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Fixed Points of Generalized Conjugations

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Conjugation, or Legendre transformation, is a basic tool in convex analysis, rational mechanics, economics and optimization. It maps a function on a linear topological space into another one, defined in the dual of the linear space by coupling these spaces by means of the duality product.

Generalized conjugation extends classical conjugation to any pair of domains, using an arbitrary coupling function between these spaces. This generalization of conjugation is now being widely used in optimal transportation problems, variational analysis and also optimization.

If the coupled spaces are equal, generalized conjugations define order reversing maps of a family of functions into itself. In this case, it is natural to ask for the existence of fixed points of the conjugation, that is, functions which are equal to their (generalized) conjugated. Here we prove that any generalized *symmetric* conjugation has fixed points. The basic tool of the proof is a variational principle involving the order reversing feature of the conjugation.

As an application of this abstract result, we will extend to real linear topological spaces a fixed-point theorem for Fitzpatrick's functions, previously proved in Banach spaces.

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1. Introduction

Fenchel-Legendre conjugation is a basic tool in convex analysis, classical mechanics and optimization [10, 1]. An extension of this conjugation, proposed by Moreau [8, 9] and known as Generalized Conjugation is now being used in variational analysis and optimal transportation [11, 6, 12, 15, 16]. In this work, using a variational principle, we shall prove existence of fixed points of any generalized (symmetric) conjugation. This result will be used to extend a fixed-point theorem in the family of Fitzpatrick's functions, previously proved in a Banach space setting [14].

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We use the notation $\bar{\mathbb{R}}$ for the extended real numbers:

$$\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}.$$

The family of extended real valued functions on a set E will be denoted by $\bar{\mathbb{R}}^E$. Let E and F be non-empty sets. A coupling function

$$\Phi : E \times F \to \mathbb{R} \tag{1}$$

induces two conjugations, $\mathcal{C}_1^\Phi$ and $\mathcal{C}_2^\Phi$, defined as follows

$$\begin{aligned} \mathcal{C}_1^\Phi &: \bar{\mathbb{R}}^E \to \bar{\mathbb{R}}^F, \qquad & \mathcal{C}_1^\Phi h\,(s) &= \sup_{r\in E}\{\Phi(r,s) - h(r)\} \\ \mathcal{C}_2^\Phi &: \bar{\mathbb{R}}^F \to \bar{\mathbb{R}}^E, \qquad & \mathcal{C}_2^\Phi f(r) &= \sup_{s\in F}\{\Phi(r,s) - f(s)\}. \end{aligned} \tag{2}$$

We refer to [11] for a comprehensive exposition of Generalized Conjugacy.

Whenever $E = F$ in the coupling function (1), both conjugations (with respect to such a coupling function) maps $\bar{\mathbb{R}}^E$ into itself. So, in this case, it does make sense to ask for the existence of fixed points of these conjugations, that is, $h \in \bar{\mathbb{R}}^E$ such that

$$\mathcal{C}_1^\Phi h = h \quad \text{or} \quad \mathcal{C}_2^\Phi h = h.$$

These fixed points will be called *self-conjugated* functions with respect to the coupling function Φ. Note that conjugation is order reversing. This feature of conjugation will allow us to study self-conjugated functions using a variational principle. This approach has already been used in the context of Fitzpatrick functions [14].

A coupling function $\Phi : E \times E \to \mathbb{R}$ is *symmetric* if

$$\Phi(r,s) = \Phi(s,r), \quad \forall\, r,s \in E.$$

Note that in the symmetric case, both conjugations in (2) coincides, that is, $\mathcal{C}_1^\Phi = \mathcal{C}_2^\Phi$. This additional feature makes the problem of finding fixed points more amenable. Surprisingly, the symmetry of the coupling function guarantees existence of self-conjugated functions. From now on, conjugation with respect to a symmetric coupling function Φ will be denoted by $\mathcal{C}^\Phi$ ($\mathcal{C}^\Phi = \mathcal{C}_1^\Phi = \mathcal{C}_2^\Phi$). Our aim is to prove

Theorem 1.1 (Main result). *Let E be a non-empty set and $\Phi : E \times E \to \mathbb{R}$ be symmetric. Take $g \in \bar{\mathbb{R}}^E$.*

1. *If $\mathcal{C}^\Phi g \le g$, then there exists $h \in \bar{\mathbb{R}}^E$ such that*

$$\mathcal{C}^\Phi g \le \mathcal{C}^\Phi h = h \le g.$$

2. *If $g \in \mathcal{C}^\Phi(\bar{\mathbb{R}}^E)$ and $g \le \mathcal{C}^\Phi g$, then there exists $h \in \bar{\mathbb{R}}^E$ such that*

$$g \le \mathcal{C}^\Phi h = h \le \mathcal{C}^\Phi g.$$

In particular, there exists an $h \in \bar{\mathbb{R}}^E$ self-conjugated, that is, $h = \mathcal{C}^\Phi h$.

The manuscript is organized as follows: In Section 2 we give some basic definitions, prove some technical results and our main theorem. In Section 3 we apply the results of Section 2 to the study of non-symmetric conjugations. In Section 4 we use the main result to extend to linear topological spaces a fixed point theorem in Fitzpatrick's family of functions, previously proved in Banach spaces.

2. Proof of the main result

From now on, E is a non-empty set and Φ is a coupling function,

$$\Phi : E \times E \to \mathbb{R}. \tag{3}$$

Both generalized conjugations (as the classical one) are order reversing, that is, for any $h, f \in \bar{\mathbb{R}}^E$,

$$h \leq f \Rightarrow \mathcal{C}_i^\Phi f \leq \mathcal{C}_i^\Phi h, \quad i = 1, 2. \tag{4}$$

Additionally, for any $h \in \bar{\mathbb{R}}^E$,

$$\mathcal{C}_2^\Phi \mathcal{C}_1^\Phi h \leq h, \qquad \mathcal{C}_1^\Phi \mathcal{C}_2^\Phi h \leq h. \tag{5}$$

The indicator function of $A \subset E$, is $\delta_A : E \to \mathbb{R} \cup \{\infty\}$,

$$\delta_A(r) = \begin{cases} 0, & \text{if } r \in A, \\ \infty, & \text{otherwise.} \end{cases} \tag{6}$$

The following technical result will be needed in the sequel.

Lemma 2.1. *For any $h \in \bar{\mathbb{R}}^E$, $r_0 \in E$ and $i \in \{1, 2\}$*

$$\mathcal{C}_i^\Phi h(r_0) \leq h(r_0) \quad \Rightarrow \quad \Phi(r_0, r_0)/2 \leq h(r_0),$$
$$\mathcal{C}_i^\Phi h(r_0) < h(r_0) \quad \Rightarrow \quad \Phi(r_0, r_0)/2 < h(r_0).$$

Proof. If $h(r_0) = \infty$, then trivially $\Phi(r_0, r_0)/2 < h(r_0)$. Now, suppose that $\mathcal{C}_i^\Phi h(r_0) \leq h(r_0) < \infty$. Then, by definition (2)

$$\Phi(r_0, r_0) - h(r_0) \leq \mathcal{C}_i^\Phi h(r_0) \leq h(r_0).$$

Therefore, $\Phi(r_0, r_0) \leq 2h(r_0)$. Analogously, if $\mathcal{C}_i^\Phi h(r_0) < h(r_0)$, then the second inequality in the above equation is strict and $\Phi(r_0, r_0) < 2h(r_0)$. □

To perform our variational analysis, we shall study the family of functions which are greater than their conjugates.

Definition 2.2. $\mathcal{H}^\Phi = \{h \in \bar{\mathbb{R}}^E \mid \mathcal{C}_1^\Phi h \leq h\}$.

Latter on we will see that conjugation with respect to the second variable, $\mathcal{C}_2^\Phi$, could be used to define the same family. Fixed points of a generalized (symmetric) conjugation will be obtained by means of a variational principle, applied on $\mathcal{H}^\Phi$.

Note that $\mathcal{H}^\Phi$ is non-empty since the function $h \equiv \infty$ belongs to $\mathcal{H}^\Phi$. Next, we shall prove existence of minimal elements of $\mathcal{H}^\Phi$. Recall that if the coupling function (3) is symmetric, then both conjugations $\mathcal{C}_1^\Phi$ and $\mathcal{C}_2^\Phi$ are identical and we use the notation $\mathcal{C}^\Phi = \mathcal{C}_1^\Phi = \mathcal{C}_2^\Phi$.

Lemma 2.3. *Suppose that the coupling function $\Phi : E \times E \to \mathbb{R}$ is symmetric. The family $\mathcal{H}^\Phi$ (Def. 2.2) is (downward)* inductively ordered, *i.e., any totally ordered family $\{h_\alpha\}_{\alpha \in \Lambda} \subset \mathcal{H}^\Phi$ has a lower bound in $\mathcal{H}^\Phi$.*

If $g \in \mathcal{H}^\Phi$, that is, $\mathcal{C}^\Phi g \le g$, then there exists a minimal $h \in \mathcal{H}^\Phi$ such that

$$\mathcal{C}^\Phi g \le h \le g.$$

In particular, $\mathcal{H}^\Phi$ has minimal elements.

Proof. Let $\{h_\alpha\}_{\alpha\in\Lambda}$ be a totally ordered subset of $\mathcal{H}^\Phi$.

First we claim that

$$\mathcal{C}^\Phi h_\alpha \le h_\beta, \quad \forall\, \alpha, \beta \in \Lambda.$$

To check this claim, take $\lambda, \mu \in \Lambda$ and suppose that $h_\lambda \le h_\mu$. Since the conjugation reverse the order, $\mathcal{C}^\Phi h_\mu \le \mathcal{C}^\Phi h_\lambda$. As $\mathcal{C}^\Phi h_\lambda \le h_\lambda$ (because $h_\lambda \in \mathcal{H}^\Phi$), we conclude that

$$\mathcal{C}^\Phi h_\mu \le \mathcal{C}^\Phi h_\lambda \le h_\lambda \le h_\mu.$$

Therefore, $\mathcal{C}^\Phi h_\lambda \le h_\mu$ and $\mathcal{C}^\Phi h_\mu \le h_\lambda$. To end the proof of the first claim, use the fact that $\{h_\alpha\}_{\alpha\in\Lambda}$ is totally ordered.

Now define

$$f = \inf_{\alpha\in\Lambda} h_\alpha.$$

Using definition (2) we get

$$\mathcal{C}^\Phi f = \sup_{\alpha\in\Lambda} \mathcal{C}^\Phi h_\alpha,$$

which, combined with the previous claim and the definition of f yields

$$\mathcal{C}^\Phi f \le f.$$

So, $f \in \mathcal{H}^\Phi$ and is a lower bound for the family $\{h_\alpha\}_{\alpha\in\Lambda}$.

To prove the second part of the lemma, use Zorn's Lemma (see [2, Theorem 2, pp. 154 and Corollary 1, pp. 155]) to conclude that for any $g \in \mathcal{H}^\Phi$ there exists a minimal $h \in \mathcal{H}^\Phi$ such that $h \le g$. Applying $\mathcal{C}^\Phi$ in this inequality we obtain $\mathcal{C}^\Phi g \le \mathcal{C}^\Phi h \le h$, where the second inequality follows from the inclusion $h \in \mathcal{H}^\Phi$. To end the proof, note that $\mathcal{H}^\Phi$ is non-empty. □

Lemma 2.4. *Suppose that the coupling function $\Phi : E \times E \to \mathbb{R}$ is symmetric. If $h = \mathcal{C}^\Phi h$ then h is a minimal element of $\mathcal{H}^\Phi$.*

Proof. Suppose that $g \in \mathcal{H}^\Phi$ and $g \le h$. Applying $\mathcal{C}^\Phi$ on this inequality gives $\mathcal{C}^\Phi h \le \mathcal{C}^\Phi g$. Therefore,

$$h = \mathcal{C}^\Phi h \le \mathcal{C}^\Phi g \le g$$

where the last inequality follows from the assumption $g \in \mathcal{H}^\Phi$. Altogether we have $g \le h$ and $h \le g$. So, $g = h$ and h is minimal in $\mathcal{H}^\Phi$. □

To prove Theorem 1.1 now, it is sufficient to prove the converse of Lemma 2.4.

Lemma 2.5. *Suppose that the coupling function $\Phi : E \times E \to \mathbb{R}$ is symmetric. Then $h \in \bar{\mathbb{R}}^E$ is a minimal element of $\mathcal{H}^\Phi$ if and only if $h = \mathcal{C}^\Phi h$.*

Proof. We already know, by Lemma 2.4, that if $h = \mathcal{C}^\Phi h$ then h is minimal in $\mathcal{H}^\Phi$. Suppose now that h is minimal in $\mathcal{H}^\Phi$. We shall prove that

$$\mathcal{C}^\Phi h(r_0) < h(r_0) \tag{7}$$

cannot hold. If this inequality holds, then by Lemma 2.1, $\Phi(r_0, r_0)/2 < h(r_0)$. Hence there exists $t_0 \in \mathbb{R}$ such that

$$\max\left\{\mathcal{C}^\Phi h(r_0), \Phi(r_0, r_0)/2\right\} \le t_0 < h(r_0). \tag{8}$$

Define

$$g = \min\{h, \delta_{r_0} + t_0\}. \tag{9}$$

We will prove that $g \in \mathcal{H}^\Phi$, and this will lead to a contradiction. Using (2), we get

$$\begin{aligned}\mathcal{C}^\Phi g(r) &= \max\{\mathcal{C}^\Phi h(r), \mathcal{C}^\Phi(\delta_{r_0} + t_0)(r)\} \\ &= \max\{\mathcal{C}^\Phi h(r), \Phi(r, r_0) - t_0\}.\end{aligned}$$

For any $r \in E$,

$$\Phi(r, r_0) - t_0 \le \Phi(r, r_0) - \mathcal{C}^\Phi h(r_0) \le (\mathcal{C}^\Phi)^2 h(r) \le h(r),$$

and $\mathcal{C}^\Phi h\,(r) \le h(r)$. Hence,

$$\mathcal{C}^\Phi g \le h.$$

As $\Phi(r_0, r_0) - t_0 \le t_0$ and $\mathcal{C}^\Phi h(r_0) \le t_0$, we also conclude that

$$\mathcal{C}^\Phi g \le \delta_{r_0} + t_0.$$

Combining the two above inequalities with (9) we obtain $\mathcal{C}^\Phi g \le g$. Therefore,

$$g \in \mathcal{H}^\Phi.$$

As $g \le h$ and h is minimal in $\mathcal{H}^\Phi$, $g = h$ and, in particular,

$$h(r_0) = g(r_0).$$

From the definition of g we have $g(r_0) = t_0 < h(r_0)$, which is a contradiction. So, (7) can not hold in any $r \in E$. As $\mathcal{C}^\Phi h \le h$, we conclude that $\mathcal{C}^\Phi h = h$. □

Proof of Theorem 1.1. Combining Lemma 2.3 with Lemma 2.5 we conclude that item *1*. holds and that there exists a self-conjugated function $h = \mathcal{C}^\Phi h$.

To prove item *2*., assume that $g = \mathcal{C}^\Phi g_0$ and $g \le \mathcal{C}^\Phi g$. Applying $\mathcal{C}^\Phi$ on this inequality we obtain $(\mathcal{C}^\Phi)^2 g \le \mathcal{C}^\Phi g$, which is equivalent to

$$\mathcal{C}^\Phi (\mathcal{C}^\Phi)^2\, g_0 \le (\mathcal{C}^\Phi)^2\, g_0.$$

Applying item *1*. to $(\mathcal{C}^\Phi)^2 g_0$ we conclude that there exists h,

$$(\mathcal{C}^\Phi)^3 g_0 \le h = \mathcal{C}^\Phi h \le (\mathcal{C}^\Phi)^2\, g_0.$$

Note that $(\mathcal{C}^\Phi)^3 g_0 = (\mathcal{C}^\Phi)^2 \mathcal{C}^\Phi\, g_0 \le \mathcal{C}^\Phi\, g_0$. Applying $\mathcal{C}^\Phi$ to the inequality $(\mathcal{C}^\Phi)^2 g_0 \le g_0$ we also have $\mathcal{C}^\Phi g_0 \le (\mathcal{C}^\Phi)^3 g_0$. Hence[1], $(\mathcal{C}^\Phi)^3 g_0 = \mathcal{C}^\Phi g_0$, which combined with the above equation yields

$$g = \mathcal{C}^\Phi g_0 \le h = \mathcal{C}^\Phi h \le (\mathcal{C}^\Phi)^2\, g_0 = \mathcal{C}^\Phi g\,.$$

□

[1] In fact, $(\mathcal{C}^\Phi)^3 = \mathcal{C}^\Phi$, which is a property of any symmetric conjugation.

3. Additional results

Here we present some additional results to Section 2 which were not necessary to prove the main theorem. Non-symmetric conjugation will also be discussed with more details.

Proposition 3.1. *For any $h \in \bar{\mathbb{R}}^E$, the following conditions are equivalent*

1. $\mathcal{C}_1^\Phi h \le h$,
2. $\mathcal{C}_2^\Phi h \le h$,
3. $\max\{\mathcal{C}_1^\Phi h, \mathcal{C}_2^\Phi h\} \le h$.

Proof. Suppose that *1.* holds, $\mathcal{C}_1^\Phi h \le h$. As $\mathcal{C}_2^\Phi$ is order reversing, applying $\mathcal{C}_2^\Phi$ on this inequality we get

$$\mathcal{C}_2^\Phi h \le \mathcal{C}_2^\Phi \mathcal{C}_1^\Phi h,$$

which, combined with the first inequality in (5) yields $\mathcal{C}_2^\Phi h \le h$. So condition *1.* implies condition *2.*. To prove that condition *2.* implies *1.* apply $\mathcal{C}_1^\Phi$ on both sides of the inequality $\mathcal{C}_2^\Phi h \le h$ and follows the same reasoning.

Condition *1.* or *2.*, being equivalent, implies condition *3.*, which is equivalent to condition *1.* and *2.*. □

We define the *symmetrization* of Φ as Φ_{sy},

$$\Phi_{\text{sy}} : E \times E \to \mathbb{R}, \qquad \Phi_{\text{sy}}(r,s) = \max\{\Phi(r,s), \Phi(s,r)\}. \tag{10}$$

Notice that Φ_{sy} is symmetric. Direct calculation gives

$$\mathcal{C}^{\Phi_{\text{sy}}} h = \max\{\mathcal{C}_1^\Phi h, \mathcal{C}_2^\Phi h\}, \tag{11}$$

which, combined with Definition 2.2 yields

$$\mathcal{H}^{\Phi_{\text{sy}}} = \{h \in \bar{\mathbb{R}}^E \mid \mathcal{C}^{\Phi_{\text{sy}}} h \le h\} = \{h \in \bar{\mathbb{R}}^E \mid \max\{\mathcal{C}_1^\Phi h, \mathcal{C}_2^\Phi h\} \le h\}.$$

Using Proposition 3.1 we obtain alternative characterizations of $\mathcal{H}^\Phi$:

$$\begin{aligned} \mathcal{H}^\Phi &= \{h \in \bar{\mathbb{R}}^E \mid \mathcal{C}_1^\Phi h \le h\} \\ &= \{h \in \bar{\mathbb{R}}^E \mid \mathcal{C}_2^\Phi h \le h\} \\ &= \{h \in \bar{\mathbb{R}}^E \mid \max\{\mathcal{C}_1^\Phi h, \mathcal{C}_2^\Phi h\} \le h\} = \mathcal{H}^{\Phi_{\text{sy}}}. \end{aligned} \tag{12}$$

With the above equation, now it is straightforward to generalize Lemma 2.3 and Lemma 2.5 to non-symmetric conjugations.

Proposition 3.2. *Let $\Phi : E \times E \to \mathbb{R}$ be a generic coupling function. Then*

1. *The family $\mathcal{H}^\Phi$ is (downward) inductively ordered.*
2. *For any $g \in \mathcal{H}^\Phi$ there exists a minimal $h \in \mathcal{H}^\Phi$, such that,*

$$\max\{\mathcal{C}_1^\Phi g, \mathcal{C}_2^\Phi g\} \le h \le g.$$

3. *The family $\mathcal{H}^\Phi$ has minimal elements*

4. $h \in \mathcal{H}^{\Phi}$ *is minimal if and only if* $\mathcal{C}^{\Phi_{sy}} h = \max\{\mathcal{C}_1^{\Phi} h, \mathcal{C}_2^{\Phi} h\} = h$.

Also in the non-symmetric case, fixed points of the conjugations $\mathcal{C}_1^{\Phi}$ or $\mathcal{C}_2^{\Phi}$ are minimal elements of $\mathcal{H}^{\Phi_{sy}}$.

Proposition 3.3. *If* $h = \mathcal{C}_1^{\Phi} h$ *or* $h = \mathcal{C}_2^{\Phi} h$*, then* $h \in \mathcal{H}^{\Phi}$ *and is minimal.*

Proof. If $h = \mathcal{C}_1^{\Phi} h$ then, in particular $\mathcal{C}_1^{\Phi} h \leq h$. Hence by (12) $h \in \mathcal{H}^{\Phi_{sy}}$ and so

$$\mathcal{C}_2^{\Phi} h \leq h = \mathcal{C}_1^{\Phi} h$$

which implies $\max\{\mathcal{C}_1^{\Phi} h, \mathcal{C}_2^{\Phi} h\} = h$ so that by (11) $\mathcal{C}^{\Phi_{sy}} h = h$. Now apply Lemma 2.4 to conclude that h is minimal in $\mathcal{H}^{\Phi_{sy}}$. The case $\mathcal{C}_2^{\Phi} h = h$ follows the same proof, interchanging $\mathcal{C}_1^{\Phi}$ and $\mathcal{C}_2^{\Phi}$. □

A natural question is whether Lemma 2.5 can be extended to a non-symmetric Φ. The answer is negative, as exposed in the next example.

Take $E = \{a, b\}$ with $a \neq b$ and $\Phi : E \times E \to \mathbb{R}$

$$\Phi(a,a) = 0, \qquad \Phi(a,b) = -3,$$
$$\Phi(b,a) = 0, \qquad \Phi(b,b) = -3.$$

For the function $h : E \to \bar{\mathbb{R}}$, $h(a) = 1$ and $h(b) = -1$, we have

$$\mathcal{C}_1^{\Phi} h(a) = 1, \qquad \mathcal{C}_1^{\Phi} h(b) = -2,$$
$$\mathcal{C}_2^{\Phi} h(a) = -1, \qquad \mathcal{C}_2^{\Phi} h(b) = -1.$$

As $h = \max\{\mathcal{C}_1^{\Phi} h, \mathcal{C}_2^{\Phi} h\}$, by (11) and Lemma 2.4, h is minimal in $\mathcal{H}^{\Phi_{sy}}$ but is not a fixed point of $\mathcal{C}_1^{\Phi}$ or $\mathcal{C}_2^{\Phi}$.

Lemma 2.1 applied to family $\mathcal{H}^{\Phi}$ yields the following result, which relates these functions $h \in \mathcal{H}^{\Phi}$ with the coupling function Φ and the generalized subdifferential.

Corollary 3.4. *For any* $h \in \mathcal{H}^{\Phi}$*:*

1. $\Phi(r,r)/2 \leq h(r)$ *for all* $r \in E$.

2. *If* $h(r_0) = \Phi(r_0, r_0)/2$*, then*

$$\mathcal{C}_1^{\Phi} h(r_0) = \mathcal{C}_2^{\Phi} h(r_0) = \Phi(r_0, r_0)/2$$

and $r_0 \in \partial_1^{\Phi} h(r_0)$, $r_0 \in \partial_2^{\Phi} h(r_0)$*, that is, for all* $r \in E$

$$h(r_0) + \left[\Phi(r, r_0) - \Phi(r_0, r_0)\right] \leq h(r),$$
$$h(r_0) + \left[\Phi(r_0, r) - \Phi(r_0, r_0)\right] \leq h(r).$$

Proof. Item *1.* follows directly from Definition 2.2 and the first implication on Lemma 2.1.

To prove item *2.*, first use the second implication on Lemma 2.1 to conclude that $\mathcal{C}_i^{\Phi} h(r_0) \geq h(r_0)$. Now use (12) to conclude that this inequality holds as an equality. As $\mathcal{C}_1^{\Phi} h(r_0) = h(r_0) = \Phi(r_0, r_0)/2$, by (2)

$$h(r_0) \geq \Phi(r, r_0) - h(r)$$

for all $r \in E$. Hence

$$\begin{aligned} h(r) &\geq \Phi(r, r_0) - h(r_0) \\ &= \Phi(r, r_0) - 2h(r_0) + h(r_0) = \Phi(r, r_0) - \Phi(r_0, r_0) + h(r_0). \end{aligned}$$

The last inequality follows from the same arguments. □

4. Self-conjugated Fitzpatrick functions, or fixed points of the $\mathcal{J}$ mapping

Now we will use Theorem 1.1 to study self-conjugated Fitzpatrick's functions.

In this section X is a real linear topological space and X^* its dual, endowed with the weak-$*$ topology. In $X \times X^*$, consider the canonical product topology. Use the notation $\langle x, x^* \rangle$ for the duality product

$$\langle x, x^* \rangle = x^*(x), \quad x \in X, \ x^* \in X^*.$$

A point to set operator $T : X \rightrightarrows X^*$ is a relation on X to X^*:

$$T \subset X \times X^*$$

and $x^* \in T(x)$ means $(x, x^*) \in T$. An operator $T : X \rightrightarrows X^*$ is *monotone* if

$$\langle x - y, x^* - y^* \rangle \geq 0, \quad \forall (x, x^*), (y, y^*) \in T.$$

The operator T is *maximal monotone* if it is monotone and maximal in the family of monotone operators of X into X^* (with respect to order of inclusion).

Fitzpatrick proved that associated to any maximal monotone operator in X there exists a family of lower semicontinuous convex functions in $X \times X^*$ which characterize the operator:

Theorem 4.1 ([5, Theorem 3.10]). *If T is a maximal monotone operator on a real linear topological space X, then $\varphi_T : X \times X^* \to \bar{\mathbb{R}}$*

$$\varphi_T(x, x^*) = \sup_{(y, y^*) \in T} \langle x - y, y^* - x^* \rangle + \langle x, x^* \rangle \tag{13}$$

is the smallest element of the family $\mathcal{F}_T$,

$$\mathcal{F}_T = \left\{ h \in \bar{\mathbb{R}}^{X \times X^*} \;\middle|\; \begin{array}{l} h \text{ is convex and lower semicontinuous} \\ \langle x, x^* \rangle \leq h(x, x^*), \ \forall (x, x^*) \in X \times X^* \\ (x, x^*) \in T \Rightarrow h(x, x^*) = \langle x, x^* \rangle \end{array} \right\} \tag{14}$$

Moreover, for any $h \in \mathcal{F}_T$,

$$(x, x^*) \in T \iff h(x, x^*) = \langle x, x^* \rangle.$$

Note that *any* $h \in \mathcal{F}_T$ fully characterizes T. Fitzpatrick's family of convex representation of maximal monotone operators was recently rediscovered [4, 7] and since then, this subject has been object of intense research. Burachik and Svaiter define [4, 3] (with a different notation), for $T : X \rightrightarrows X^*$ maximal monotone (see [4, Corollary 4.1] or [3])

$$\mathcal{S}_T = \sup_{h \in \mathcal{F}_T} h,$$

and proved that $\mathcal{S}_T \in \mathcal{F}_T$ and is the biggest element of this family.

We will use the notation π for the duality product in $X \times X^*$

$$\pi : X \times X^* \to \mathbb{R}, \qquad \pi(x, x^*) = \langle x, x^* \rangle.$$

Burachik and Svaiter prove that if X is a Banach space, then ([4, Eq. 35], [3, Eq. 29])

$$\mathcal{S}_T = \operatorname{cl}\operatorname{conv}(\pi + \delta_T),$$

where cl conv stands for the lower semicontinuous convex closure.

Proposition 4.2. *Let T be a maximal monotone operator on a real linear topological space X. There exists a (unique) maximum element $\mathcal{S}_T \in \mathcal{F}_T$,*

$$\mathcal{S}_T = \sup_{h \in \mathcal{F}_T} \{h\}.$$

Proof. The family $\mathcal{F}_T$ is closed under the sup operation. □

The maximal representation $\mathcal{S}_T$ and the structure of its epigraph were studied on a Banach space setting in [4, 3, 13].

Fenchel-Legendre conjugate of $f : X \to \bar{\mathbb{R}}$ is defined as $f^* : X^* \to \bar{\mathbb{R}}$,

$$f^*(x^*) = \sup_{x \in X} \langle x, x^* \rangle - f(x). \tag{15}$$

Define, as in [4]

$$\mathcal{J} : \bar{\mathbb{R}}^{X \times X^*} \to \bar{\mathbb{R}}^{X \times X^*}, \qquad (\mathcal{J}h)(x, x^*) = h^*(x^*, x). \tag{16}$$

Hence, for all $(x, x^*) \in X \times X^*$,

$$\begin{aligned} \mathcal{J}h\ (x, x^*) &= \sup_{(y, y^*) \in X \times X^*} \big\langle (y, y^*)\,,\ (x^*, x) \big\rangle - h(y, y^*) \\ &= \sup_{(y, y^*) \in X \times X^*} \langle x, y^* \rangle + \langle y, x^* \rangle - h(y, y^*). \end{aligned} \tag{17}$$

Direct use of (16) or (17) and (15) shows that

$$\mathcal{J}^2 f = f^{**} \le f, \quad \forall f \in \bar{\mathbb{R}}^{X \times X^*}. \tag{18}$$

The family $\mathcal{F}_T$ is invariant under $\mathcal{J}$ in a Banach space setting [4]. Here we extend this result to linear topological spaces. Note that if X is not Hausdorff, any lower semicontinuous function must assume only one value at each family of non-separable points. So, in dealing with lower semicontinuous functions, whenever we need X to be Hausdorff, we can work in $X/(X^*)^\dagger$, where $(X^*)^\dagger$ is the annihilator of X^*.

Theorem 4.3. *Let T be a maximal monotone operator on a real linear topological space X. The application $\mathcal{J}$ maps $\mathcal{F}_T$ into itself. If X is locally convex, $\mathcal{J}$ maps $\mathcal{F}_T$* onto *itself.*

Proof. Take $h \in \mathcal{F}_T$. By Theorem 4.1

$$\varphi_T \leq h \leq \delta_T + \pi.$$

As $\mathcal{J}$ is order reversing, applying this mapping on both terms of this inequalities we obtain

$$\mathcal{J}(\delta_T + \pi) \leq \mathcal{J}h \leq \mathcal{J}\varphi_T.$$

Direct use of (17) and (13) yields $\mathcal{J}(\delta_T+\pi) = \varphi_T$, which applied to the above inequality yields

$$\varphi_T \leq \mathcal{J}h \leq \mathcal{J}^2(\delta_T + \pi).$$

Again by Theorem 4.1 $\pi \leq \varphi_T$. Combining this result with the above inequalities and (18) we obtain

$$\pi \leq \mathcal{J}h \leq \delta_T + \pi.$$

According to the above equation, $\langle x, x^* \rangle \leq \mathcal{J}h(x, x^*)$ for all $(x, x^*) \in X \times X^*$, with equality if $(x, x^*) \in T$. By definition (16) or (17), $\mathcal{J}h$ is convex and lower semicontinuous. Therefore, $\mathcal{J}h \in \mathcal{F}_T$.

Assume now that X is locally convex. Take $h \in \mathcal{F}_T$. As h is convex and lower semicontinuous,

$$\mathcal{J}(\mathcal{J}h) = h^{**} = h.$$

As $\mathcal{J}h \in \mathcal{F}_T$, we obtain $h = \mathcal{J}^2 h \in \mathcal{J}(\mathcal{F}_T)$. □

Now we are ready to extend the fixed point theorem in [14] to linear topological spaces.

Theorem 4.4. *Let T be a maximal monotone operator on a real linear topological space X.*

1. *If $g \in \mathcal{F}_T$ and $\mathcal{J}g \leq g$ then there exists $h \in \mathcal{F}_T$ such that*

$$\mathcal{J}g \leq \mathcal{J}h = h \leq g.$$

2. *If $g \in \mathcal{J}(\mathcal{F}_T)$ and $g \leq \mathcal{J}g$ then there exists $h \in \mathcal{F}_T$ such that*

$$g \leq h = \mathcal{J}h \leq \mathcal{J}g.$$

In particular, there exists $h \in \mathcal{F}_T$ such that $h = \mathcal{J}h$.

Proof. Take $E := X \times X^*$ and consider the coupling function Φ,

$$\Phi : (X \times X^*) \times (X \times X^*) \to \mathbb{R}, \qquad \Phi\big(\,(x, x^*)\,,\ (y, y^*)\,\big) := \langle x, y^* \rangle + \langle y, x^* \rangle\ .$$

Note that Φ is symmetric. Moreover, using (15), (16) and (2) we have

$$\mathcal{C}^{\Phi} = \mathcal{J}.$$

If $g \in \mathcal{F}_T$ and $\mathcal{J}g \leq g$, this means $\mathcal{C}^\Phi g \leq g$. Using item *1.* of Theorem 1.1 we conclude that there exists $h \in \mathcal{H}^\Phi$ such that

$$\mathcal{J}g \leq \mathcal{J}h = h \leq g.$$

Now we must show that $h \in \mathcal{F}_T$. As $\mathcal{J}h = h$ is the supremum of a family of continuous affine functionals on $X \times X^*$, we conclude that h is convex and lower semicontinuous. Since $\mathcal{J}g \in \mathcal{F}_T$, for any $(x, x^*) \in X \times X^*$,

$$\langle x, x^* \rangle \leq \mathcal{J}g(x, x^*) \leq h(x, x^*) \leq g(x, x^*).$$

In particular, $\langle x, x^* \rangle \leq h(x, x^*)$. If $(x, x^*) \in T$ then, as $g \in \mathcal{F}_T$, $g(x, x^*) = \langle x, x^* \rangle$, the above inequalities hold as equalities and $h(x, x^*) = \langle x, x^* \rangle$. Therefore, $h \in \mathcal{F}_T$ and item *1.* holds.

To prove item *2.* use item *2.* of Theorem 1.1 and repeat the reasoning used to prove item *1.*.

To end the proof, we must show that there exists a fixed point of $\mathcal{J}$ in $\mathcal{F}_T$. As $\mathcal{S}_T$ is maximal in $\mathcal{F}_T$ and $\mathcal{J}\mathcal{S}_T \in \mathcal{F}_T$, we conclude that $\mathcal{J}\mathcal{S}_T \leq \mathcal{S}_T$. Now, apply item *1.* of the theorem. □

Corollary 4.5. *Let T be a maximal monotone operator on a real linear locally convex topological space X. If $g \in \mathcal{F}_T$ and $g \leq \mathcal{J}g$ then there exists $h \in \mathcal{F}_T$ such that*

$$g \leq \mathcal{J}h = h \leq \mathcal{J}g.$$

Proof. First use Theorem 4.3 to conclude that $g \in \mathcal{J}(\mathcal{F}_T)$ and then apply item *2.* of Theorem 1.1. □

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