theorem. □

Remark 2.12. If A° is continuous at the point u_0 then the above theorem is implied by Theorem 2.6 but it is not necessarily the case otherwise since if we take a sequence such that $\xi_n \to u_0$, the sequence $\|A^\circ \xi_n\|$ may not converge to $\|A^\circ u_0\|$.

Corollary 2.13. *Let A and u_0 be as in the assumptions of Theorem* 2.6*. Assume, additionally, that there exists $p > 3$ such that*

$$\|A^\circ(\alpha u_0)\| \leq \alpha^{p-1} \|A^\circ u_0\|, \quad \textit{for any } \alpha \in \left(\frac{1}{2}, 1\right).$$

Then

$$T(u_0) \geq \frac{2^{p-1}(p-2)^{p-2}}{(p-1)^{p-1}} \frac{\|u_0\|}{\|A^\circ u_0\|}, \tag{23}$$

with

$$\frac{2^{p-1}(p-2)^{p-2}}{(p-1)^{p-1}} > 1.$$

Proof. By taking $\xi = \alpha u_0$ with $\frac{1}{2} < \alpha < 1$, we get from Theorem 2.11

$$T(u_0) \geq \frac{2\alpha - 1}{\alpha^{p-1}} \frac{\|u_0\|}{\|A^\circ u_0\|}.$$

The maximum of the function $\alpha \mapsto \frac{2\alpha}{\alpha^{p-1}}$ is attained for $\alpha = \frac{p-1}{2(p-2)}$ and this value belongs to $\left(\frac{1}{2}, 1\right)$ once we assume $p > 3$. □

Theorem 2.6 can be improved when we assume that A is the subdifferential of a convex function.

Theorem 2.14. *Assume that $A = \partial\varphi$ for some convex, l.s.c. and proper function $\varphi : D(\varphi) \subset H \to (-\infty, +\infty]$ such that*

$$\varphi(0) = 0 \quad \textit{and} \quad \varphi(u) > 0 \ \ \forall u \neq 0. \tag{24}$$

Let $u_0 \in D(\varphi)$ with $u_0 \neq 0$. Then if $T(u_0)$ denotes the extinction time associated to the corresponding solution of (1) *we have*

$$T(u_0) \geq \frac{\|u_0\|^2}{\varphi(u_0)}. \tag{25}$$

Moreover, this inequality is optimal in the sense of Theorem 2.3 *and*

$$\frac{\|u_0\|^2}{\varphi(u_0)} \geq \frac{\|u_0\|}{\|A^\circ u_0\|}. \tag{26}$$

Proof. Thanks to Theorem 3.6 of [17] we have

$$\frac{\|u(t) - u_0\|}{\sqrt{t}} \leq \sqrt{\varphi(u_0) - \varphi(u(t))},$$

for any $t \in (0, T(u_0))$. Passing to the limit as $t \uparrow T(u_0)$ it shows (25) thanks to (24). Moreover the example built in the proof of Theorem 2.6 remains being true for $A = \partial\varphi$ with $\varphi(u) = (u_0, u)$. Indeed, by the subdifferential definition we have

$$v \in A^\circ u_0 \ \text{ if and only if } \forall \xi \in H,\ \varphi(\xi) \geq \varphi(u_0) + (A^\circ u_0, \xi - u_0),$$

which yields (taking $\xi = 0$) to

$$\varphi(u_0) \leq (A^\circ u_0, u_0) \leq \|A^\circ u_0\| \ \|u_0\|,$$

after using Cauchy-Schwarz inequality. □

Remark 2.15. Under the assumptions of the previous theorem, we have that

$$T(u_0) \geq t + \frac{\|u(t)\|^2}{\varphi(u(t))},$$

for any $t \in (0, T(u_0))$. As a consequence,

$$\lim_{t \uparrow T(u_0)} \frac{\varphi(u(t))}{\|u(t)\|^2} = +\infty,$$

when $T(u_0)$ is finite.

Remark 2.16. For some other results on the finite extinction time property we send the reader to the expositions [2], [24], [23] and its references.

2.3. Both sides estimates for the extinction time

We start by generalizing Proposition 2.10 to the case of $A = \partial\varphi$. We point out that the following series of results gives also some estimates on the asymptotic decay $u(t) \to 0$ as $t \to +\infty$ when $T(u_0) = +\infty$.

Proposition 2.17. *Let u be a solution of* (1) *where $A = \partial\varphi$ for some convex, l.s.c. and proper function φ satisfying* (24). *Let $u_0 \in D(\varphi)$ with $u_0 \neq 0$ and let us assume that there exists $\xi \in D(A)$ and $T > 0$ such that $\varphi(u(t)) \neq \varphi(\xi)$ for any $t \in [0, T)$. Then we have*

$$\varphi(u(t)) \leq \varphi(\xi) + \frac{\varphi(u_0) - \varphi(\xi)}{1 + (\varphi(u_0) - \varphi(\xi)) \displaystyle\int_0^t \frac{1}{\|u(\tau) - \xi\|^2} d\tau}, \tag{27}$$

and

$$\varphi(u(t)) \leq \frac{\varphi(u_0)}{1 + \varphi(u_0) \displaystyle\int_0^t \frac{1}{\|u(\tau)\|^2} d\tau}. \tag{28}$$

Moreover, if we assume that

$$\exists q \geq 0 \quad \textit{such that}\, (A^\circ u, u) \geq (1+q)\, \varphi(u) \quad \forall u \in D(A), \tag{29}$$

then

$$\varphi(u(t)) \leq \frac{\varphi(u_0)}{1 + \varphi(u_0)(1+q)^2 \displaystyle\int_0^t \frac{1}{\|u(\tau)\|^2} d\tau}. \tag{30}$$

Proof. By Theorem 3.6 of [17], for any $t \in (0, T(u_0))$, we have

$$\frac{d^+}{dt}\varphi(u(t)) = -\left\|\frac{d^+ u}{dt}\right\|^2 = -\left\|A^\circ u(t)\right\|^2.$$

From the convexity of φ

$$\varphi(u(t)) \leq \varphi(\xi) + (A^\circ u(t), u(t) - \xi) \leq \varphi(\xi) + \|A^\circ u(t)\| \, \|u(t) - \xi\| \,,$$

and by the Cauchy-Schwarz inequality

$$(\varphi(u(t)) - \varphi(\xi))^2 \leq \|A^\circ u(t)\|^2 \ \|u(t) - \xi\|^2 = - \|u(t) - \xi\|^2 \ \frac{d^+}{dt}\varphi(u(t)).$$

Hence,

$$\frac{\frac{d^+}{dt}(\varphi(u(t)) - \varphi(\xi))}{(\varphi(u(t)) - \varphi(\xi))^2} + \frac{1}{\|u(t) - \xi\|^2} \leq 0,$$

since by assumption, $\varphi(u(t)) \neq \varphi(\xi)$. Let $0 < t_0 < t < T$, then by the mean value theorem,

$$\frac{1}{\varphi(u(t_0)) - \varphi(\xi)} - \frac{1}{\varphi(u(t)) - \varphi(\xi)} + \int_{t_0}^{t} \frac{1}{\|u(\tau) - \xi\|^2} d\tau \leq 0.$$

Passing to the limit as $t_0 \to 0$ we get

$$\frac{1}{\varphi(u_0) - \varphi(\xi)} - \frac{1}{\varphi(u(t)) - \varphi(\xi)} + \int_{0}^{t} \frac{1}{\|u(\tau) - \xi\|^2} d\tau \leq 0,$$

(since all the quantities have a finite limit) and the proof of (27) is completed. For the proof of (28) it is enough to take $\xi = 0$. Finally, to prove (30) it suffices to replace $\|u(t)\|$ by $\frac{\|u(t)\|}{1+q}$. □

We get a simpler bound for $\varphi(u(t))$ by using that the function $t \mapsto \|u(t)\|$ is non increasing.

Corollary 2.18. *Under the assumptions of Proposition* 2.17*, for all* $t \geq 0$*,*

$$\varphi(u(t)) \leq \frac{\varphi(u_0)}{1 + \varphi(u_0)\dfrac{t}{\|u_0\|^2}},$$

and if (29) *holds*

$$\varphi(u(t)) \leq \frac{\varphi(u_0)}{1 + \varphi(u_0)(1+q)^2\dfrac{t}{\|u_0\|^2}}.$$

We can refine the previous estimate:

Corollary 2.19. *Let* $A = \partial\varphi$ *for some convex, l.s.c. and proper function* φ *satisfying* (24)*. Let* $u_0 \in D(\varphi)$ *with* $u_0 \neq 0$ *and assume 5. Define*

$$\Gamma(s) = \int_{s}^{s_0} \frac{dh}{\lambda_1(h)}, \tag{31}$$

for a fixed $s_0 > 0$. *Then*

$$\varphi(u(t)) \leq \frac{\varphi(u_0)}{1+\varphi(u_0)\displaystyle\int_0^t \frac{1}{\Gamma^{-1}(2\tau+\Gamma(\|u_0\|^2))}d\tau},$$

for any $t \in [0, T(u_0))$. *Moreover, if we assume* (29)

$$\varphi(u(t)) \leq \frac{\varphi(u_0)}{1+\varphi(u_0)(1+q)^2\displaystyle\int_0^t \frac{1}{\Gamma^{-1}(2\tau+\Gamma(\|u_0\|^2))}d\tau}.$$

Proof. From Theorem 2.6,

$$\Gamma(\|u(t)\|^2) - \Gamma(\|u_0\|^2) \geq 2t,$$

so, since $s \mapsto \Gamma(s)$ is a decreasing function,

$$\|u(t)\|^2 \leq \Gamma^{-1}(2t + \Gamma(\|u_0\|^2)),$$

which leads to the conclusion. □

Some better estimates can be obtained by connecting expressions $\varphi(u)$ and $(A^\circ u, u)$ through some additional condition as the following one:

$$\exists p > 1 \text{ such that } p\,\varphi(u) \geq (A^\circ u, u) \text{ for any } u \in D(A). \tag{32}$$

Theorem 2.20. *Let* $A = \partial\varphi$ *for some convex, l.s.c. and proper function* φ *satisfying* (24). *Let* $u_0 \in D(\varphi)$ *with* $u_0 \neq 0$ *and assume* (5) *and* (32). *Then*

$$\begin{aligned}\Gamma^{-1}(2t+\Gamma(\|u_0\|^2)) \geq\ & \|u(t)\|^2 \\ \geq\ & \|u_0\|^2 - 2p\int_0^t \frac{\varphi(u_0)}{1+\varphi(u_0)\displaystyle\int_0^\tau \frac{1}{\Gamma^{-1}(2\theta+\Gamma(\|u_0\|^2))}d\theta}d\tau,\end{aligned} \tag{33}$$

for any $t \in [0, T(u_0))$. *Moreover, if we assume* (29)

$$\begin{aligned}\Gamma^{-1}(2t+\Gamma(\|u_0\|^2)) \geq\ & \|u(t)\|^2 \\ \geq\ & \|u_0\|^2 - 2p\int_0^t \frac{\varphi(u_0)}{1+\varphi(u_0)(1+q)^2\displaystyle\int_0^\tau \frac{1}{\Gamma^{-1}(2\theta+\Gamma(\|u_0\|^2))}d\theta}d\tau,\end{aligned} \tag{34}$$

Proof. The left-hand side comes directly from Theorem 2.6. For the other side, we have

$$-\frac{d^+}{dt}\|u(t)\|^2 = 2(A^\circ u(t), u(t)) \leq 2p\,\varphi(u(t)).$$

for all $t \in [0, T(u_0))$ and it is enough to apply the mean value theorem and the previous corollary. □

Our aim now is to estimate the extinction time when it is finite.

Corollary 2.21. *Under the assumptions of Theorem* 3.4 *and condition* (10) *then we have the estimates*

$$T(u_0) \leq \frac{1}{2} \int_0^{\|u_0\|^2} \frac{dh}{\lambda_1(h)} = \frac{1}{2} \left(\Gamma(0) - \Gamma(\|u_0\|^2) \right), \tag{35}$$

and

$$2p \int_0^{T(u_0)} \frac{\varphi(u_0)}{1 + \varphi(u_0)(1+q)^2 \displaystyle\int_0^\tau \frac{1}{\Gamma^{-1}(2\theta + \Gamma(\|u_0\|^2))} d\theta} d\tau \geq \|u_0\|^2. \tag{36}$$

Proof. It is enough to apply Theorems 2.1 and 2.20. □

Remark 2.22. Conversely to Corollary 2.21, from Theorem 2.20 we can deduce a "quasi-necessary condition" for the existence of the finite extinction time. Indeed, property

$$2p \int_0^{+\infty} \frac{\varphi(u_0)}{1 + \varphi(u_0)(1+q)^2 \displaystyle\int_0^\tau \frac{1}{\Gamma^{-1}(2\theta + \Gamma(\|u_0\|^2))} d\theta} d\tau \leq \|u_0\|^2,$$

never happens when

$$\int_0^1 \frac{dh}{\lambda_1(h)} = +\infty.$$

Now we pass to the question about how to estimate $\lambda_1(h)$ and therefore the study of the associated function $\Gamma(s)$ (see (31)). We start by considering the special case in which

$$\exists C > 0 \text{ such that } \lambda_1(h) \geq Ch \text{ for any } h \in (0, \|u_0\|^2]. \tag{37}$$

We shall see later (see estimate (78)) that this holds in many cases.

Corollary 2.23. *Let $A = \partial\varphi$ for some convex, l.s.c. and proper function φ satisfying* (24)*. Let $u_0 \in D(\varphi)$ with $u_0 \neq 0$ and assume* (5), (32) *and* (29)*. Let q be such that $q < \sqrt{2p} - 1$, with p given in* (32)*. Then, if we assume* (37) *we have*

$$T(u_0) \geq \frac{1}{2C} \ln \left(\frac{\varphi(u_0)(1+q)^2 - 2C\|u_0\|^2}{\varphi(u_0)(1+q)^2 - 2C\|u_0\|^2 \exp\left(\frac{\varphi(u_0)(1+q)^2 - 2C\|u_0\|^2}{2p\varphi(u_0)} \right)} \right), \tag{38}$$

if $\varphi(u_0)(1+q)^2 \neq 2C\|u_0\|^2$, and

$$T(u_0) \geq \frac{1}{2C} \ln \frac{2p}{2p - (1+q)^2}, \tag{39}$$

if $\varphi(u_0)(1+q)^2 = 2C\|u_0\|^2$.

Proof. We take $s_0 = \|u_0\|^2$ in the definition of $\Gamma(s)$. Then

$$\Gamma(s) \leq \frac{1}{C} \ln\left(\frac{\|u_0\|^2}{s}\right),$$

for $0 < s \leq 1$. So, $\Gamma^{-1}(s) \leq \|u_0\|^2 e^{-Cx}$. As consequence,

$$\int_0^\tau \frac{1}{\Gamma^{-1}(2\theta + \Gamma(\|u_0\|^2))} d\theta = \int_0^\tau \frac{1}{\Gamma^{-1}(2\theta)} d\theta \geq \frac{1}{2C\|u_0\|^2}\left(e^{2C\tau} - 1\right),$$

for $\tau \in (0, T(u_0))$. Hence, for all $t \in (0, T(u_0))$,

$$\begin{aligned}
&2p \int_0^t \frac{\varphi(u_0)}{1 + \varphi(u_0)(1+q)^2 \displaystyle\int_0^\tau \frac{1}{\Gamma^{-1}(2\theta + \Gamma(\|u_0\|^2))} d\theta} d\tau \\
&\leq 4Cp \|u_0\|^2 \varphi(u_0) \int_0^t \frac{1}{2C\|u_0\|^2 + \varphi(u_0)(1+q)^2\left(e^{2C\tau} - 1\right)} d\tau \\
&= 4Cp \|u_0\|^2 \varphi(u_0) \int_0^t \frac{e^{-2C\tau}}{(2C\|u_0\|^2 - \varphi(u_0)(1+q)^2)e^{-2C\tau} + \varphi(u_0)(1+q)^2} d\tau \\
&= \frac{2p\|u_0\|^2 \varphi(u_0)}{2C\|u_0\|^2 - \varphi(u_0)(1+q)^2} \ln\left(\frac{2C\|u_0\|^2}{\left(2C\|u_0\|^2 - \varphi(u_0)(1+q)^2\right)e^{-2Ct} + \varphi(u_0)(1+q)^2}\right),
\end{aligned}$$

for $\varphi(u_0)(1+q)^2 \neq 2C\|u_0\|^2$. Moreover,

$$\frac{2p\varphi(u_0)}{2C\|u_0\|^2 - \varphi(u_0)(1+q)^2} \ln\left(\frac{2C\|u_0\|^2}{\left(2C\|u_0\|^2 - \varphi(u_0)(1+q)^2\right)e^{-2CT(u_0)} + \varphi(u_0)(1+q)^2}\right) \geq 1,$$

which yields for $\varphi(u_0)(1+q)^2 > 2C\|u_0\|^2$,

$$\begin{aligned}
&\ln\left(\frac{-\left(\varphi(u_0)(1+q)^2 - 2C\|u_0\|^2\right)e^{-2CT(u_0)} + \varphi(u_0)(1+q)^2}{2C\|u_0\|^2}\right) \\
&\geq \frac{\varphi(u_0)(1+q)^2 - 2C\|u_0\|^2}{2p\varphi(u_0)},
\end{aligned}$$

$$\begin{aligned}
&\left(\varphi(u_0)(1+q)^2 - 2C\|u_0\|^2\right)e^{-2CT(u_0)} \\
&\leq \varphi(u_0)(1+q)^2 - 2C\|u_0\|^2 \exp\left(\frac{\varphi(u_0)(1+q)^2 - 2C\|u_0\|^2}{2p\varphi(u_0)}\right).
\end{aligned}$$

Finally,

$$e^{2CT(u_0)} \geq \frac{\varphi(u_0)(1+q)^2 - 2C\|u_0\|^2}{\varphi(u_0)(1+q)^2 - 2C\|u_0\|^2 \exp\left(\frac{\varphi(u_0)(1+q)^2 - 2C\|u_0\|^2}{2p\varphi(u_0)}\right)}.$$

When $\varphi(u_0)(1+q)^2 = 2C\|u_0\|^2$, we get

$$\begin{aligned}
&4Cp\|u_0\|^2\varphi(u_0)\int_0^t \frac{e^{-2C\tau}}{(2C\|u_0\|^2-\varphi(u_0)(1+q)^2)e^{-2C\tau}+\varphi(u_0)(1+q)^2}d\tau\\
&= 4Cp\|u_0\|^2\varphi(u_0)\int_0^t \frac{e^{-2C\tau}}{\varphi(u_0)(1+q)^2}d\tau = \frac{2p\|u_0\|^2}{(1+q)^2}\left(1-e^{-2Ct}\right),
\end{aligned}$$

which leads to the conclusion. □

Remark 2.24. It is always possible to let C to zero. In that case, we get the estimate

$$T(u_0) \geq \frac{\|u_0\|^2}{\varphi(u_0)}\left(\frac{\exp\left(\frac{(1+q)^2}{2p}\right)-1}{(1+q)^2}\right).$$

In some case, this inequality is sharper than (25).

Corollary 2.25 (Critical case). *Let $A=\partial\varphi$ as in Corollary 2.23 but replacing condition (37) by*

$$\exists C>0,\ \forall h\in(0,\|u_0\|^2],\quad \lambda_1(h)\geq Ch(-\ln h). \tag{40}$$

Then

$$T(u_0)\geq \frac{1}{2C(-\ln\|u_0\|^2)} \tag{41}$$

$$\ln\left(\frac{\varphi(u_0)(1+q)^2-2C(-\ln\|u_0\|^2)\|u_0\|^2}{\varphi(u_0)(1+q)^2-2C(-\ln\|u_0\|^2)\|u_0\|^2\exp\left(\frac{\varphi(u_0)(1+q)^2-2C(-\ln\|u_0\|^2)\|u_0\|^2}{2p\varphi(u_0)}\right)}\right),$$

when $\varphi(u_0)(1+q)^2\neq 2C(-\ln\|u_0\|^2)\|u_0\|^2$.

Proof. For $h\leq\|u_0\|^2$, we have $\lambda_1(h)\geq Ch(-\ln h)\geq Ch(-\ln\|u_0\|^2)$. It suffices now to replace C by $C(-\ln\|u_0\|^2)$ in the previous corollary and we get the result. □

2.4. Application to some partial differential operators

Let us start with the case of the linear Laplacian and bi-Laplacian operators. Let us take Ω a bounded domain of $\mathbb{R}^N$ ($N\geq 1$), $H=L^2(\Omega)$, $A=-\Delta$ (respectively $A=\Delta(\Delta)$) with $D(A)=H^2(\Omega)\cap H_0^1(\Omega)$ (respectively $D(A)=H^4(\Omega)\cap H_0^1(\Omega)$). We can take $q=1$ (non absorption) and $p>2$ fixed since $2\varphi(u)=(Au,u)$. The constant C can be taken as the first eigenvalue $\lambda_{1,\Omega}$. We choose for u_0 the first eigenfunction such that $\|u_0\|=1$ so $\varphi(u_0)(1+q)^2=2C\|u_0\|^2$. As consequence,

$$T(u_0)\geq\frac{1}{2C}\ln\frac{2p}{2p-4}.$$

Passing to the limit $p\downarrow 2$ it gives that $T(u_0)=+\infty$.

Now we pass to the application to the so called p-Laplacian operator. Let Ω be a regular bounded open connected subset of $\mathbb{R}^N$ $(N \geq 1)$. We consider the solution of

$$(Pp) \begin{cases} u_t - \Delta_p u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u(x,0) = u_0(x) & \text{on } \Omega, \end{cases}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ for $p > 1$. The finite time extinction of the solutions of (Pp) when $2N/(N+2) \leq p < 2$ and $N \geq 2$ was proved in [7], and, for $1 < p < 2N/(N+2)$, in [31]. In both cases, additional assumptions on the initial datum $u_0 \in L^2(\Omega)$ must be supposed. Now we shall apply our abstract results to the case of $H = L^2(\Omega)$ and $Au = -\Delta_p u$ with domain $D(A) = \{u \in W_0^{1,p}(\Omega) : \Delta_p u \in L^2(\Omega)\}$. The operator A is the subdifferential of function φ given by

$$\varphi(u) = \frac{1}{p} \int_\Omega |\nabla u(x)|^p \, dx, \text{ if } u \in D(\varphi) = W_0^{1,p}(\Omega). \tag{42}$$

In that case

$$\lambda_1 = \inf_{\|u\|^2=1} (A^\circ u, u) = \inf_{\|u\|^2_{L^2(\Omega)}=1} \int_\Omega |\nabla u(x)|^p \, dx, \tag{43}$$

and consequently, since A is homogenous,

$$\lambda_1(h) = \inf_{\|u\|^2=h} (Au, u) = \lambda_1 h^{\frac{p}{2}}. \tag{44}$$

Corollary 2.26. *Assume $1 < p < 2$ then if $u_0 \in L^2(\Omega)$ there is a finite extinction time and*

$$T(u_0) \leq \frac{\|u_0\|^{2-p}_{L^2(\Omega)}}{(2-p)\lambda_1}. \tag{45}$$

Moreover, for $u_0 \in W_0^{1,p}(\Omega)$

$$T(u_0) \geq \frac{p \, \|u_0\|^2_{L^2(\Omega)}}{\int_\Omega |\nabla u_0(x)|^p dx}. \tag{46}$$

Proof. Applying Theorem 2.1 (notice that $\overline{D(A)} = L^2(\Omega)$) we get

$$\frac{1}{2} \int_0^{\|u_0\|^2} \frac{dh}{\lambda_1(h)} = \frac{\|u_0\|^{2-p}}{(2-p)\lambda_1}.$$

The second estimate holds by Theorem 2.14. □

The same arguments are applicable to the higher order version of this operator. Consider the problem

$$(Hp) \begin{cases} u_t + \Delta(|\Delta u|^{p-2} \Delta u) = 0 & \text{in } \Omega, \\ \Delta u = u = 0 & \text{on } \partial\Omega, \\ u(x,0) = u_0(x) & \text{on } \Omega. \end{cases}$$

The finite time extinction of the solutions of (Hp), for $1 < p < 2$, was proved in [2] under some additional assumptions on the initial datum $u_0 \in L^2(\Omega)$. Now we shall apply our abstract results to the case of $H = L^2(\Omega)$ and $Au = \Delta(|\Delta u|^{p-2}\,\Delta u)$. In that case, the operator A is the subdifferential of function φ given by

$$\varphi(u) = \frac{1}{p}\int_\Omega |\Delta u(x)|^p\,dx, \text{ if } u \in D(\varphi) = \{u \in W_0^{1,p}(\Omega) : \Delta u \in L^p(\Omega) \cap W_0^{1,1}(\Omega)\} \quad (47)$$

In that case

$$\lambda_1 = \inf_{\|u\|^2=1} (A^\circ u, u) = \inf_{\|u\|^2_{L^2(\Omega)}=1} \int_\Omega |\Delta u(x)|^p\,dx, \quad (48)$$

and since since A is homogenous, $\lambda_1(h) = \inf_{\|u\|^2=h}(Au,u) = \lambda_1 h^{\frac{p}{2}}$ we get

Corollary 2.27. *Assume $1 < p < 2$ then if $u_0 \in L^2(\Omega)$ there is a finite extinction time and*

$$T(u_0) \le \frac{\|u_0\|^{2-p}_{L^2(\Omega)}}{(2-p)\lambda_1}. \quad (49)$$

Moreover, for $u_0 \in D(\varphi)$

$$T(u_0) \ge \frac{p\,\|u_0\|^2_{L^2(\Omega)}}{\int_\Omega |\Delta u_0(x)|^p dx}. \quad (50)$$

Our last application, in this subsection, deals with the so called nonlinear diffusion operator (sometimes called also as the porous media operator). Let Ω be a regular domain of $\mathbb{R}^N$ and $\beta(u) = |u|^{m-1}u$ for $m > 0$. From Corollary 31 of [16], for all $u_0 \in H^{-1}(\Omega)(= H_0^1(\Omega)')$, there exists a unique solution u of the equation

$$\begin{cases} u_t - \Delta(\beta(u)) = 0 & \text{in } \Omega \times (0,+\infty), \\ \beta(u(x,t)) = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x) & \text{on } \Omega, \end{cases}$$

satisfying $u \in \mathcal{C}([0,+\infty) : H^{-1}(\Omega))$, $u(x,t) \in L^1(\Omega)$ and $\beta(u(x,t)) \in H_0^1(\Omega)$, $\forall t > 0$. Moreover, $\beta = \partial j$ where j is a convex l.s.c. function on the space $H = H^{-1}(\Omega)$ and $A = \partial\varphi$ with

$$\varphi(u) = \begin{cases} \int_\Omega j(u(x))dx & \text{if } u \in L^1(\Omega) \text{ and } j(u) \in L^1(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

From Theorem 17 in [16], $f \in H^{-1}(\Omega)$, $f \in \partial\varphi(u)$ if and only if $(\Lambda^{-1}f)(x) \in \beta(u(x,t))$ a.e. on Ω where $\Lambda = -\Delta$ is the canonical isomorphism from $H_0^1(\Omega)$ onto $H^{-1}(\Omega)$. The extinction in finite time property, for $m \in (0,1)$, was first proved in [15] for very smooth initial datum and later generalized in [21]. In that case,

$$\lambda_1(h) = \inf_{\|u\|^2_{H^{-1}(\Omega)}=h} (A^\circ u, u) = \frac{\lambda_1(1)}{h^{\frac{m+1}{2}}}.$$

By setting $\lambda_1(1) = \lambda_1$, we have

Corollary 2.28. *Assume $m \in (0,1)$ and let $u_0 \in H^{-1}(\Omega)$. Then there is extinction in finite time and*

$$T(u_0) \leq \frac{\|u_0\|_{H^{-1}(\Omega)}^{1-m}}{(1-m)\lambda_1}. \tag{51}$$

Moreover, if $u_0 \in L^{m+1}(\Omega)$ then

$$T(u_0) \geq \frac{(m+1)\,\|u_0\|_{H^{-1}(\Omega)}^2}{\int_\Omega |u_0(x)|^{m+1}dx}. \tag{52}$$

Proof. It is similar to the one to the case of the p-Laplacian operator (the density of $D(A)$ in $H^{-1}(\Omega)$ was proved in [16]). $\square$

3. Application to a semilinear parabolic equation

In that section we shall apply and improve the results of the above section to the case of the semilinear problem (9)

$$\begin{cases} u_t - \Delta u + a(x)u^q\chi_{u>0} = 0 & \text{in } \Omega \times (0,+\infty), \\ u = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x) & \text{on } \Omega, \end{cases}$$

where $-1 < q < 1$ and a a nonnegative, measurable and bounded function on Ω (a regular bounded open connected subset of $\mathbb{R}^N$, $N \geq 1$). Some studies for the parabolic singular case can be found in [20] and [34]. The associated singular elliptic equation was considered in [14], [25] and [26].

As in the case of the abstract results, our general task will be to find a lower bound for the expression

$$\int_\Omega |\nabla u|^2 + a(x)u^{1+q}\,dx,$$

with the help of the function

$$\lambda_1(h) = \inf\left\{\int_\Omega |\nabla v|^2 + a(x)|v|^q v\,dx, v \in H_0^1(\Omega), \|v\|_{L^2(\Omega)}^2 = h\right\},$$

for $h > 0$. It is easy to see that this infimum is reached by a nonnegative function v_h and $\lambda_1(h)$ is related to an eigenvalue problem via the Euler-Lagrange equation, i.e., there exists a real number $\nu(h)$ and a function for all $v_h \in W_0^{1,1}(\Omega)$, with $a(x)v_h^q\chi_{v_h>0} \in L^1(\Omega)$ such that

$$\int_\Omega \nabla v_h.\nabla w + \frac{1+q}{2}a(x)v_h^q\chi_{v_h>0}\,w\,dx = \nu(h)\int_\Omega v_h\,w\,dx. \tag{53}$$

for all $w \in W_0^{1,\infty}(\Omega)$. In what follows, we denote by ω_N the volume of the unit-ball in $\mathbb{R}^N$.

3.1. A general upper bound for $\lambda_1(h)$

To obtain finer estimates on the extinction time we shall need some sharp information on the dependence of $\lambda_1(h)$ with respect to h. We start by getting an upper bound on $\lambda_1(h)$ proving that $\frac{\lambda_1(h)^{\frac{4+(1-q)N}{4}}}{h^{\frac{2(1+q)+(1-q)N}{4}}}$ tends to zero when $h \to 0$.

Lemma 3.1. *Let Ω be a general domain (not necessarily bounded) of R^N, $N \geq 1$ and let a be a measurable function on Ω such that*

$$a \in L^\infty(U) \quad \text{for some open set } U \subset \Omega. \tag{54}$$

Let $q > -1$ such that $\frac{1-q}{4+N(1-q)} > 0$. Let O be an arbitrary oint of U. Let ω be a bounded domain of R^N and let φ be a nonnegative function of $H_0^1(\omega)$ with $\|\varphi\|_{L^2(\omega)} = 1$. Then there exists $h_0 > 0$ depending on ω, q, N, O and ∂U such that for all $0 < h \leq h_0$,

$$\lambda_1(h) \leq \left(\int_\omega |\nabla\varphi|^2 \, dx + \|a\|_{L^\infty(U)} \int_\omega \varphi^{1+q} \, dx \right) h^{\frac{2(1+q)+(1-q)N}{4+(1-q)N}}. \tag{55}$$

Remark 3.2. Assumption (54) is very weak, only "pathological" functions do not satisfy it (modify Example 3.5 page 25 of [9] by taking $\Omega = [0, 1]$).

Proof. We use an argument based on homothetic domains. We denote by ω_r the image of ω by the homothecy of center O and ratio $r > 0$. We set for all $x \in \omega_r$, $\varphi_r(x) = \varphi(x/r)$. By a change of variables $x = ry$, we get

$$\int_{\omega_r} |\nabla\varphi_r(x)|^2 \, dx = r^{N-2} \int_\omega |\nabla\varphi(y)|^2 \, dy,$$

and for all $\alpha > 0$,

$$\int_{\omega_r} \varphi_r(x)^\alpha \, dx = r^N \int_\omega \varphi(y)^\alpha \, dy.$$

Let $h > 0$ small enough and $v = \left(\frac{h}{r^N}\right)^{\frac{1}{2}} \varphi_r$. From the last assertion, $\|v\|^2_{L^2(\omega)} = h$. We use v in the definition of $\lambda_1(h)$ since for r small enough, $\omega_r \subset U$. Then

$$\lambda_1(h) \leq \frac{h}{r^2} \int_\omega |\nabla\varphi(y)|^2 \, dy + \|a\|_{L^\infty(\omega_r)} \left(\frac{h}{r^N}\right)^{\frac{1+q}{2}} r^N \int_\omega \varphi(y)^{1+q} \, dy. \tag{56}$$

Notice that $\|a\|_{L^\infty(\omega_r)}$ is bounded by $\|a\|_{L^\infty(U)}$. We choose r such that

$$\frac{h}{r^2} = \left(\frac{h}{r^N}\right)^{\frac{1+q}{2}} r^N,$$

that is $r = h^{\frac{1-q}{4+N(1-q)}}$. Then we get

$$\lambda_1(h) \leq \left(\int_\omega |\nabla\varphi(y)|^2 \, dy + \|a\|_{L^\infty(U)} \int_\omega \varphi(y)^{1+q} \, dy \right) h^{\frac{4+(N-2)(1-q)}{4+N(1-q)}}.$$

It remains to prove that $\omega_r \subset U$. We define $r_0 = \sup\{r > 0, \omega_r \subset U\}$, and $h_0 = r_0^{\frac{4+N(1-q)}{1-q}}$.Then for $h \leq h_0$, $r \leq r_0$ and $\omega_r \subset U$. Notice that r_0 depends on ω and the position of O with respect to ∂U. $\square$

Remark 3.3. If U is unbounded (when Ω is unbounded), r_0 could be infinite but (55) still remains valid for all $h > 0$.

Theorem 3.4. *Let Ω be a domain of R^N, $N \geq 1$, assume* (54) *and let $q \in (-1, 1)$. Let O be any point of U. Let B be the unit-ball of R^N and let $\lambda_{1,B}$ be the first eigenvalue of $-\Delta$ in $H_0^1(B)$. Then*

$$\lambda_1(h) \leq \left(\lambda_{1,B} + \|a\|_{L^\infty(U)}\, \omega_N^{\frac{1-q}{2}}\right) h^{\frac{4+(N-2)(1-q)}{4+N(1-q)}}, \tag{57}$$

for all $0 < h \leq h_0$, *where*

$$h_0 = d(O, \partial U)^{\frac{4+N(1-q)}{1-q}}. \tag{58}$$

Proof. We apply the previous lemma by taking as φ the first eigenfunction of $-\Delta$ in $H_0^1(B)$ with $\|\varphi\|_{L^2(B)} = 1$. So,

$$\int_B |\nabla\varphi|^2 \, dx = \lambda_{1,B}, \tag{59}$$

and by Hölder's inequality,

$$\int_B \varphi^{1+q} \, dx \leq \left(\int_B \varphi^2 \, dx\right)^{\frac{1+q}{2}} \operatorname{meas}(B)^{\frac{1-q}{2}} = \omega_N^{\frac{1-q}{2}}. \tag{60}$$

h_0 is given by the lemma. □

Remark 3.5. It is possible to obtain better upper bounds by following the methods developed in [11] or [9].

Theorem 3.6. *Let Ω be a bounded regular domain of R^N, $N \geq 1$, and let a be a nonnegative measurable function on Ω. Assume that*

$$\textit{there exists a sequence } U_n \subset \Omega \textit{ such that} \tag{61}$$

$$\|a\|_{L^\infty(U_n)} \to 0, \; n \to +\infty.$$

Then,

$$\lim_{h\to 0} h^{-\frac{2(1+q)+(1-q)N)}{4}} \lambda_1(h)^{\frac{4+(1-q)N}{4}} = 0. \tag{62}$$

Proof. Let $\varepsilon > 0$. We choose a ball B_r with a radius r large enough such that the first eigenvalue $\lambda_{1,B_r} \leq \varepsilon$. We denote by φ the positive first eigenfunction of $-\Delta$ in $H_0^1(B_r)$ with $\|\varphi\|_{L^2(B_r)} = 1$. Lemma 3.1 yields

$$\lambda_1(h) \leq \left(\varepsilon + \|a\|_{L^\infty(U_n)} \int_{B_r} \varphi^{1+q}(x) \, dx\right) h^{\frac{2(1+q)+(1-q)N}{4+(1-q)N}}.$$

for n in $\mathbb{N}$ and h small enough. By Hölder's inequality,

$$\lambda_1(h) \leq \left(\varepsilon + \|a\|_{L^\infty(U_n)} \operatorname{meas}(B_r)^{1-q}\right) h^{\frac{2(1+q)+(1-q)N}{4+(1-q)N}}.$$

We fix now n large enough such that

$$\|a\|_{L^\infty(U_n)}\,\mathrm{meas}(B_r)^{1-q} < \varepsilon.$$

For that n and for h small enough we have

$$0 \le h^{-\frac{2(1+q)+(1-q)N}{4+(1-q)N}}\lambda_1(h) \le 2\varepsilon,$$

which implies

$$\lim_{h\to 0} h^{-\frac{2(1+q)+(1-q)N}{4+(1-q)N}}\lambda_1(h) = 0,$$

and the proof is complete. □

Remark 3.7. (61) is equivalent to the following condition: "the function a has an almost nonempty interior" (see [9]).

3.2. Sharp estimate on the constants in the regularizing effects

In this subsection we assume $N \ge 3$. We denote by $C(N,\Omega)$ the best constant in the Sobolev-Gagliardo-Nirenberg injection, i.e.,

$$C(N,\Omega) = \sup\{C>0:\ \forall\psi\in H_0^1(\Omega),\ \|\nabla\psi\|^2_{L^2(\Omega)} \ge C\ \|\psi\|^2_{L^{2^*}(\Omega)}\}, \tag{63}$$

where

$$\frac{1}{2^*} = \frac{1}{2} - \frac{1}{N}. \tag{64}$$

Lemma 3.8. *Let Ω be a C^1 bounded domain of R^N, $N\ge 3$ and let a be a bounded measurable function on Ω. Let v_h be defined by (53) for $h>0$. Let $\theta\in[1,\frac{2^*}{2}]$ and γ_0 be real numbers such that $\frac{\gamma_0}{\theta} > \frac{3}{2}$. We denote by $(\gamma_n)_{n\ge 0}$ the sequence $\gamma_n = \gamma_0\left(\frac{2^*}{2\theta}\right)^n$. Then for all $n\ge 1$,*

$$\|v_h\|_{L^{\gamma_n}(\Omega)} \le \left[\prod_{k=0}^{n-1}\left(\frac{\gamma_k^2}{\gamma_k-\theta}\right)^{\frac{\theta}{\gamma_k}}\right]\left(\frac{m(\theta)}{4\theta\, C(N,\Omega)}\right)^{\sum_{k=0}^{n-1}\frac{\theta}{\gamma_k}}\|v_h\|_{L^{\gamma_0}(\Omega)}, \tag{65}$$

where

$$m(\theta) = \left(\int_{\{x:\nu(h)>\frac{1+q}{2}\frac{a(x)}{v_h^{1-q}}\}}\left(\nu(h) - \frac{1+q}{2}\frac{a(x)}{v_h^{1-q}}\right)^{\frac{\theta}{\theta-1}} dx\right)^{\frac{\theta-1}{\theta}}, \tag{66}$$

for $\theta\in(1,\frac{2^*}{2}]$, with $\nu(h)$ given in (53) and

$$m(1) = \nu(h). \tag{67}$$

Proof. v_h^r belongs to $H_0^1(\Omega)$ for $r>\frac{1}{2}$ so

$$r\int_\Omega |\nabla v_h|^2 v_h^{r-1}\,dx + \int_\Omega \frac{1+q}{2}a(x)v_h^{q+r}\,dx = \nu(h)\int_\Omega v_h^{r+1}\,dx,$$

which yields

$$\frac{4r}{(r+1)^2}\int_\Omega |\nabla(v_h^{\frac{r+1}{2}})|^2\,dx+\int_\Omega \frac{1+q}{2}a(x)v_h^{q+r}\,dx=\nu(h)\int_\Omega v_h^{r+1}\,dx.$$

Using the Gagliardo-Nirenberg-Sobolev injections in the left-hand side it gives

$$\frac{4r}{(r+1)^2}C(N,\Omega)\,\|v_h\|^{r+1}_{L^{(r+1)\frac{2^*}{2}}(\Omega)}\le\int_{\{x:\nu(h)>\frac{1+q}{2}\,\frac{a(x)}{v_h^{1-q}}\}} v_h^{r+1}\left(\nu(h)-\frac{1+q}{2}\,\frac{a(x)}{v_h^{1-q}}\right)\,dx.$$

By Hölder's inequality,

$$\frac{4r}{(r+1)^2}C(N,\Omega)\,\|v_h\|^{r+1}_{L^{(r+1)\frac{2^*}{2}}(\Omega)}\le\left(\int_{\{x:\nu(h)>\frac{1+q}{2}\,\frac{a(x)}{v_h^{1-q}}\}} v_h^{(r+1)\theta}\,dx\right)^{\frac{1}{\theta}} m(\theta). \tag{68}$$

Consequently,

$$\|v_h\|^{r+1}_{L^{(r+1)\frac{2^*}{2}}(\Omega)}\le\frac{(r+1)^2}{4r}\frac{m(\theta)}{C(N,\Omega)}\,\|v_h\|^{r+1}_{L^{(r+1)\theta}(\Omega)}\,. \tag{69}$$

If $\gamma_{n-1}=(r+1)\theta$ then $\gamma_n=(r+1)\frac{2^*}{2}$ and we can rewrite (69) under the form

$$\|v_h\|^{\frac{\gamma_{n-1}}{\theta}}_{L^{\gamma_n}(\Omega)}\le\frac{\left(\frac{\gamma_{n-1}}{\theta}\right)^2}{\left(\frac{\gamma_{n-1}}{\theta}-1\right)}\frac{m(\theta)}{4\,C(N,\Omega)}\,\|v_h\|^{\frac{\gamma_{n-1}}{\theta}}_{L^{\gamma_{n-1}}(\Omega)}\,.$$

The choice $\gamma_{n-1}=(r+1)\theta$ is possible since $\gamma_0>\frac{3}{2}\theta$. By iteration, we get the result. □

Lemma 3.9. *Under the same assumptions than in the previous lemma, for* $n\ge 1$ *and* $\theta\in[1,\frac{2^*}{2})$ *we have*

$$\sum_{k=0}^{n-1}\frac{\theta}{\gamma_k}=\frac{\theta}{\gamma_0}\frac{1-\left(\frac{2\theta}{2^*}\right)^n}{1-\frac{2\theta}{2^*}}=\frac{N\theta}{\gamma_0}\frac{1-\left(\frac{(N-2)\theta}{N}\right)^n}{N-\theta(N-2)}, \tag{70}$$

$$\sum_{k=0}^{n-1}k\left(\frac{2\theta}{2^*}\right)^k=\frac{\frac{2\theta}{2^*}}{\left(1-\frac{2\theta}{2^*}\right)^2}+\frac{\frac{2\theta}{2^*}}{\left(1-\frac{2\theta}{2^*}\right)^2}\left(-n\left(\frac{2\theta}{2^*}\right)^{n-1}+(n-1)\left(\frac{2\theta}{2^*}\right)^n\right) \tag{71}$$

$$=\frac{N(N-2)\theta}{(N-\theta(N-2))^2}+\frac{N(N-2)\theta}{(N-\theta(N-2))^2}\left(-n\left(\frac{(N-2)\theta}{N}\right)^{n-1}+(n-1)\left(\frac{(N-2)\theta}{N}\right)^n\right),$$

$$\sum_{k=0}^{n-1}\left(\frac{2\theta}{2^*}\right)^{2k}=\frac{1-\left(\frac{2\theta}{2^*}\right)^{2n}}{1-\left(\frac{2\theta}{2^*}\right)^2}=\frac{N}{N^2-\theta^2(N-2)^2}\left(1-\left(\frac{\theta(N-2)}{N}\right)^{2n}\right), \tag{72}$$

$$\prod_{k=0}^{n-1}\left(\frac{\gamma_k^2}{\gamma_k-\theta}\right)^{\frac{\theta}{\gamma_k}}\le\gamma_0^{\frac{N\theta}{\gamma_0}\frac{1-\left(\frac{(N-2)\theta}{N}\right)^n}{N-\theta(N-2)}}\left(\frac{(N-2)\theta}{N}\right)^{\frac{\theta}{\gamma_0}\sum_{k=0}^{n-1}k\left(\frac{2\theta}{2^*}\right)^k} \tag{73}$$

$$\left(\frac{\gamma_0}{\gamma_0-\theta}\right)^{\frac{\theta}{\gamma_0}\frac{N}{N^2-\theta^2(N-2)^2}\left(1-\left(\frac{\theta(N-2)}{N}\right)^{2n}\right)}.$$

Proof. (71) comes from the equality

$$\sum_{k=0}^{n-1} kx^k = \frac{x}{(1-x)^2} + \frac{x}{(1-x)^2}\left(-nx^{n-1} + (n-1)x^n\right),$$

for all $x \neq 1$.

$$\prod_{k=0}^{n-1}\left(\frac{\gamma_k^2}{\gamma_k-\theta}\right)^{\frac{\theta}{\gamma_k}} = \exp\left(\sum_{k=0}^{n-1}\frac{\theta}{\gamma_0}\left(\frac{2\theta}{2^*}\right)^k\left(\ln\gamma_k - \ln\left(1-\frac{\theta}{\gamma_k}\right)\right)\right).$$

Now, we use the concavity of ln. Fix $k \geq 0$. We find α such that

$$1-\frac{\theta}{\gamma_k} = \alpha + (1-\alpha)\left(1-\frac{\theta}{\gamma_0}\right),$$

i.e.,

$$\alpha = 1-\left(\frac{2\theta}{2^*}\right)^k.$$

Then,

$$\ln\left(1-\frac{\theta}{\gamma_k}\right) \geq \left(\frac{2\theta}{2^*}\right)^k \ln\left(1-\frac{\theta}{\gamma_0}\right).$$

Hence,

$$\begin{aligned}
\prod_{k=0}^{n-1}\left(\frac{\gamma_k^2}{\gamma_k-\theta}\right)^{\frac{\theta}{\gamma_k}} &\leq \exp\left(\sum_{k=0}^{n-1}\frac{\theta}{\gamma_0}\left(\frac{2\theta}{2^*}\right)^k\left(\ln\gamma_k - \left(\frac{2\theta}{2^*}\right)^k\ln\left(1-\frac{\theta}{\gamma_0}\right)\right)\right)\\
&\leq \exp\left(\frac{\theta}{\gamma_0}(\ln\gamma_0)\sum_{k=0}^{n-1}\left(\frac{2\theta}{2^*}\right)^k + \frac{\theta}{\gamma_0}\ln\left(\frac{2\theta}{2^*}\right)\sum_{k=0}^{n-1}k\left(\frac{2\theta}{2^*}\right)^k\right.\\
&\qquad\left.-\frac{\theta}{\gamma_0}\ln\left(1-\frac{\theta}{\gamma_0}\right)\sum_{k=0}^{n-1}\left(\frac{2\theta}{2^*}\right)^{2k}\right)\\
&= \gamma_0^{\frac{N\theta}{\gamma_0}\frac{1-\left(\frac{(N-2)\theta}{N}\right)^n}{N-\theta(N-2)}}\left(\frac{(N-2)\theta}{N}\right)^{\frac{\theta}{\gamma_0}\sum_{k=0}^{n-1}k\left(\frac{2\theta}{2^*}\right)^k}\\
&\qquad\left(\frac{\gamma_0}{\gamma_0-\theta}\right)^{\frac{\theta}{\gamma_0}\frac{N}{N^2-\theta^2(N-2)^2}\left(1-\left(\frac{\theta(N-2)}{N}\right)^{2n}\right)},
\end{aligned}$$

which completes the proof. $\square$

Now we shall use the above lemmas to get a sharper estimate.

Proposition 3.10. *Let Ω be a C^1 bounded domain of R^N, $N \geq 3$ and assume that a is bounded, nonnegative and measurable on Ω. Let v_h be defined by (53) for $h > 0$. Then,*

$$\|v_h\|_{L^\infty(\Omega)} \leq \left(\frac{\nu(h)}{C(N,\Omega)}\right)^{\frac{N}{4}} K(N) \|v_h\|_{L^2(\Omega)}, \tag{74}$$

with

$$K(N) = 2^{\frac{-(N-2)}{4}} \left(\frac{N-2}{N}\right)^{\frac{(N-2)(N-4)}{8}} \left(\frac{2N}{N+2}\right)^{\frac{N-2}{8(N-1)}}. \tag{75}$$

Proof. We take $\theta = 1$, $\gamma_0 = 2$ and so $\gamma_1 = 2^*$. Then

$$\|v_h\|_{L^{2^*}(\Omega)} \leq \left(\frac{\nu(h)}{C(N,\Omega)}\right)^{\frac{1}{2}} \|v_h\|_{L^2(\Omega)}.$$

We use now (65), always for $\theta = 1$ but with $\gamma_0 = 2^*$. Letting n to infinity it gives

$$\|v_h\|_{L^\infty(\Omega)} \leq \left[\prod_{k=0}^{+\infty} \left(\frac{\gamma_k^2}{\gamma_k - 1}\right)^{\frac{1}{\gamma_k}}\right] \left(\frac{\nu(h)}{4C(N,\Omega)}\right)^{\sum_{k=0}^{+\infty} \frac{1}{\gamma_k}} \|v_h\|_{L^{2^*}(\Omega)},$$

with

$$\sum_{k=0}^{+\infty} \frac{1}{\gamma_k} = \frac{N}{2.2^*} = \frac{N-2}{4},$$

and

$$\prod_{k=0}^{+\infty} \left(\frac{\gamma_k^2}{\gamma_k - 1}\right)^{\frac{1}{\gamma_k}} \leq (2^*)^{\frac{N-2}{4}} \left(\frac{N-2}{N}\right)^{\frac{1}{2^*}\frac{N(N-2)}{4}} \left(\frac{2^*}{2^*-1}\right)^{\frac{1}{2^*}\frac{N}{4(N-1)}},$$

i.e.,

$$\prod_{k=0}^{+\infty} \left(\frac{\gamma_k^2}{\gamma_k - 1}\right)^{\frac{1}{\gamma_k}} \leq 2^{\frac{N-2}{4}} \left(\frac{N-2}{N}\right)^{\frac{(N-2)(N-4)}{8}} \left(\frac{2N}{N+2}\right)^{\frac{N-2}{8(N-1)}}.$$

Hence,

$$\|v_h\|_{L^\infty(\Omega)} \leq 2^{\frac{N-2}{4}} \left(\frac{N-2}{N}\right)^{\frac{(N-2)(N-4)}{8}} \left(\frac{2N}{N+2}\right)^{\frac{N-2}{8(N-1)}} \left(\frac{\nu(h)}{4C(N,\Omega)}\right)^{\frac{N-2}{4}} \|v_h\|_{L^{2^*}(\Omega)}, \tag{76}$$

which yields

$$\|v_h\|_{L^\infty(\Omega)} \leq 2^{\frac{-(N-2)}{4}} \left(\frac{N-2}{N}\right)^{\frac{(N-2)(N-4)}{8}} \left(\frac{2N}{N+2}\right)^{\frac{N-2}{8(N-1)}} \left(\frac{\nu(h)}{C(N,\Omega)}\right)^{\frac{N}{4}} \|v_h\|_{L^2(\Omega)},$$

and the proof is complete. □

Remark 3.11. A simpler expression is provided in the radial case. It seems also possible to have an estimate of $\lambda_1(h)$ for high order operators [13]. It depends only on an elliptic equation related to the parabolic one.

3.3. Sufficient condition for the extinction in finite time

The key-stone of this section is the following :

Corollary 3.12. *If $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ with $a(x)u_0^q\chi_{u_0>0} \in L^2(\Omega)$ and if we assume (10) then the solution of (9) satisfies the finite extinction time property and*

$$T(u_0) \leq \frac{1}{2}\int_0^{\|u_0\|^2} \frac{dh}{\lambda_1(h)}. \tag{77}$$

Proof. It is a direct application of Theorem 2.11. □

Now, it suffices to estimate $\lambda_1(h)$. The first step is to find a lower bound depending on v_h.

Lemma 3.13. *Assume that a is a bounded nonnegative measurable function and that Ω is a C^1 bounded domain of R^N, $N \geq 3$. Then, for all $h > 0$,*

$$(C(N,\Omega))^{\frac{N}{2}} \leq \int_{\{x:v_h>0\}\cap\{x:\lambda_1(h)\, v_h^{1-q}>h\, a(x)\}} \left(\frac{\lambda_1(h)}{h} - \frac{a(x)}{v_h^{1-q}}\right)^{\frac{N}{2}} dx. \tag{78}$$

Proof. We know that the infimum is reached by a nonnegative function v_h so

$$\lambda_1(h) = \int_\Omega |\nabla v_h|^2 + a(x)v_h^{1+q}\, dx.$$

The main difference with respect the case $q \in (0,1)$ (considered in [11]) is that v_h may vanish on a subset of positive measure of Ω if $q < 0$ (see [25] and [26]). Hence

$$\int_\Omega |\nabla v_h|^2 = \int_{\{x:v_h>0\}} v_h^2 \left(\frac{\lambda_1(h)}{h} - \frac{a(x)}{v_h^{1-q}}\right) dx,$$

since $\|v_h\|^2_{L^2(\Omega)} = h$. On the left-hand side, we can use the Sobolev-Gagliardo-Nirenberg continuous injection and in the right-hand side, the Hölder's inequality to get

$$C(N,\Omega) \leq \left(\int_{\{x:v_h>0\}\cap\{x:\frac{\lambda_1(h)}{h}>\frac{a(x)}{v_h^{1-q}}\}} \left(\frac{\lambda_1(h)}{h} - \frac{a(x)}{v_h^{1-q}}\right)^{\frac{N}{2}} dx\right)^{\frac{2}{N}},$$

which completes the proof. □

As a second step, thanks to the regularizing effects of the equation, we can find a lower bound for $\lambda_1(h)$ which does not depend on v_h.

Proposition 3.14. *Under the assumption of the previous Lemma,*

$$(C(N,\Omega))^{\frac{N}{2}} \leq \left(\frac{\lambda_1(h)}{h}\right)^{\frac{N}{2}} \tag{79}$$

$$meas\left(\{x : v_h > 0\} \cap \left\{x : \frac{\lambda_1(h)^{\frac{4+(1-q)N}{4}}}{h^{\frac{2(1+q)+(1-q)N}{4}}}\, \frac{K(N)^{1-q}}{(C(N,\Omega))^{\frac{(1-q)N}{4}}} > a(x)\right\}\right).$$

Proof. We shall simplify the lower bound (78). It is clear that

$$(C(N,\Omega))^{\frac{N}{2}} \leq \left(\frac{\lambda_1(h)}{h}\right)^{\frac{N}{2}} \operatorname{meas}(\{x : v_h > 0\} \cap \{x : \lambda_1(h)\ v_h^{1-q} > h\ a(x)\}).$$

Since $q \neq 1$, v_h appears in the inequality, but the set $\{x : \lambda_1(h)\ v_h^{1-q} > h\ a(x)\}$ is included in the set $\{x : \lambda_1(h)\ \|v_h\|_{L^\infty(\Omega)}^{1-q} > h\ a(x)\}$. Hence,

$$(C(N,\Omega))^{\frac{N}{2}} \leq \left(\frac{\lambda_1(h)}{h}\right)^{\frac{N}{2}} \operatorname{meas}(\{x : v_h > 0\} \cap \{x : \lambda_1(h)\ \|v_h\|_{L^\infty(\Omega)}^{1-q} > h\ a(x)\}).$$

As shown in Proposition 3.10

$$\|v_h\|_{L^\infty(\Omega)}^{1-q} \leq \left(\frac{\nu(h)}{C(N,\Omega)}\right)^{\frac{(1-q)N}{4}} K(N)^{1-q} h^{\frac{1-q}{2}},$$

since $\|v_h\|_{L^2(\Omega)}^{1-q} = h^{\frac{1-q}{2}}$. Taking $w = v_h$ in (53) yields,

$$h\nu(h) = \int_\Omega |\nabla v_h|^2 + \frac{1+q}{2} a(x) v_h^{1+q}\, dx \leq \lambda_1(h).$$

Therefore, we obtain

$$(C(N,\Omega))^{\frac{N}{2}} \leq \left(\frac{\lambda_1(h)}{h}\right)^{\frac{N}{2}}$$

$$\operatorname{meas}\left(\{x : v_h > 0\} \cap \left\{x : \lambda_1(h)\ \left(\frac{\lambda_1(h)}{C(N,\Omega)h}\right)^{\frac{(1-q)N}{4}} K(N)^{1-q} > h^{\frac{1+q}{2}}\ a(x)\right\}\right),$$

which leads to (79). □

Remark 3.15. We can also express a lower bound for $\lambda_1(h)$ thank to the increasing rearrangement (see [33] and [22]).

The following result gives a sufficient condition for the existence of a finite extinction time in the spirit of Theorem 3.1 in [11].

Theorem 3.16. *Let $N \geq 3$ and let Ω be a C^1 bounded domain of R^N. Assume $-1 < q < 1$ and that function a is measurable, positive a.e. and bounded on Ω such that*

$$\left|\ln \frac{1}{a}\right|^s \in L^1(\Omega), \tag{80}$$

for some $s > N/2$. Then, given $u_0 \in L^2(\Omega)$ such that $u_0(x) \geq 0$ a.e. $x \in \Omega$ the solution of (9) satisfies the property of the finite time, i.e. $T(u_0) < +\infty$. Moreover, for any $M \geq \|a\|_{L^\infty(\Omega)}$ such that

$$\frac{\lambda_1(h)^{\frac{4+(1-q)N}{4}}}{h^{\frac{2(1+q)+(1-q)N}{4}}} \frac{K(N)^{1-q}}{(C(N,\Omega))^{\frac{(1-q)N}{4}}} < M, \quad \forall h \in \left(0, \|u_0\|_{L^2(\Omega)}^2\right],$$

we have the estimate

$$T(u_0) \leq \frac{1}{2} \int_0^{\|u_0\|^2_{L^\infty(\Omega)}} \frac{dh}{f(h)}, \tag{81}$$

where

$$f(h) = h \left((\gamma - \beta) \ln h - \ln K - C \frac{2\beta}{\alpha} \ln \left((\gamma - \beta) \ln \left(\frac{h}{\|u_0\|^2_{L^\infty(\Omega)}} \right) + x_0 \right) \right)^{\frac{2}{\alpha}}$$

$$\alpha = \frac{N}{s}, \qquad \beta = \frac{4 + (1-q)N}{4}, \qquad \gamma = \frac{2(1+q) + (1-q)N}{4},$$

$$C = C(N,\Omega)^{\frac{N}{2s}} \left(\int_\Omega \left(\ln \frac{M}{a(x)} \right)^s dx \right)^{\frac{1}{s}}, \qquad K = \frac{K(N)^{1-q}}{M\left(C(N,\Omega)\right)^{\frac{(1-q)N}{4}}},$$

and

$$x_0 + C \frac{2\beta}{\alpha} \ln x_0 = (\gamma - \beta) \ln \left(\|u_0\|^2_{L^2(\Omega)} \right) - \ln K.$$

We point out that although the condition $\left|\ln \frac{1}{a}\right|^s \in L^1(\Omega)$ for the extinction in a finite time was already been proved in [11] the estimate given in Theorem 3.16 is new.

Remark 3.17. If $\|u_0\|_{L^2(\Omega)}$ is small enough, we can take $M = \max(\|a\|_{L^\infty(\Omega)}, \lambda_{1,B} + \|a\|_{L^\infty(\Omega)} \omega_N^{\frac{1-q}{2}})$ where $\lambda_{1,B}$ is the first eigenvalue of $-\Delta$ in $H_0^1(B)$ with B unit-ball of R^N. Generally speaking, M depends on the L^∞-norm of a and the shape of Ω.

For the proof of Theorem 3.16 we shall need two lemmas.

Lemma 3.18. *Let x, y, x_0, δ be four positive real numbers such that $x \geq x_0$ and*

$$y = x + \delta \ln x.n$$

Then

$$x \geq y - \delta \ln \left(y - \delta \ln x_0 \right).$$

Proof. $x \geq x_0$ implies that $y \geq x + \delta \ln x_0$ and so $x \leq y - \delta \ln x_0$. We conclude with $y \leq x + \delta \ln \left(y - \ln x_0 \right)$. □

Lemma 3.19. *Let h_0, α, β, γ, C and K be five positive real numbers such that $\gamma < \beta$ and $\alpha < 2$. Let $h \mapsto \lambda_1(h)$ be a measurable function defined on $(0, h_0]$ with values in $(0, +\infty)$ such that for all $h \in (0, h_0]$,*

$$\gamma \ln h - \beta \ln \lambda_1(h) - \ln K > 0,$$

and

$$\lambda_1(h) \geq h \, C^{\frac{2}{\alpha}} \left(\gamma \ln h - \beta \ln \lambda_1(h) - \ln K \right)^{\frac{2}{\alpha}}.$$

Then $\int_0^{h_0} \frac{dh}{\lambda_1(h)}$ *converges and*

$$\begin{aligned}&\frac{1}{2}\int_0^{h_0} \frac{dh}{\lambda_1(h)}\\ &\leq \frac{1}{2}\int_0^{h_0} \frac{dh}{h\left((\gamma-\beta)\ln h - \ln K - C\frac{2\beta}{\alpha}\ln\left((\gamma-\beta)\ln h - \ln K - C\frac{2\beta}{\alpha}\ln x_0\right)\right)^{\frac{2}{\alpha}}},\end{aligned}$$

with

$$x_0 + C\frac{2\beta}{\alpha}\ln x_0 = (\gamma-\beta)\ln h_0 - \ln K.$$

Proof. If $X(h) = \lambda_1(h)^{\frac{\alpha}{2}}$ and $Y(h) = h^{\frac{\alpha}{2}}$,

$$X(h) \geq Y(h)\, C\, \left(\frac{2\gamma}{\alpha}\ln Y(h) - \frac{2\beta}{\alpha}\ln X(h) - \ln K\right),$$

which gives

$$\frac{X(h)}{Y(h)} + C\frac{2\beta}{\alpha}\ln\frac{X(h)}{Y(h)} \geq \frac{2(\gamma-\beta)}{\alpha}\ln Y(h) - \ln K.$$

$Y(h) = h^{\frac{\alpha}{2}}$ tends to zero when $h \to 0$ so $\frac{2(\gamma-\beta)}{\alpha}\ln Y(h)$ tends to $+\infty$ since $\gamma < \beta$ and so $\frac{X(h)}{Y(h)}$. Asymptotically, there exists a positive constant C' such that for h small enough,

$$\frac{X(h)}{Y(h)} \geq C' \ln Y(h),$$

or equivalently,

$$\lambda_1(h) \geq C''\, h\,(-\ln h)^{\frac{2}{\alpha}}, \quad C'' > 0,$$

which implies that $\int_0^{h_0} \frac{dh}{\lambda_1(h)}$ converges since $\alpha < 2$. Now, we estimate the integral. For

$$y(h) = \frac{2(\gamma-\beta)}{\alpha}\ln Y(h) - \ln K,$$

we have an $x(h) > 0$ such that $x(h) + C\frac{2\beta}{\alpha}\ln x(h) = y(h)$ since $x \mapsto x + C\frac{2\beta}{\alpha}\ln x$ is an increasing continuous function. Moreover, $\frac{X(h)}{Y(h)} \geq x(h)$. For

$$x_0 = \inf\{x(h), h \leq h_0\} = x(h_0),$$

(notice that $h \mapsto x(h)$ is a continuous increasing function which tends to $+\infty$ when $h \to 0$) i.e.,

$$x_0 + C\frac{2\beta}{\alpha}\ln x_0 = (\gamma-\beta)\ln h_0 - \ln K,$$

we obtain from Lemma 3.18,

$$x(h) \geq y(h) - C\frac{2\beta}{\alpha}\ln\left(y(h) - C\frac{2\beta}{\alpha}\ln x_0\right),$$

which yields

$$\frac{X(h)}{Y(h)} \geq \frac{2(\gamma-\beta)}{\alpha} \ln Y(h) - \ln K - C\frac{2\beta}{\alpha} \ln\left(\frac{2(\gamma-\beta)}{\alpha} \ln Y(h) - \ln K - C\frac{2\beta}{\alpha} \ln x_0\right).$$

Finally,

$$\lambda_1(h) \geq h\left((\gamma-\beta)\ln h - \ln K - C\frac{2\beta}{\alpha}\ln\left((\gamma-\beta)\ln h - \ln K - C\frac{2\beta}{\alpha}\ln x_0\right)\right)^{\frac{2}{\alpha}},$$

and the proof is complete. $\square$

Proof of Theorem 3.16. By Lemma 3.1, such a positive number M always exists under our assumptions (such a M will be necessary to ensure that all the quantities involved here are strictly greater that 1). We start with (79).

$$(C(N,\Omega))^{\frac{N}{2}} \leq \left(\frac{\lambda_1(h)}{h}\right)^{\frac{N}{2}}$$

$$\operatorname{meas}\left(\left\{x : \left(\ln \frac{M}{\left(\frac{\lambda_1(h)^{\frac{4+(1-q)N}{4}}}{h^{\frac{2(1+q)+(1-q)N}{4}}} \frac{K(N)^{1-q}}{(C(N,\Omega))^{\frac{(1-q)N}{4}}}\right)}\right)^s < \left(\ln\frac{M}{a(x)}\right)^s\right\}\right).$$

We use the following property: for all nonnegative $b \in L^1(\Omega)$ and for all $t > 0$,

$$\operatorname{meas}(\{x : b(x) \geq t\}) \leq \frac{1}{t}\int_{\{x:b(x)\geq t\}} b(x)\,dx \leq \frac{1}{t}\int_\Omega b(x)\,dx. \tag{82}$$

Then,

$$(C(N,\Omega))^{\frac{N}{2}} \leq \left(\frac{\lambda_1(h)}{h}\right)^{\frac{N}{2}}$$

$$\left(\ln \frac{M}{\left(\frac{\lambda_1(h)^{\frac{4+(1-q)N}{4}}}{h^{\frac{2(1+q)+(1-q)N}{4}}} \frac{K(N)^{1-q}}{(C(N,\Omega))^{\frac{(1-q)N}{4}}}\right)}\right)^{-s} \left(\int_\Omega \left(\ln\frac{M}{a(x)}\right)^s dx\right),$$

which gives $\frac{\lambda_1(h)}{h} \geq C(N,\Omega)$

$$\left(-\ln\left(\frac{\lambda_1(h)^{\frac{4+(1-q)N}{4}}}{h^{\frac{2(1+q)+(1-q)N}{4}}} \frac{K(N)^{1-q}}{M\,(C(N,\Omega))^{\frac{(1-q)N}{4}}}\right)\right)^{\frac{2s}{N}} \left(\int_\Omega \left(\ln\frac{M}{a(x)}\right)^s dx\right)^{\frac{2}{N}}.$$

So,

$$\lambda_1(h) \geq h\,C(N,\Omega)\left(\int_\Omega \left(\ln\frac{M}{a(x)}\right)^s dx\right)^{\frac{2}{N}}$$

$$\left(\left(\frac{2(1+q)+(1-q)N}{4}\right)\ln h - \left(\frac{4+(1-q)N}{4}\right)\ln\lambda_1(h) - \ln\left(\frac{K(N)^{1-q}}{M\,(C(N,\Omega))^{\frac{(1-q)N}{4}}}\right)\right)^{\frac{2s}{N}}.$$

M is such that the quantity above in the parenthesis is always positive. $\alpha = \frac{N}{s} < 2$ and $\beta > \gamma$ so by Lemma 3.9, we get the conclusion. $\square$

Remark 3.20. (82) is actually an estimate of Marcinkiewicz type (see [11], Remark 3.1). One can also consider it as an application of the Bienaimé-Chebycheff inequality for the random variable b with probability measure $P(E) = \frac{\text{meas}(E)}{\text{meas}(\Omega)}$.

3.4. Improvement estimates on the extinction time

As we have seen, the main difficulty is to estimate $\text{meas}\{x : a(x) < t\}$ for all $t > 0$, but this task is specially easy if the function a is assumed to be radially continuous and increasing so that, $\text{meas}\{x : a(x) < t\} = \omega_N t^N$ for t small enough. We give two examples of this type.

Corollary 3.21. *Let $N \geq 3$ and let Ω be a C^1 bounded domain of R^N large enough with $O \in \Omega$. For $a(x) = A|x|^\alpha$ with $\alpha > 0$ and $A > 0$, the solutions of (9) vanish in a finite time and the extinction time $T(u_0)$ is estimated by*

$$T(u_0) \leq \frac{1}{2}\left(\frac{K(N)^{4(1-q)}}{A^4\, C(N,\Omega)^{2\alpha+(1-q)N}}\omega_N{}^{\frac{4\alpha}{N}}\right)^{\frac{1}{2\alpha+4+(1-q)N}} \frac{2\alpha+4+(1-q)N}{2(1-q)} \tag{83}$$

$$\|u_0\|_{L^2(\Omega)}^{\frac{2(1-q)}{2\alpha+4+(1-q)N}}.$$

Proof. We drop the set $\{x : v_h > 0\}$, more precisely, it is included in Ω. From Theorem 3.6 $\frac{\lambda_1(h)^{\frac{4+(1-q)N}{4}}}{h^{\frac{2(1+q)+(1-q)N}{4}}}$ tends to zero when $h \to 0$. We have

$$C(N,\Omega) \leq \frac{\lambda_1(h)}{h}\omega_N^{\frac{2}{N}}\left(\frac{\lambda_1(h)^{\frac{4+(1-q)N}{4}}}{h^{\frac{2(1+q)+(1-q)N}{4}}}\,\frac{K(N)^{1-q}}{A\ (C(N,\Omega))^{\frac{(1-q)N}{4}}}\right)^{\frac{2}{\alpha}},$$

$$A^{\frac{2}{\alpha}}\frac{C(N,\Omega)^{1+\frac{(1-q)N}{2\alpha}}}{K(N)^{\frac{2(1-q)}{\alpha}}}\omega_N^{\frac{-2}{N}} \leq \frac{\lambda_1(h)^{1+\frac{1}{\alpha}(\frac{4+(1-q)N}{2})}}{h^{1+\frac{1}{\alpha}(\frac{2(1+q)+(1-q)N}{2})}},$$

$$A^{\frac{2}{\alpha}}\frac{C(N,\Omega)^{\frac{2\alpha+(1-q)N}{2\alpha}}}{K(N)^{\frac{2(1-q)}{\alpha}}}\omega_N^{\frac{-2}{N}} \leq \frac{\lambda_1(h)^{\frac{2\alpha+4+(1-q)N}{2\alpha}}}{h^{\frac{2\alpha+2(1+q)+(1-q)N}{2\alpha}}},$$

which implies that the integral $\int_0^1 \frac{dh}{\lambda_1(h)}$ converges. Indeed,

$$\lambda_1(h) \geq \left(A^4\ \frac{C(N,\Omega)^{2\alpha+(1-q)N}}{K(N)^{4(1-q)}}\omega_N^{\frac{-4\alpha}{N}}\right)^{\frac{1}{2\alpha+4+(1-q)N}} h^{\frac{2\alpha+2(1+q)+(1-q)N}{2\alpha+4+(1-q)N}},$$

gives

$$\frac{2\alpha+2(1+q)+(1-q)N}{2\alpha+4+(1-q)N} < 1.$$

We end the proof by a simple computation. □

Remark 3.22. The assumption "Ω large enough" means that the ball of center O and radius

$$\sup_{0<h\leq\|u_0\|^2_{L^2(\Omega)}}\left(\frac{\lambda_1(h)^{\frac{4+(1-q)N}{4}}}{h^{\frac{2(1+q)+(1-q)N}{4}}}\,\frac{K(N)^{1-q}}{A\ (C(N,\Omega))^{\frac{(1-q)N}{4}}}\right)^{\frac{1}{\alpha}}$$

is a subset of Ω. Otherwise, we have to take into account the boundary of Ω.

Remark 3.23. Notice that when α tends to infinity, the upper bound of $T(u_0)$ also tends to infinity.

Corollary 3.24. *Let $N \geq 3$ and let Ω be a C^1 bounded domain of R^N large enough with $O \in \Omega$. For $a(x) = \exp(\frac{-1}{|x|^\alpha})$ with $\alpha > 0$, the solutions of (9) vanish in a finite time for $\alpha < 2$ and the extinction time T can be estimated by*

$$T \leq \frac{1}{2} \int_0^{\|u_0\|^2} \frac{dh}{h\left((\gamma-\beta)\ln h - \ln K - C\frac{2\beta}{\alpha}\ln\left((\gamma-\beta)\ln\left(\frac{h}{\|u_0\|^2_{L^2(\Omega)}}\right)+x_0\right)\right)^{\frac{2}{\alpha}}}, \tag{84}$$

where

$$\beta = \frac{4+(1-q)N}{4}, \qquad \gamma = \frac{2(1+q)+(1-q)N}{4},$$

$$K = \frac{K(N)^{1-q}}{(C(N,\Omega))^{\frac{(1-q)N}{4}}}, \qquad C = \left(\frac{C(N,\Omega)}{\omega_N^{\frac{2}{N}}}\right)^{\frac{\alpha}{2}},$$

and

$$x_0 + C\frac{2\beta}{\alpha}\ln x_0 = (\gamma-\beta)\ln\left(\|u_0\|^2_{L^2(\Omega)}\right) - \ln K.$$

Proof. From Theorem 3.6, $\frac{\lambda_1(h)^{\frac{4+(1-q)N}{4}}}{h^{\frac{2(1+q)+(1-q)N}{4}}}$ tends to zero when $h \to 0$. From Proposition 3.14, we have for h small enough

$$\lambda_1(h) \geq h\, C^{\frac{2}{\alpha}}\, (\gamma \ln h - \beta \ln \lambda_1(h) - \ln K)^{\frac{2}{\alpha}}.$$

By Lemma 3.19, we get the extinction in finite time for $\alpha < 2$ and the upper bound for the time extinction. "h small enough" means that all the h we consider are such that the ball of center O and radius

$$\frac{1}{(\gamma \ln h - \beta \ln \lambda_1(h) - \ln K)^{\frac{1}{\alpha}}},$$

is included in Ω. By setting $h_0 = \|u_0\|^2_{L^2(\Omega)}$, "$h$ small enough" means that all the balls for $h \leq h_0$ are subsets of Ω. $\square$

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