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A Universal Compactification of Topological Positively Convex Sets

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Dedicated to Helmut Röhrl on the occasion of his 80th birthday.

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A topological positively convex set is a positively convex subset of a topological real linear space with the induced topology. Topological positively convex modules are a canonical generalization defined without the requirement to be a subset of a linear space. For any topological positively convex module or set there is a universal continuous positively affine mapping to a regularly ordered Saks space yielding the universal compactification.

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1. Introduction

The embedding of topological convex sets, cones and caps of cones with a topology into locally convex (ordered) real linear spaces has been investigated by many authors (cp. e.g. [1], [4], [9], [10] [11], [20], [22], [23]) under varying assumptions. The results of this paper render many of these results as special cases.

The central tool for the following investigations is the notion of a topological positively convex module in Definition 2.1; the cap of a cone in a topological real linear space is a paradigmatic case of a topological positively convex module. In [2] Doberkat has investigated the Eilenberg-Moore algebras of the modified Giry Monad induced by the subprobability functor of Polish spaces. These are of importance for probabilistic methods in theoretical computer science and Doberkat uses topological positively convex sets for their characterization. For an explicit characterization of these Eilenberg-Moore algebras an as complete as possible knowledge of the relations and properties of the (canonical) topologies on the spaces of finite Borel measures on Polish spaces is required. In a forthcoming paper D. Pallaschke and the author will use the results and methods of this paper to classify the (canonical) topologies on the space of finite Borel measures of a (semi-) metric space.

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2. Topological positively convex modules

All linear spaces in this paper are real linear spaces. A *positively convex subset* X of a real, linear space E is a non-empty subset of E closed under positively convex operations, i.e. $x_i \in X,\ 1 \le i \le n$, and $\alpha_i \ge 0,\ 1 \le i \le n, \sum_{i=1}^n \alpha_i \le 1$ implies $\sum_{i=1}^n \alpha_i x_i \in X$. A positively convex operation may be described as an element of $\Omega_{pc} := \{\hat{\alpha} | \hat{\alpha} = (\alpha_1, \dots, \alpha_n),\ n \in \mathbb{N},\ \alpha_i \ge 0,\ 1 \le i \le n$, and $\sum_{i=1}^n \alpha_i \le 1\}$. One introduces the following canonical generalization:

Definition 2.1 (cf. [12], [19]). *A* positively convex module P is a non-empty set together with a family of mappings $\hat{\alpha}_P : P^n \to P,\ \hat{\alpha} = (\alpha_1, \dots, \alpha_n) \in \Omega_{pc}, n \in \mathbb{N}$. With the notation

$$\sum_{i=1}^n \alpha_i x_i := \hat{\alpha}_P(x_1, \dots, x_n)$$

for $\hat{\alpha} = (\alpha_1, \dots, \alpha_n) \in \Omega_{pc}, x_i \in P,\ 1 \le i \le n$, also the following equations have to be satisfied:

$$\sum_{i=1}^n \delta_{ik} x_i = x_k, \tag{PC 1}$$

$x_i \in P,\ 1 \le i \le n$, and $\delta_{ik},\ 1 \le i, k \le n$, the Kroneckersymbol.

$$\sum_{i=1}^n \alpha_i \left(\sum_{k \in K_i} \beta_{ik} x_k \right) = \sum_{k=1}^m \left(\sum_{\substack{i=1 \\ k \in K_i}} \alpha_i \beta_{ik} \right) x_k, \tag{PC 2}$$

where $(\alpha_1, \dots, \alpha_n),\ (\beta_{ik} \mid k \in K_i) \in \Omega_{pc},\ 1 \le i \le n,\ x_k \in P$, for $k \in \bigcup_{i=1}^n K_i$. Moreover, $\bigcup_{i=1}^n K_i = \mathbb{N}_m := \{k \mid k \in \mathbb{N}$ and $1 \le k \le n\}$ and in the "sum" $\sum_{k \in K_i} \beta_{ik} x_k$ the summands are supposed to be written in the natural order of the $k's$.

A number of computational rules follow from (PC 1) and (PC 2) (cf. [12], [13]), e.g. the fact that $\sum_{i=1}^n \alpha_i x_i$ is not changed by adding or omitting summands with zero coefficients. Hence, (PC 2) takes the more simple form

$$\sum_{i=1}^n \alpha_i \left(\sum_{k=1}^m \beta_{ik} x_i \right) = \sum_{k=1}^m \left(\sum_{i=1}^n \alpha_i \beta_{ik} \right) x_k.$$

Obviously a positively convex set in a linear space is a positively convex module. The converse does not hold as is shown by the set $\{0, 1\}$ and the positively convex operations defined by

$$\sum_{i=1}^n \alpha_i x_i := \begin{cases} 1 & \text{if, in } \mathbb{R},\ \sum_{i=1}^n \alpha_i x_i = 1, \\ 0 & \text{else.} \end{cases}$$

A positively convex module is a convex module, the converse does not hold. Important examples of positively convex sets are the universal caps of cones ([27]).

A map $f : P_1 \to P_2$, P_i positively convex modules, $i = 1, 2$, is called *positively affine*, if

$$f\left(\sum_{i=1}^{n} \alpha_i x_i\right) = \sum_{i=1}^{n} \alpha_i f(x_i)$$

for all $(\alpha_1, \ldots, \alpha_n) \in \Omega_{pc}$ and all $x_i \in P_1, 1 \le i \le n$.

One may also represent Ω_{pc} as

$$\Omega_{pc} = \bigcup_{n=1}^{\infty} \Omega_{pc}^n$$

with $\Omega_{pc}^n := \{\hat{\alpha} \mid \hat{\alpha} = (\alpha_1, \ldots, \alpha_n) \subset \Omega_{pc}\}, n \in \mathbb{N}$. If

$$\mu_P^n : \Omega_{pc}^n \times P^n \longrightarrow P \tag{$*$}$$

is defined, for a positively convex module P and $n \in \mathbb{N}$, by

$$\mu_P^n(\hat{\alpha}, \mathbf{x}) := \sum_{i=1}^{n} \alpha_i x_i,$$

$\hat{\alpha} \in \Omega_{pc}^n, \mathbf{x} = (x_1, \ldots, x_n) \in P^n$, we may define the mapping

$$\mu_P : \bigcup_{n=1}^{\infty} (\Omega_{pc}^n \times P^n) \to P$$

by $\mu_P(\hat{\alpha}, \mathbf{x}) := \mu_P^n(\hat{\alpha}, x)$, for $(\hat{\alpha}, \mathbf{x}) \in \Omega_{pc}^n \times P^n$.

The positively convex modules together with the positively affine mappings form the *category* **PC** of *positively convex modules* with the usual forgetful functor U : **PC** $\to$ **Set**, **Set** the category of sets. U has a left adjoint, **PC** is monadic and is the category of Eilenberg-Moore algebras of the functor $\Delta : \mathbf{Vec}_1^+ \to \mathbf{Set}$ defined by $\Delta(E) := \mathrm{O}(E) \cap \mathrm{C}(E)$, for $E \in \mathbf{Vec}_1^+, \mathrm{O}(E)$ being the closed unit ball and $\mathrm{C}(E)$ the ordering cone of E, where $\mathbf{Vec}_1^+$ denotes the category of regularly ordered normed linear spaces ([12], [19]).

In a (real) Banach space, there are positively convex sets like, for instance, the closed or the open unit ball, which are even closed under a larger class of operations, namely the so-called *positively superconvex operations*

$$\Omega_{psc} := \left\{ \hat{\alpha} \mid \hat{\alpha} = (\alpha_i, i \in \mathbb{N}), \alpha_i \ge 0, i \in \mathbb{N}, \text{ and } \sum_{i=1}^{\infty} \alpha_i \le 1 \right\}.$$

This leads to the

Definition 2.2 ([12], [19]). A non-empty set P is called a *positively superconvex module*, if there are mappings $\hat{\alpha}_P : P^{\mathbb{N}} \to P, \hat{\alpha} \in \Omega_{psc}$, s.th. with the notation

$$\sum_{i=1}^{\infty} \alpha_i x_i := \hat{\alpha}_P(x_i \mid i \in \mathbf{N}),$$

$x_i \in P$, $i \in \mathbb{N}$, (PC 1) and (PC 2) in 2.1 are satisfied for the $\hat{\alpha} \in \Omega_{psc}$. Analogously, positively superaffine mappings between positively superconvex modules are mappings preserving these positively superconvex "sums". These mappings and the positively superconvex modules form the *category* **PSC** of *positively superconvex modules.* **PSC** is the category of Eilenberg-Moore algebras of the functor $\Delta : \mathbf{Ban}_1^+ \to \mathbf{Set}, \mathbf{Ban}_1^+$ the category of regularly ordered Banach spaces (cp. [12]). For $P \in \mathbf{PSC}$ the representation analogous to $(*)$ is

$$\mu_P : \Omega_{psc} \times P^{\mathbf{N}} \to P$$

defined as $\mu_P(\hat{\alpha}, \mathbf{x}) := \hat{\alpha}_P(\mathbf{x}), \hat{\alpha} \in \Omega_{psc}, \mathbf{x} \in P^{\mathbf{N}}$.

If $P \subset E$ is a positively convex set in a topological linear space E, the mappings $\mu_P^n : \Omega_{pc}^n \times P^n \to P$ are continuous, if P carries the subspace topology, Ω_{pc}^n the l_1-topology and $\Omega_{pc}^n \times P^n$ and P^n the product topology. This follows from the continuity of the linear operations $\cdot : \mathbb{R} \times E \to E$ and $+ : E \times E \to E$. This leads to the

Definition 2.3. A topological positively convex module is a positively convex module P with a topology $\mathcal{T}_P$ such that

$$\mu_P^n : \Omega_{pc}^n \times P^n \to P$$

is continuous, $n \in \mathbb{N}$, if Ω_{pc}^n carries the l_1-topology and P^n and $\Omega_{pc}^n \times P^n$ the product topology. The topological positively convex modules with the continuous positively affine mappings as morphisms form the *category* **TopPC** of *topological positively convex modules.* The *category* **TopPSC** of *topological positively superconvex modules* is defined analogously demanding the continuity of μ_P.

The following result is very useful in investigating topological positively convex sets.

Lemma 2.4 (cp. [21], Lemma 2, p. 51). *Let E be a Hausdorff topological linear space. $P \subset E$ positively convex, $\overline{P}$ the closure and P° the interior of P. Then, for all $\alpha \in]0,1[$, any $x \in P^\circ$ and any $y \in \overline{P}$*

$$\alpha x + (1-\alpha) y \in \overset{\circ}{P}$$

holds.

Some canonical cases of topological positively superconvex sets are given in

Proposition 2.5. *If E is a Hausdorff topological linear space and $P \subset E$, then the following statements hold:*

- (i) *If P is a positively superconvex set in E and for any $x_i \in P$, $i \in \mathbb{N}$, and any $\hat{\alpha} \in \Omega_{psc}$, $\sum_{i=1}^{\infty} \alpha_i x_i = \lim_{n\to\infty} \sum_{i=1}^{n} \alpha_i x_i$ holds in E, then P is bounded and a topological positively superconvex set.*
- (ii) *If P is positively convex, bounded and*
  - (a) *sequentially complete, or*
  - (b) *E is sequentially complete and P is open or $P^\circ = P \setminus \{0\}$, then P is a topological positively superconvex set.*

Proof. (i): That P is bounded follows from Jameson [7], because any positively superconvex sets is superconvex. To show the continuity of μ_P consider a neighborhood U of 0. Then there is a circled neighborhood V of 0 with $4V \subset U$. As P is bounded there is an $\varepsilon_V > 0$ with $\varepsilon_\nu P \subset V$ and there is a $k(V) \in \mathbb{N}$, s.th.

$$\sum_{n>k(V)} \alpha_n < \frac{1}{2}\varepsilon_V$$

holds. Moreover, assume that $\hat{\beta} - \hat{\alpha} \in O_{\frac{1}{2}\varepsilon_V}(l_1(\mathbb{N}))$, if, for a normed linear space E and $r > 0, \mathrm{O}_r(E)$ denotes here and in the following the closed ball with radius r and center 0. This yields

$$\sum_{n>k(V)} \beta_n < \varepsilon_V.$$

Also because of the continuity of the linear space operations, for any $k_0 \in \mathbb{N}$, there is a $W(V, k_0) \in \mathbb{U}(0), \mathbb{U}(0)$ the neighborhood system of 0, with

$$\sum_{n \leq k_0} \beta_n(y_n - x_n) \in V$$

for $(x_1 - y_1, \dots, x_{k_0} - y_{k_0}) \in W^{k_0}(V, k_0)$. Putting, for $i \in \mathbb{N}$,

$$W_i := \begin{cases} W(V, k(V)), & \text{for } 1 \leq i \leq k(V), \\ E, & \text{for } i = k(V). \end{cases}$$

one gets for $\mathbf{y} - \mathbf{x} \in \mathbf{X}_{i=1}^{\infty} W_i$,

$$\sum_{n>k(V)} \beta_n(y_n - x_n) \in \varepsilon_V P - \varepsilon_V P \subset V - V \subset 2V,$$

hence

$$\sum_n \beta_n(y_n - x_n) \in V + 2V = 3V.$$

$\|\hat{\beta} - \hat{\alpha}\|_1 \leq 2^{-1}\varepsilon_V$ now yields

$$\sum_n (\beta_n - \alpha_n)x_n = \sum_{\substack{n \\ \beta_n > \alpha_n}} (\beta_n - \alpha_n)x_n - \sum_{\substack{n \\ \beta_n \leq \alpha_n}} (\alpha_n - \beta_n)x_n,$$

$$\sum_n (\beta_n - \alpha_n)x_n \in \frac{1}{2}\varepsilon_V P - \frac{1}{2}\varepsilon_V P \subset V.$$

For any $(\hat{\beta}, \mathbf{y}) - (\hat{\alpha}, \mathbf{x}) \in [\mathrm{O}_{2^{-1}\varepsilon_V}(l_1(\mathbb{N})) \times \mathbf{X}_{i=1}^{\infty} W_i]$ and $(\hat{\alpha}, \mathbf{y}), (\hat{\beta}, \mathbf{x}) \in \Omega_{psc} \times P^{\mathbb{N}}$, these results yield

$$\mu_P(\hat{\beta}, \mathbf{y}) \in \mu_P(\hat{\alpha}, \mathbf{x}) + 3V + V \subset \mu_P(\hat{\alpha}, \mathbf{x}) + U,$$

i.e. μ_P is continuous.

(ii), (a): For $U \in \mathbb{U}(0)$ there is an $\varepsilon_U > 0$ with $\varepsilon_U P \subset U$, because P is bounded. Also, for ε_U there exists a $n_0(\varepsilon_U) \in \mathbb{N}$, s.th. for any $n \geq n_0(\varepsilon_U)$

$$\sum_{i=n}^{\infty} \alpha_i < \varepsilon_U$$

holds. Hence, for any $q > p \geq n_0(\varepsilon_U)$ and any $x_i \in P,\ i \in \mathbb{N}$, and $\hat{\alpha} \in \Omega_{psc},\ \hat{\alpha} = (\hat{\alpha}_i \mid i \in \mathbb{N})$,

$$\sum_{i=p+1}^{q} \alpha_i x_i \in \varepsilon_U P \subset U$$

follows, i.e. $(\sum_{i=1}^{n} \alpha_i x_i \mid n \in \mathbb{N})$ is a Cauchy sequence in P and the definition

$$\sum_{i=1}^{\infty} \alpha_i x_i := \lim_{n\to\infty} \sum_{i=1}^{\infty} \alpha_i x_i$$

makes P positively superconvex and the assertion follows from (i).

(ii), (b): Any positively convex set (or module) is trivially also convex (cf. e.g. [16]) and, conversely, any convex set which contains 0 is trivially positively convex. Lemma 2.4 yields that P°, the interior of P, is convex, because $P^\circ \subset P \subset \overline{P}$. As P is bounded so is P°, i.e. for any $U \in \mathbb{U}(0)$ there is an $\varepsilon_U > 0$ with $\varepsilon_U P^\circ \subset \varepsilon_U P \subset U$. Hence, as the argument in the proof of (ii), (a) shows, $(\sum_{i=1}^{n} \alpha_i x_i \mid n \in \mathbb{N})$ is a Cauchy sequence in P and

$$\sum_{i=1}^{\infty} \alpha_i x_i = \lim_{n\to\infty} \sum_{i=1}^{n} \alpha_i x_i$$

exists in E for any $\hat{\alpha} \in \Omega_{psc}$ and any $x_i \in P^\circ,\ i \in \mathbb{N}$. We may assume that the support of $\hat{\alpha}$ is infinite and denote by $i_0 \in \mathbb{N}$ the smallest natural number s.th. $\alpha_{i_0} > 0$. This equation together with (PC 1) and (PC 2) yields

$$\sum_{i=1}^{\infty} \alpha_i x_i = \alpha_{i_0} x_{i_0} + (1 - \alpha_{i_0}) \sum_{i=i_0+1}^{\infty} \frac{\alpha_i}{1 - \alpha_{i_0}} x_i$$

with $\sum_{i=i_{0+1}}^{\infty} \alpha_i (1 - \alpha_{i_0})^{-1} x_i \in \overline{P^\circ} \subset \overline{P}$, so by 2.4

$$\sum_{i=1}^{\infty} \alpha_i x_i \in \overset{\circ}{P}, \qquad (*)$$

and the assertion is proved, if $P = P^\circ$. In any case $(*)$ shows P° to be superconvex.

If, however, $0 \notin P^\circ$ and $P = P^\circ \cup \{0\}$ holds and only finitely many $x_{i_\nu}, 1 \leq \nu \leq k$, in $\sum_i \alpha_i x_i$ are in P°, then $\sum_{i=1}^{\infty} \alpha_i x_i = \sum_{\nu=1}^{k} \alpha_{i_\nu} x_{i_\nu} \in P$. If infinitely many of the $x_i,\ i \in \mathbb{N}$, are in P°, we may assume that all $x_i \in P^\circ,\ i \in \mathbb{N}$, and $(*)$ holds again, i.e.

$$\sum_{i=1}^{\infty} \alpha_i x_i \in \overset{\circ}{P} \subset P$$

and P is also positively superconvex.

3. Reguarly ordered Saks spaces.

Different types of Saks spaces have already been used in the investigation of several types of convex sets and modules (cf. [17], [18]). In the following we need the new notion of a regularly ordered Saks space.

A real normed linear space E is called *regularly ordered* if it is ordered by a proper, closed cone $\mathrm{C}(E)$, i.e. $\mathrm{C}(E) \cap (-\mathrm{C}(E)) = \{0\}$, and its norm $\|\Box\|$ is the Riesz norm defined by $\mathrm{C}(E)$:

$$\|x\| = \inf\{\|c\| \mid -c \leq x \leq c\}$$

([27]). From now on the closed unit ball of E will be denoted by $\mathrm{O}(E)$ and the open unit ball by $\mathring{\mathrm{O}}(E)$. A morphism $f : E_1 \to E_2$ between regularly ordered (normed) linear space is a positive, linear contraction. These morphisms and the regularly ordered spaces form the category $\mathbf{Vec}_1^+$. The *regularly ordered Banach spaces* constitute the full subcategory $\mathbf{Ban}_1^+$.

Saks spaces were introduced by Cooper [3] as a triple $(E, \|\Box\|, \mathcal{T})$, with E a linear space with norm $\|\Box\|$ and an additional locally convex Hausdorff topology $\mathcal{T}$, s.th. $\mathrm{O}(E)$ is $\mathcal{T}$-bounded and $\mathcal{T}$-closed. A morphism between Saks spaces is a linear contraction $f : E_1 \to E_2$ which, when restricted to the unit balls, $\mathrm{O}(f) : \mathrm{O}(E_1) \to \mathrm{O}(E_2)$, is $\mathcal{T}_1 - \mathcal{T}_2$-continuous, if $\mathcal{T}_i$ is the l.c. topology on $E_i, i = 1, 2$. These morphisms and the Saks spaces form the *category* $\mathbf{Saks}_1$ of *Saks space* ([17]). The notion of a Saks space was specialized to the notion of a base normed Saks space in [18]. Here, we need still another special type, namely regularly ordered Saks spaces.

Definition 3.1. A regularly ordered Saks spaces is a quadruple $(E, \|\Box\|, \mathrm{C}, \mathcal{T})$ with a regularly ordered linear space E with norm $\|\Box\|$ and ordering cone C. Moreover, $\mathcal{T}$ is a locally convex and locally order convex (locally 0-convex in [27]) Hausdorff topology on E, such that C is $\mathcal{T}$-closed and the cap $\Delta(E) := \mathrm{O}(E) \cap \mathrm{C}$ is $\mathcal{T}$-bounded and $\mathcal{T}$-closed. In the following, the norm of regularly ordered Saks space will be noted by $\|\Box\|_E$ if necessary, the topology by $\mathcal{T}_E$ and the cone by $\mathrm{C}(E)$. A regularly ordered Saks space will be denoted by a single letter, e.g. E, to simplify notation.

A *morphism* $f : E_1 \to E_2$ between *regularly ordered Saks spaces* is a $\mathbf{Vec}_1^+$ morphism, the restriction of which to the caps $\Delta(f) : \Delta(E_1) \to \Delta(E_2)$ is $\mathcal{T}_{E_1} - \mathcal{T}_{E_2}$-continuous. These morphisms together with the regular ordered Saks spaces form the category $\mathbf{Saks}_1^+$ of regularly ordered Saks spaces.

$E \in \mathbf{Saks}_1^+$ is called *complete*, if $\Delta(E)$ is $\mathcal{T}_E$-complete and *compact* if $\Delta(E)$ is $\mathcal{T}_E$-compact. This leads to the full subcategories $\mathbf{CompSaks}_1^+$ and $\mathbf{ComplSaks}_1^+$ of compact or complete objects of $\mathbf{Saks}_1^+$ respectively.

Regularly ordered Saks spaces abound as the following examples show:

Examples 3.2. (i) If E is a base normed Saks space ([18], [27]) with base $\mathrm{Bs}(E)$, then $\mathrm{C}(E) = \mathbb{R}_+\mathrm{B}s(E), \mathbb{R}_+ := [0, \infty[$ ([6], 3.8.3) is $\mathcal{T}_E$-closed, hence $\Delta(E)$ is, too, and $\Delta(E)$ is $\mathcal{T}_E$-bounded because of $\Delta(E) = [0, 1]\mathrm{Bs}(E)$. Therefore, E is in $\mathbf{Saks}_1^+$ if $\mathcal{T}_E$ is locally order convex.

(ii) For $E \in \mathbf{Vec}_1^+, \mathrm{O}(E)$ and $\mathrm{C}(E)$ are $\sigma(E, E')$-closed hence $\mathrm{O}(E)$ and $\Delta(E) =$

$\mathrm{O}(E) \cap \mathrm{C}(E)$ are bounded and closed, because $\sigma(E, E')$ is compatible with the norm topology. $(E, \sigma(E, E'))$ is locally convex and even locally order convex, as follows easily from the equality $E' = \mathrm{C}(E') - C(E')$ (cp. [27], (6.12)). This proves E with $\sigma(E, E')$ to be in $\mathbf{Saks}_1^+$.

(iii) For $E \in \mathbf{Vec}_1^+$, E' is also in $\mathbf{Vec}_1^+$ with the Riesz norm generated by $\mathrm{C}(E')$, $E' = \mathrm{C}(E') - \mathrm{C}(E')$, and $\mathrm{O}(E')$ is $\sigma(E', E)$-compact, in other words, weakly $*$-compact. $\mathrm{C}(E')$ is $\sigma(E', E)$-closed, hence $\Delta(E')$ is $\sigma(E', E)$-compact. E' is a locally convex space and even locally order convex, which follows by a reasoning analogous to (ii). This implies that E' with $\sigma(E', E)$ is in $\mathbf{CompSaks}_1^+$.

(iv) ([3]) Let E be a regularly ordered Saks space and assume that

(a) $\Delta(E)$ is superconvex or
(b) $\Delta(E)$ is $\mathcal{T}_E$-sequentially complete

then E with $\|\square\|_E$ is Banach space. In particular, if E is $\mathcal{T}_E$ - complete, then E is also $\|\square\|_E$-complete.

Proof of (iv). (a) Consider an increasing Cauchy sequence with respect to the norm $\|\square\|$ of $E, c_n \in \mathrm{C}(E), n \in \mathbb{N}$. We may assume that $c_n < c_{n+1}$ and $\|c_{n+1} - c_n\| < 2^{-n}$ for $n \in \mathbb{N}$. Put $e_n := 2^n(c_{n+1} - c_n) > 0,\ n \in \mathbb{N}$, and one has $\|e_n\| < 1$, i.e. $e_n \in \Delta(E),\ n \in \mathbb{N}$. Define

$$e_0 := \sum_{n=1}^{\infty} \frac{1}{2^n} e_n \in \Delta(E)$$

and

$$s_n := \sum_{i=1}^{n} \frac{1}{2^i} e_i = c_{n+1} - c_1.$$

Now, the computation rules for superconvex combinations ([16]), [18]) yield

$$\begin{aligned} e_0 &= (1 - 2^{-n}) \sum_{i=1}^{n} 2^{-i}(1 - 2^{-n}) e_i + 2^{-n} \sum_{i=n+1}^{\infty} 2^{n-i} e_i \\ &= \sum_{i=1}^{n} 2^{-i} e_i + 2^{-n} e_n^* = c_{n+1} - c_1 + 2^{-n} e_n^*, \end{aligned}$$

with

$$e_n^* := \sum_{i=n+1}^{\infty} 2^{n-i} e_i \in \Delta(E),$$

i.e. $\|e_n^*\| \le 1$. Hence, $\lim_{n\to\infty} c_n = e_0 + c_1 \in \mathrm{C}(E)$ i.e. $\mathrm{C}(E)$ is $\|\square\|$-closed and the assertion follows from [6], 3.5.11.

(b): Because of 2.5, (ii), (a), $\Delta(E)$ is superconvex and the assertion follows from (a).

4. The compactification

For $E \in \mathbf{Saks}_1^+$, $\Delta(E)$ is a topological positively convex set with the topology induced by $\mathcal{T}_E$. If E is even a complete or compact regularly ordered Saks space $\Delta(E)$ is a topological positively superconvex set. Thus functors $\Delta : \mathbf{Saks}_1^+ \to \mathbf{TopPC}$, $\Delta : \mathbf{ComplSaks}_1^+ \to \mathbf{TopPSC}$, $\Delta : \mathbf{CompSaks}_1^+ \to \mathbf{TopPSC}$ are defined, induced by $\Delta(E), E \in \mathbf{Saks}_1^+$.

Lemma 4.1 (cf. [7]). *If E is a Hausdorff topological linear space and $P \in$ **TopPSC**, then the following statements are equivalent for a continuous mapping $f : P \to E$:*

(i) *f is positively affine,*
(ii) *f is positively superaffine.*

Moreover, a continuous positively superaffine mapping $f : P \to E$ is bounded and $f(P)$ is a topological positively superconvex set.

Proof. $(ii) \Rightarrow (i)$ is trivial. For $(i) \Rightarrow (ii)$ consider an $\hat{\alpha} \in \Omega_{psc}$ and $x_i \in P$, $i \in \mathbb{N}$. As $\sum_{i=1}^{\infty} \alpha_i x_i$, is a limit of the sequence $(\sum_{i=1}^{n} \alpha_i x_i \mid n \in \mathbb{N})$ in P, $f(\sum_{i=1}^{\infty} \alpha_i x_i)$ is the limit of $(\sum_{i=1}^{n} a_i f(x_i) \mid n \in \mathbb{N})$ in E, i.e.

$$f\left(\sum_{i=1}^{\infty} \alpha_i x_i\right) = \sum_{i=1}^{\infty} \alpha_i f(x_i).$$

The last assertions follow from 2.5, (i).

Definition 4.2. For $P \in$ **TopPC** one defines

$$\mathrm{Aff}_c^+(P) := \{f \mid f : P \to \mathbb{R}, \text{ positively affine, bounded and continuous.}\}$$

From 4.1 it follows that $\mathrm{Aff}_c^+(P)$ consists exactly of all positively superaffine and continuous mappings for $P \in$ **TopPSC**.

$\mathrm{Aff}_c^+(P)$ is a Banach space with the supremum norm $\|\Box\|_\infty$ and ordered by the proper cone $\mathrm{C}(P) := \{f \in \mathrm{Aff}_c^+(P) \mid f(x) \geq 0 \text{ for } x \in P\}$, which is obviously closed. One puts

$$\mathrm{E}(P) := \mathrm{C}(P) - \mathrm{C}(P)$$

and for $g \in \mathrm{E}(P)$

$$\|g\| := \inf\{\|f\|_\infty \mid -f \leq g \leq f\}.$$

$\|g\|$ is well-defined and $\|g\| < \infty$, also $\|g\|_\infty \leq \|g\|$ holds, i.e. $\|\Box\|$ is a norm. As $\|g\| = \|g\|_\infty$ for $g \in \mathrm{C}(P)$, $\|\Box\|$ is a Riesz norm on $\mathrm{E}(P)$. For a set $M \subset \mathrm{E}(P)$, $\overline{M}$ will denote the closure of M with respect to $\|\Box\|$ and $\overline{M}_\infty$ the closure with respect to $\|\Box\|_\infty$. $\|\Box\|_\infty \leq \|\Box\|$ yields

$$\mathrm{C}(P) \subset \overline{\mathrm{C}(P)} \subset \overline{\mathrm{C}(P)}_\infty = \mathrm{C}(P)$$

i.e. $\mathrm{C}(P) = \overline{\mathrm{C}(P)}$ and $\mathrm{E}(P)$ with $\mathrm{C}(P)$ and $\|\Box\|$ is a regularly ordered linear space.

It was shown in 3.2, (iii), that $\mathrm{E}'(P)$ is a compact regular ordered Saks space, ordered by $\mathrm{C}'(P) := \mathrm{C}(\mathrm{E}'(P))$ and supplied with the weak $*$-topology $\sigma(\mathrm{E}'(P), \mathrm{E}(P))$. Besides, $\mathrm{E}'(P)$ with its Riesz norm defined by $\mathrm{C}'(P)$ is a regularly ordered Banach space.

Lemma 4.3. *For $P \in$ **TopPC**, the evaluation mapping $\hat{\tau}_P : P \longrightarrow \Delta(E'(P))$ defined by $\hat{\tau}_P(x)(f) := f(x)$, $x \in P$, $f \in E(P)$, is positively affine, $\mathcal{T}_P$ - weakly $*$-continuous and induces a natural transformation.*

Proof. $\hat{\tau}_P$ is trivially positively affine. As the weak $*$-topology is induced as the initial topology by $\{\hat{f} \mid f \in \mathrm{C}(P)\}$ with $\hat{f}(\lambda) = \lambda(f)$ for $\lambda \in \mathrm{E}'(P)$ the definition of $\hat{\tau}_P$ yields $\hat{f}\hat{\tau}_P = f$, for $f \in \mathrm{C}(P)$, and $\hat{\tau}_P$ is $\mathcal{T}_P$ - weakly $*$-continuous. Also $\|\hat{\tau}_P(x)\| \leq 1$ for $x \in P$ follows from the definition. Hence, one gets $\hat{\tau}_P(P) \subset \Delta(\mathrm{E}'(P))$ because $\hat{\tau}_P(x)(f) \geq 0$ for $f \geq 0$.

In the following, for a set $M \subset \mathrm{E}'(P)$, $\mathrm{cl}_*(M)$ will denote the weak $*$-closure of M. Define the linear space

$$\mathrm{S}(P) := \mathbb{R}_+\mathrm{cl}_*(\hat{\tau}_P(P)) - \mathbb{R}_+\mathrm{cl}_*(\hat{\tau}_P(P))$$

for $P \in$ **TopPC**. $\mathcal{T}_P^*$ or simply $\mathcal{T}^*$ will denote the topology on $\mathrm{S}(P)$ induced by the weak $*$-topology, i.e. by $\sigma(\mathrm{E}'(P), \mathrm{E}(P))$.

In the following proposition the notion of a solid set is needed. It is therefore recalled for the reader's convenience.

Definition 4.4 (cf. [27]). Let E be an ordered linear space. A subset $M \subset \mathrm{E}$ is called

(i) absolutely dominated if for any $x \in M$ there exists a $y \in M$ with $-y \leq x \leq y$;
(ii) absolutely order convex if for any $a \in M$ with $a \geq 0$ for the order interval $[-a, a] = \{x \mid -a \leq x \leq a\} \subset M$ holds.

M is called *solid* if it is absolutely dominated and absolutely order convex. For any subset $M \subset E$

$$\mathrm{sol}(M) := \bigcup\{[-m, m] \mid m \in M, m \geq 0\}.$$

is easily shown to be solid, possibly empty, of course. M is absolutely dominated if $M \subset \mathrm{sol}(M)$ holds and absolutely order convex iff $\mathrm{sol}(M) \subset M$.

Proposition 4.5. *If $P \in$ **TopPC**, then $\mathrm{S}(P)$ is a compact regularly ordered Saks space with cone $\mathrm{C}_0(P) = \mathbb{R}_+\mathrm{cl}_*(\hat{\tau}_P(P))$, norm $\|z\|_s := \inf\{\alpha > 0 \mid z \in \alpha\,\mathrm{sol}(\mathrm{cl}_*(\hat{\tau}_P(P)))\}$ and the topology $\mathcal{T}_P^*$. Moreover, $\mathrm{O}(\mathrm{S}(P)) = \mathrm{sol}(\mathrm{cl}_*(\hat{\tau}_P(P)))$ holds.*

Proof. $\mathrm{cl}_*(\hat{\tau}_P(P))$ is positively convex because $\hat{\tau}_P(P)$ is. As $\hat{\tau}_P(P) \subset \Delta(\mathrm{E}'(P))$ and $\Delta(\mathrm{E}'(P))$ is weakly $*$-compact (cf. 3.2, (iii)), $\mathrm{cl}_*(\hat{\tau}_P(P)) \subset \Delta(\mathrm{E}'(P))$ follows and $\mathrm{cl}_*(\hat{\tau}_P(P))$ is weakly $*$-compact. Also,

$$\mathrm{sol}(\mathrm{cl}_*(\hat{\tau}_P(P))) \subset \mathrm{sol}(\Delta(\mathrm{E}'(P)) \subset \mathrm{O}(\mathrm{E}'(P)) \qquad (*)$$

holds obviously. If $\zeta_i \in \mathrm{sol}(\mathrm{cl}_*(\hat{\tau}_P(P))$, $i \in I$, is a weakly $*$-convergent net, $x_0 = \lim_i \zeta_i$, then, for any $i \in I$, there is an $k_i \in \mathrm{cl}_*(\hat{\tau}_P(P))$ with $-k_i \leq \zeta_i \leq k_i$. As $\mathrm{cl}_*(\hat{\tau}_P(P))$ is weakly $*$-compact there is a weakly $*$-convergent subnet $k_{i_\nu}, \nu \in N$, of $\{k_i \mid i \in I\}$ and we put $k_0 := \lim_\nu k_{i_\nu} \in \mathrm{cl}_*(\hat{\tau}_P(P))$. This implies $x_0 \in [-k_0, k_0] \subset \mathrm{sol}(\mathrm{cl}_*(\hat{\tau}_P(P)))$, i.e. $\mathrm{sol}(\mathrm{cl}_*(\hat{\tau}_P(P))$ is weakly $*$-closed hence also weakly $*$-compact.

$\mathrm{sol}(\mathrm{cl}_*(\hat\tau_P(P)))$ is absolutely convex because $\mathrm{cl}_*(\hat\tau_P(P))$ is positively convex and it is absorbing. If $|\Box\|'$ denotes the norm in $\mathrm{E}'(P)$ $(*)$ implies $\|z\|' \le \|z\|_s,\ z \in \mathrm{S}(P)$, hence $\|\Box\|_s$ is a norm on $\mathrm{S}(P)$. It is even a Riesz norm as one easily shows, i.e.

$$\|z\|_s = \inf\{\|c\|_s \mid -c \le z \le c\}.$$

Let now $c_i \in \mathrm{C}_0(P), i \in I$, be a weakly $*$-convergent net, $x_0 = \lim_i c_i$ and assume $x_0 \neq 0$. One has $c_i = \alpha_i k_i$ with $\alpha_i \ge 0$ and $k_i \in \mathrm{cl}_*(\hat\tau_P(P)), i \in I$. There is a convergent subnet $k_j, j \in J$, of $k_i, i \in I$, and $k_0 := \lim_j k_j \in \mathrm{cl}_*(\hat\tau_P(P))$. As $\mathcal{T}_P^*$ is the topology of pointwise convergence, there is a $f_0 > 0, f_0 \in \mathrm{C}(P)$, with $x_0(f_0) \neq 0$. Hence, there is a $j_0(f_0)$ s.th., for $j \ge j_0(f_0), c_j(f_0) > 0$ holds and

$$\alpha_j = \frac{c_j(f_0)}{k_j(f_0)}$$

is defined. This implies the existence of $\alpha_0 := \lim_j \alpha_j$ and $x_0 = \alpha_0 k_0 \in \mathrm{C}_0(P)$, i.e. $\mathrm{C}_0(P)$ is weakly$*$-closed.

For $z \in \mathrm{S}(P)$ with $\|z\|_s = 1$ and $n \in \mathbb{N}$, $n > 0$,

$$\left\|\left(1 - \frac{1}{n}\right) z\right\|_s < 1$$

follows and $(1 - n^{-1})z \in \mathrm{sol}(\mathrm{cl}_*(\hat\tau_P(P)))$ because of the definition of $\|\Box\|_s$. This implies $z = \lim_n (1 - n^{-1})z \in \mathrm{sol}(\mathrm{cl}_*(\hat\tau_P(P)))$ and consequently (cf. $(*)$)

$$\mathrm{O}(\mathrm{S}(P)) = \mathrm{sol}(\mathrm{cl}_*(\hat\tau_P(P))),$$

i.e. $\mathrm{O}(\mathrm{S}(P))$ is weakly $*$-compact. Because of $\Delta(\mathrm{S}(P)) = \mathrm{O}(\mathrm{S}(P)) \cap \mathrm{C}_0(P)$ this means weak $*$-compactness of $\Delta(\mathrm{S}(P))$ and the assertion.

Theorem 4.6. *The* $\mathrm{S}(P)$, $P \in \mathbf{TopPC}$, *induce a functor* $\mathrm{S} : \mathbf{TopPC} \to \mathbf{CompSaks}_1^+$ *which is left adjoint to the full and faithful functor* $\Delta : \mathbf{CompSaks}_1^+ \to \mathbf{TopPC}$ *by the restriction* $\tau_P : P \to \Delta \circ \mathrm{S}(P)$ *of* $\hat\tau_P$.

Proof. A straightforward calculation shows that the $\mathrm{S}(P), P \in \mathbf{TopPC}$, induce a functor and that τ_P is a natural transformation, see Lemma 4.3.

Now, let F be a compact regularly ordered Saks space and put $\tau_0 := \tau_{\Delta(F)}$ for brevity's sake. Any $f \in \mathrm{Aff}_c^+(\Delta(F))$ is bounded because $\Delta(F)$ is compact. $f \ge 0$ can be uniquely extended to a cone morphism $\tilde f : \mathrm{C}(F) \to \mathbb{R}_+$ and then uniquely to a linear form $\lambda_f \in (F, \mathrm{C}(F), \|\Box\|)'$, hence any $f \in \mathrm{E}(\Delta(F))$ can be uniquely extended to an $ext(f) \in (F, \mathrm{C}(F), \|\Box\|)'$. On the other hand, any $\mu \in (F, \mathrm{C}(F), \mathcal{T}_F)'$ restricted to $\Delta(F)$ yields a $rest(\mu) \in \mathrm{E}(\Delta(F))$. This yields two canonical injections

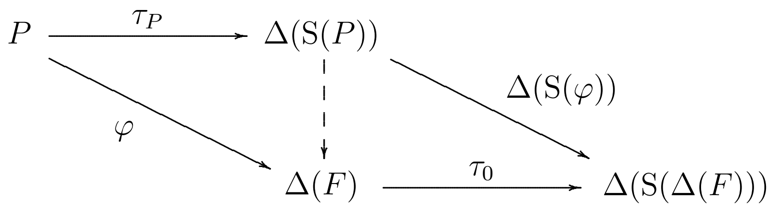
$$(F, \mathrm{C}(F), \mathcal{T}_P)' \overset{rest}{\to} \mathrm{E}(\Delta(F)) \overset{ext}{\to} (F, \mathrm{C}(F), \|\Box\|)'$$

If $x, y \in \Delta(F)$ with $\hat\tau_0(x) = \hat\tau_0(y)$, then, for any $\lambda \in (F, \mathrm{C}(F), \mathcal{T}_F)', \lambda(x) = \lambda(y)$ follows, i.e. $x = y$ and $\hat\tau_0 : \Delta(F) \to \mathrm{E}'(\Delta(F))$ is injective. $\hat\tau_0(\Delta(F))$ is compact, hence, in particular, closed and one has $\mathrm{C}_0(\Delta(F)) = \mathbb{R}_+ \tau_0(\Delta(F))$. $\tau_0 : \Delta(F) \longrightarrow$

$\Delta(\mathrm{S}(\Delta(F))$ can be uniquely extended to a cone morphism $\tilde{\tau}_0 : \mathrm{C}(F) \longrightarrow \mathrm{C}_0(\Delta(F))$ by the definition $\tilde{\tau}_0(\alpha e) := \alpha\tilde{\tau}_0(e)$, $\alpha \geq 0$ and $e \in \Delta(F)$. $\tilde{\tau}_0$ is obviously a cone isomorphism hence preserves the order. As $\mathrm{O}(\mathrm{S}(\Delta(F)) = \mathrm{sol}(\tau_0(\Delta(F)))$holds, $d \in \Delta(\mathrm{S}(\Delta(F)))$ implies $0 \leq d \leq \tau_0(e)$ with $e \in \Delta(F)$. This leads to $0 \leq \tilde{\tau}_0^{-1}(d) \leq e$, i.e. $\tilde{\tau}_0^{-1}(d) \in \Delta(F)$ and $d \in \tau_0(\Delta(F))$ which proves $\tau_0(\Delta(F)) = \Delta(\mathrm{S}(\Delta(F))$, i.e. τ_0 is an isomorphism in **TopPC**.

Now, τ_0 and τ_0^{-1} can be both extended uniquely first to the cones, like above, and then in an obvious way uniquely to the spaces F and $\mathrm{S}(\Delta(F))$ in $\mathbf{CompSaks}_1^+$. This yields an isomorphism $t_0 : F \to \mathrm{S}(\Delta(F))$ in $\mathbf{CompSaks}_1^+$. In exactly the same way one proves that the functor Δ is full and faithful.

To show the adjointness we regard a morphism $\varphi : P \longrightarrow \Delta(F)$ in **TopPC** with $F \in \mathbf{CompSaks}_1^+$:



As τ_0 is an isomorphism $\tau_0^{-1}\Delta(\mathrm{S}(\varphi))$ is in **TopPC** and $\tau_0^{-1}\Delta(\mathrm{S}(\varphi)) = \Delta(t_0^{-1}\mathrm{S}(\varphi))$ $= \Delta(g)$, $g := t_0^{-1}\mathrm{S}(\varphi)$, follows. This implies $\varphi = \Delta(g)\tau_P$. g is uniquely determined by this equation: If $h : \mathrm{S}(P) \to F$ in $\mathbf{CompSaks}_1^+$ also satisfies it, then, for any $x \in P$,

$$\varphi(x) = g(\tau_P(x)) = h(\tau_P(x))$$

follows. As g and h are continuous on $\Delta(\mathrm{S}(P))$ the coincide on $\mathrm{cl}_*(\tau_P(P))$ and hence on $\mathrm{C}(P)$. This implies $g = h$ which proves the assertion.

Actually $\Delta : \mathbf{CompSaks}_1^+ \longrightarrow \mathbf{TopPC}$ takes its values in **TopPSC** because the weakly $*$-compact positively convex set $\Delta(\mathrm{S}(P))$ is bounded and, in particular, sequentially complete (cf. 2.5 (*ii*), (*a*)). If, by abuse of notation, we denote the restriction of S to the subcategory **TopPSC** of **TopPC** again by S, we get at once the following

Corollary 4.7. *The functor* $\mathrm{S} : \mathbf{TopPSC} \longrightarrow \mathbf{CompSaks}_1^+$ *is left adjoint to* $\Delta : \mathbf{CompSaks}_1^+ \longrightarrow \mathbf{TopPSC}$ *with the adjunction morphism* $\tau_P : P \longrightarrow \Delta \circ \mathrm{S}(P)$.

In analogy to Definition 4.5 in [19] we introduce separated topological positively convex modules and also compact ones.

Definition 4.8 (cf. [8], [15], [19]). $P \in \mathbf{TopPC}$ is called *separated* if **TopPC** $(P, [0,1])$ is point separating.

$P \in \mathbf{TopPC}$ is called *compact* iff P is separated and the topology T_P of P makes P a compact topological space.

The separated topological positively convex modules generate the full subcategory $\mathbf{TopPC}_{\mathrm{sep}}$ and the compact ones generate the full subcategory **CompPC** of $\mathbf{TopPC}_{\mathrm{sep}}$. The corresponding subcategories of **TopPSC** are defined analogously.

Proposition 4.9.

(i) $P \in \mathbf{TopPc}$ *is separated iff* τ_P *is injective.*

(ii) $P \in \mathbf{TopPC}$ *is compact iff the restriction* $\overline{\tau}_P$ *of* τ_P *to* $\tau_P(P)$ *is an isomorphism.*

(iii) *If* $P \in \mathbf{TopPC}$ *is compact then* $P \in \mathbf{TopPSC}$ *holds.*

Proof. (i): The follows completely analogous to the proof of the injectivity of τ_0 in 4.6.

(ii): Let P be compact then τ_P is injective hence $\overline{\tau}_P : P \longrightarrow \tau_P(P)$ is bijective and a homomorphism and, finally, an isomorphism in **TopPC**.

(iii): Because of 2.5, (ii), (a), $\tau_P(P)$ is a topological positively superconvex set which implies the assertion because of (ii).

Theorem 4.10. *Put* $K(P) := cl_*(\tau_P(P))$ *for* $P \in \mathbf{TopPC}$. *Then with the weak* $*$*-topology,* $\mathrm{K}(P) \in \mathbf{CompPC}$ *holds. The* $\mathrm{K}(P), P \in \mathbf{TopPC}$*, induce a functor* $\mathrm{K} : \mathbf{TopPC} \longrightarrow \mathbf{CompPC}$ *which is an epireflection with the reflection morphism* $\kappa_P : P \longrightarrow \mathrm{K}(P)$*, where* κ_P *is the restriction of* τ_P *to* $\mathrm{K}(P)$.

Proof. Let $\varphi : P_1 \to P_2$ be a morphism in **TopPC**. If one defines $\mathrm{K}(\varphi) := \mathrm{K}(P_2) \setminus \Delta(\mathrm{S}(\varphi))/\mathrm{K}(P_1)$, the definition of τ_P at one yields $\mathrm{K}(\varphi)\kappa_{P_1} = \kappa_{P_2}\varphi$, hence K is a functor and κ a natural transformation.

For $Q \in \mathbf{CompPC}$ $\kappa_Q = \overline{\tau}_Q$ is an isomorphism because of 4.9, (ii). For a morphism $\varphi : P \to Q$ in **TopPC** put $\varphi_0 := \kappa_Q^{-1}\mathrm{K}(\varphi)$, then $\varphi = \varphi_0\kappa_P$ follows. This equation determines φ_0 uniquely , because if $\varphi = \psi\kappa_P$ with some $\psi : P \to Q$, then we have $\varphi(x) = \psi(\tau_P(x)) = \varphi_0(\tau_P(x))$ for any $x \in P$. This implies $\psi = \varphi_0$ on $\tau_P(P)$ hence $\psi = \varphi_0$ which proves the assertion.

References

[1] L. Asimov: Universally well-capped cones, Pac. J. Math. 26 (1968) 421–431.

[2] E.-E. Doberkat: Eilenberg-Moore algebras for stochastic relations, Inf. Comput. 204 (2006) 1756–1781.

[3] J. B. Cooper: Saks Spaces and Applications to Functional Analysis, North Holland, Amsterdam (1978).

[4] D. A. Edwards: On the homeomorphic affine embedding of a locally compact cone into a Banach dual space endowed with the vague topology, Proc. Lond. Math. Soc., III. Ser. 14 (1984) 399–414.

[5] M. Giry: A categorical approach to probability theory, in: Categorical Aspects of Topology and Analysis (Ottawa, 1980), B. Banaschewski (ed.), Lect. Notes Math. 915, Spinger, Berlin (1981) 68–85.

[6] G. Jameson: Ordered Linear Spaces, Lect. Notes Math. 141, Springer, Berlin (1970).

[7] G. Jameson: Convex series, Proc. Camb. Philos. Soc. 72 (1972) 37–47.

[8] R. Kemper: Positively convex spaces, Appl. Categ. Struct. 6 (1998) 333–344.

[9] H. Kleisli, H.-P. Künzi: Topological totally convex spaces I, Appl. Categ. Struct. 2 (1994) 45–55.

[10] H. Kleisli, H.-P. Künzi: Topological totally convex spaces II, Cah. Topologie Géom. Différ. Catég. 36 (1995) 11–52.

[11] J. D. Lawson: Embeddings of compact convex sets and locally compact cones, Pac. J. Math. 66 (1976) 443–453.

[12] D. Pumplün: Regularly ordered Banach spaces and positively convex spaces, Result. Math. 7 (1984) 85–112.

[13] D. Pumplün, H. Röhrl: Banach spaces and totally convex spaces I, Commun. Algebra 12 (1984) 953–1019.

[14] D. Pumplün, H. Röhrl: Banach spaces and totally convex spaces II, Commun. Algebra 13 (1985) 1047–1113.

[15] D. Pumplün, H. Röhrl: Convexity Theories. IV, Klein-Hilbert parts in convex modules, Appl. Categ. Struct. 3 (1985) 173–200.

[16] D. Pumplün: Banach spaces and superconvex modules, in: Proc. 2nd Gauss Sympos., Conference A: Mathematics and Theoretical Physics (Munich, 1993), M. Behara et al. (ed.), de Gruyter, Berlin (1995) 323–338.

[17] D. Pumplün: Absolutely convex modules and Saks spaces, J. Pure Appl. Algebra 155 (2001) 257–270.

[18] D. Pumplün: The universal compactification of topological convex sets and modules, J. Convex Analysis 8 (2001) 255–267.

[19] D. Pumplün: Positively convex modules and ordered normed linear spaces, J. Convex Analysis 10 (2003) 109–127.

[20] J. W. Roberts: The embedding of compact convex sets in locally convex spaces, Can. J. Math. 30 (1978) 449–454.

[21] H. Rummler: Bemerkungen und Beispiele zum Begriff der Superkonvexität, Seminarberichte, FB Mathematik Nr. 41, Fernuniversität Hagen (1981).

[22] Z. Semadeni: Free compact convex sets, Bull. Acad. Pol. Sci., Sér. Sci. Math. Astron. Phys. 13 (1965) 141–146.

[23] Z. Semadeni: Categorical methods in convexity, Proc. Colloq. Convexity, Copenhagen 1965 (1967) 281–307.

[24] Z. Semadeni: Monads and their Eilenberg-Moore algebras in functional analysis, Queen's Papers Pure Appl. Math. 33 (1973).

[25] T. Swirszcz: Monadic functors and convexity, Bull. Acad. Pol. Sci., Sér. Sci. Math. Astron. Phys. 22 (1974) 39–42.

[26] T. Swirszcz: Monadic functors and categories of convex sets, Preprint 70, Inst. Math., Polish Acad. Sci., Warsaw (1975).

[27] Y.-C. Wong, K.-F. Ng: Partially Ordered Topological Vector Spaces, Clarendon Press, Oxford (1973).