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The Fitzpatrick Function - A Bridge between Convex Analysis and Multivalued Stochastic Differential Equations*

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Using the Fitzpatrick function, we characterize the solutions for different classes of deterministic and stochastic differential equations driven by maximal monotone operators (or in particular subdifferential operators) as the minimum point of a suitably chosen convex lower semicontinuous function. Such technique provides a new approach for the existence of the solutions for the considered equations.

Keywords: Maximal monotone operators, Fitzpatrick function, Skorohod problem, stochastic differential equations

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1. Preliminaries. Notations

The Fitzpatrik function proved to be a very useful tool of the convex analysis in the study of maximal monotone operators. In our paper this function is used for deterministic and stochastic differential equations driven by multivalued maximal monotone operators. We will show how we can reduce the existence problem for stochastic differential equations of the following types:

- *forward case*

$$\begin{cases} dX_t + A\left(X_t\right)(dt) \ni F\left(t, X_t\right) dt + G\left(t, X_t\right) dW_t\,, \\ X_0 = \xi,\ t \in [0,T] \quad \text{and} \end{cases} \tag{1}$$

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- *backward case*

$$\begin{cases} -dY_t + A(Y_t)\,dt \ni H(t, Y_t, Z_t)\,dt - Z_t dW_t\,, \\ Y_T = \xi,\ t \in [0,T] \end{cases} \tag{2}$$

to a minimizing problem for convex lower semicontinuous functions.

Usually, existence results are obtained via a penalized problem with Yosida's approximation operator $A_\varepsilon := [I - (I + \varepsilon A)^{-1}]/\varepsilon$.

For the forward equation (1), by studying first a generalized Skorohod problem

$$\begin{cases} dx(t) + A(x(t))(dt) \ni f(t)\,dt + dm(t)\,, \\ x(0) = x_0,\ t \in [0,T]\,. \end{cases}$$

the existence of the solution is obtained (see Bensoussan & Răşcanu [4], Răşcanu [16], or Asiminoaei & Răşcanu [1]) in the general case of a maximal monotone operator.

For backward stochastic differential equations the existence problem (see Pardoux & Rascanu [13]) is solved only in the case of $A = \partial\varphi$ (the subdifferential of a lower semicontinuous convex function) and it is an open problem in the general case. That is *the reason and the main motivation* to find an approach via convex analysis.

In 1988, in the paper [10], Fitzpatrick proved that any maximal monotone operator can be represented by a convex function; he explicitly defined the minimal convex representation. The connection between maximal monotone operators and convex functions was also approached 13 years later by Martinez-Legaz & Théra in [12], Burachik & Svaiter in [7] and Burachik & Fitzpatrick in [6]. Since these last three papers, Fitzpatrick's results have been the subject of intense research (J. P. Revalski, M. Théra, R. S. Burachik, B. F. Svaiter, J.-P. Penot, S. Simons, C. Zălinescu, J.-E. Martinez-Legaz etc.). Their results stay in the domain of nonlinear operators: properties, characterizations, new classes of monotone operators.

Using the idea of Fitzpatrick function we can reduce the existence problems for stochastic equations of the form (1) or (2) to a minimizing problem of a convex lower semicontinuous function. Inspired by the studies of Gyöngy & Martínez [11], we present a new approach for solving the existence problem for stochastic differential equations with maximal monotone operator. In this paper we will identify the solutions of different types of forward and backward multivalued stochastic differential equations with the minimum points of a suitably chosen convex lower semicontinuous functionals.

The paper is organized as follows. In the first section we present some basic properties of the Fitzpatrick's function and we will introduce the stochastic framework that will be used. The next section contains a Fitzpatrick function approach for the study of a generalized Skorohod problem as well of forward and backward stochastic differential equations, while Section 3 is dedicated to the case of forward and backward stochastic variational inequalities.

1.1. On Fitzpatrick's function

Let $(\mathbb{X}, \|.\|)$ be a real Banach space and $(\mathbb{X}^*, \|.\|_*)$ be its dual. For $x^* \in \mathbb{X}^*$ and $x \in \mathbb{X}$ we denote $x^*(x)$ (the value of x^* in x) by $\langle x, x^* \rangle$ or $\langle x^*, x \rangle$.

If $A : \mathbb{X} \rightrightarrows \mathbb{X}^*$ is a point-to-set operator (from $\mathbb{X}$ to the family of subsets of $\mathbb{X}^*$), then $\mathrm{Dom}(A) := \{x \in \mathbb{X} : A(x) \neq \emptyset\}$ and $R(A) = \{x^* : \exists x \in \mathrm{Dom}(A) \text{ s.t. } x^* \in A(x)\}$. We shall always assume that the operator A is proper, i.e. $\mathrm{Dom}(A) \neq \emptyset$. Usually the operator A is identified with its graph $\mathrm{gr}(A) = \{(x, x^*) \in \mathbb{X} \times \mathbb{X}^* : x \in \mathrm{Dom}(A),\ x^* \in A(x)\}$.

The operator $A : \mathbb{X} \rightrightarrows \mathbb{X}^*$ is a monotone operator ($A \subset \mathbb{X} \times \mathbb{X}^*$ is a monotone set) if

$$\langle x - y, x^* - y^* \rangle \geq 0, \quad \forall (x, x^*), (y, y^*) \in A.$$

A monotone operator (set) is maximal monotone if it is not properly contained in any other monotone operator (set). Clearly if A is maximal monotone and $(y, y^*) \in \mathbb{X} \times \mathbb{X}^*$ then

$$\inf_{(u,u^*) \in A} \langle y - u, y^* - u^* \rangle \geq 0 \iff (y, y^*) \in A.$$

Given a function $\psi : \mathbb{X} \to]-\infty, +\infty]$ we denote $\mathrm{Dom}(\psi) := \{x \in \mathbb{X} : \psi(x) < \infty\}$. We say that ψ is proper if $\mathrm{Dom}(\psi) \neq \emptyset$. The subdifferential $\partial\psi : \mathbb{X} \rightrightarrows \mathbb{X}^*$ is defined by

$$(x, x^*) \in \partial\psi \ \text{ if } \ \langle y - x, x^* \rangle + \psi(x) \leq \psi(y), \quad \forall y \in \mathbb{X}.$$

It is well known that: if ψ is a proper convex l.s.c. function, then $\partial\psi : \mathbb{X} \rightrightarrows \mathbb{X}^*$ is a maximal monotone operator.

Let $\psi : \mathbb{X} \to]-\infty, +\infty]$ be a proper function. The conjugate of ψ is the function $\psi^* : \mathbb{X}^* \to]-\infty, +\infty]$,

$$\psi^*(x^*) := \sup\{\langle u, x^* \rangle - \psi(u) : u \in \mathbb{X}\}.$$

Remark that, if $h : \mathbb{X} \times \mathbb{X}^* \to]-\infty, +\infty]$, then $h^* : \mathbb{X}^* \times \mathbb{X}^{**} \to]-\infty, +\infty]$ and, for any $(x^*, x) \in \mathbb{X}^* \times \mathbb{X}$, $h^*(x^*, x)$ is well defined by identifying $\mathbb{X}$ with its image under canonical injection of $\mathbb{X}$ into $\mathbb{X}^{**}$, that is, every $x \in \mathbb{X}$ can be seen as a function $x : \mathbb{X}^* \to \mathbb{R}$ defined by $x(x^*) = x^*(x) = \langle x, x^* \rangle$. For a complete study on maximal monotone operators, one can consult Barbu [2] or Brézis [5].

Definition 1.1. Given a monotone operator $A : \mathbb{X} \rightrightarrows \mathbb{X}^*$, the associated Fitzpatrick function is defined as $\mathcal{H} = \mathcal{H}_A : \mathbb{X} \times \mathbb{X}^* \to]-\infty, +\infty]$,

$$\begin{aligned} \mathcal{H}(x, x^*) &:= \langle x, x^* \rangle - \inf\{\langle x - u, x^* - u^* \rangle : (u, u^*) \in A\} \\ &= \sup\{\langle u, x^* \rangle + \langle x, u^* \rangle - \langle u, u^* \rangle : (u, u^*) \in A\} \end{aligned} \tag{3}$$

Clearly $\mathcal{H}(x, x^*) \leq \langle x, x^* \rangle$, for all $(x, x^*) \in A$ and, as supremum of convex strongly (and (w, w^*)) continuous functions, $\mathcal{H} = \mathcal{H}_A : \mathbb{X} \times \mathbb{X}^* \to]-\infty, +\infty]$ is a proper convex strongly (and (w, w^*)) l.s.c. function. Usually, we shall consider on $\mathbb{X}$ the strong topology and, on $\mathbb{X}^*$ the w^*-topology; in this case, $\mathcal{H}$ is also a l.s.c. function. Whenever is necessary, we will consider the Fitzpatrick function $\mathcal{H}$ restricted at $\mathbb{U} \times \mathbb{V}$, with $\mathbb{U} \subset \mathbb{X}$ and $\mathbb{V} \subset \mathbb{X}^*$.

Let $(x^*, x) \in \partial\mathcal{H}(u, u^*)$. Then, from the definition of a subdifferential operator, we have

$$\langle (x^*, x), (z, z^*) - (u, u^*)\rangle + \mathcal{H}(u, u^*) \leq \mathcal{H}(z, z^*), \quad \forall (z, z^*) \in \mathbb{X}^{**} \times \mathbb{X}^*,$$

or, equivalently,

$$\begin{aligned} &\langle u - x, u^* - x^*\rangle - \inf\{\langle u - y, u^* - y^*\rangle : (y, y^*) \in A\} \\ \leq\ &\langle z - x, z^* - x^*\rangle - \inf\{\langle z - y, z^* - y^*\rangle : (y, y^*) \in A\}, \quad \forall (z, z^*) \in \mathbb{X}^{**} \times \mathbb{X}^*. \end{aligned} \tag{4}$$

Since the operator A is a maximal monotone one, then

$$\begin{aligned} &\inf\{\langle u - y, u^* - y^*\rangle : (y, y^*) \in A\} \leq 0 \quad \text{and} \\ &\inf\{\langle z - y, z^* - y^*\rangle : (y, y^*) \in A\} = 0, \quad \forall (z, z^*) \in A; \end{aligned}$$

consequently, we have

$$(x^*, x) \in \partial\mathcal{H}(u, u^*) \implies \langle u - x, u^* - x^*\rangle \leq \inf\{\langle z - x, z^* - x^*\rangle : (z, z^*) \in A\}. \tag{5}$$

Also, by the monotonicity of A, from (4) follows

$$(x, x^*) \in A \implies (x^*, x) \in \partial\mathcal{H}(x, x^*).$$

Hence, if $A : \mathbb{X} \rightrightarrows \mathbb{X}^*$ is a maximal monotone operator, then $\mathcal{H}_A$ characterizes A as follows.

Theorem 1.2 (Fitzpatrick, see Fitzpatrick [10], Simons & Zălinescu [17]). *Let $A : \mathbb{X} \rightrightarrows \mathbb{X}^*$ be a maximal monotone operator and $\mathcal{H}$ its associated Fitzpatrick function. Then, for all $(x, x^*) \in \mathbb{X} \times \mathbb{X}^*$,*

$$\mathcal{H}(x, x^*) \geq \langle x, x^*\rangle.$$

Moreover, the following assertions are equivalent:

(a) $(x, x^*) \in A$;
(b) $\mathcal{H}(x, x^*) = \langle x, x^*\rangle$;
(c) $\mathcal{H}^*(x^*, x) = \langle x, x^*\rangle$;
(d) $\exists (u, u^*) \in \mathrm{Dom}(\partial\mathcal{H})$ *such that* $(x^*, x) \in \partial\mathcal{H}(u, u^*)$ *and* $\langle u - x, u^* - x^*\rangle = 0$;
(e) $(x^*, x) \in \partial\mathcal{H}(x, x^*)$.

Proof. It is not difficult to show that $(b) \Leftrightarrow (a) \Rightarrow (e) \Rightarrow (d) \Rightarrow (a)$. Moreover, using the Fenchel equality:

$$(x^*, x) \in \partial\mathcal{H}(x, x^*) \Rightarrow \mathcal{H}(x, x^*) + \mathcal{H}^*(x^*, x) = \langle (x, x^*), (x^*, x)\rangle,$$

we obtain that $(e) \,\&\, (b) \Rightarrow (c)$. The point (c) yields (a) by using the equivalent form of the definition of $\mathcal{H}^*$:

$$\mathcal{H}^*(x^*, x) = \langle x, x^*\rangle - \inf_{(u,u^*)\in\mathbb{X}\times\mathbb{X}^*} \{\langle x - u, x^* - u^*\rangle + \mathcal{H}(u, u^*) - \langle u, u^*\rangle\}.$$

□

Remark 1.3. The function $\mathcal{H}_A$ is minimal in the family of convex functions $f : \mathbb{X} \times \mathbb{X}^* \to]-\infty, +\infty]$ with the properties: $f(x, x^*) \geq \langle x, x^* \rangle$ for all $(x, x^*) \in \mathbb{X} \times \mathbb{X}^*$ and $f(x, x^*) = \langle x, x^* \rangle$ for all $(x, x^*) \in A$.

Using the above tools, in the paper [17], Simons and Zălinescu give a nice proof of the famous Rockafellar's characterization of a maximal monotone operator.

Let $\mathbb{H}$ be a real separable Hilbert space and $A : \mathbb{H} \rightrightarrows \mathbb{H}$ be a maximal monotone operator. Denote for $\varepsilon > 0$, $J_\varepsilon, A_\varepsilon : \mathbb{H} \to \mathbb{H}$, the (1-, resp. $1/\varepsilon$ -) Lipschitz continuous functions $J_\varepsilon(x) = (I + \varepsilon A)^{-1}(x)$ and

$$A_\varepsilon(x) = \frac{x - J_\varepsilon(x)}{\varepsilon} \in A(J_\varepsilon(x)).$$

Let

$$BV_0([0,T];\mathbb{H}) = \{k : [0,T] \to \mathbb{H} : \updownarrow k \updownarrow_T < \infty,\ k(0) = 0\},$$

where $\updownarrow k \updownarrow_T := \|k\|_{BV([0,T];\mathbb{H})}$. If we consider on $C([0,T];\mathbb{H})$ the usual norm

$$\|y\|_{C([0,T];\mathbb{H})} = \|y\|_T = \sup\{|y(s)| : 0 \leq s \leq T\},$$

then $(C([0,T];\mathbb{H}))^* = BV_0([0,T];\mathbb{H})$. We denote the duality between these spaces by

$$\langle\langle z, g \rangle\rangle := \int_0^T \langle z(t), dg(t) \rangle.$$

Denote by $\mathcal{A}$ the realization on $C([0,T];\mathbb{H})$ of the maximal monotone operator $A : \mathbb{H} \rightrightarrows \mathbb{H}$, that is the operator $\mathcal{A} : C([0,T];\mathbb{H}) \rightrightarrows BV_0([0,T];\mathbb{H})$ defined as follows: $(x,k) \in \mathcal{A}$ if $x \in C([0,T];\mathbb{R}^d)$, $k \in BV_0([0,T];\mathbb{H})$ and one of the following equivalent conditions are satisfied:

(d_1) for all $0 \leq s \leq t \leq T$, $\int_s^t \langle x(r) - z, dk(r) - z^* dr \rangle \geq 0$, $\forall (z, z^*) \in A$;

(d_2) for all $0 \leq s \leq t \leq T$ and for all $u, u^* \in C([0,T];\mathbb{H})$ such that $(u(r), u^*(r)) \in A$, $\forall r \in [s,t]$,

$$\int_s^t \langle x(r) - u(r), dk(r) - u^*(r)dr \rangle \geq 0;$$

(d_3) for all $u, u^* \in C([0,T];\mathbb{H})$ such that $(u(r), u^*(r)) \in A$, $\forall r \in [0,T]$,

$$\int_0^T \langle x(r) - u(r), dk(r) - u^*(r)dr \rangle \geq 0.$$

$\mathcal{A}$ is a maximal monotone operator since, setting

$$u(r) = J_\varepsilon\left(\frac{x(r) + y(r)}{2}\right) = \frac{x(r) + y(r)}{2} - \varepsilon A_\varepsilon\left(\frac{x(r) + y(r)}{2}\right);$$

$$u^*(r) = A_\varepsilon\left(\frac{x(r) + y(r)}{2}\right)$$

in (d_2) written for $(x,k)\in\mathcal{A}$ and respectively for $(y,\ell)\in\mathcal{A}$ and taking then $\varepsilon\to 0$, we infer (since $\varepsilon A_\varepsilon \to 0$ as $\varepsilon\to 0$) that

$$\int_s^t \langle x(r)-y(r), dk(r)-d\ell(r)\rangle \geq 0, \quad \forall\, 0\leq s\leq t\leq T. \tag{6}$$

The maximality clearly follows from the definition of $\mathcal{A}$.

For the realization of the operator A on $L^r(0,T;\mathbb{H})$, $r\geq 1$, we use the same notation $\mathcal{A}$ without risk of confusion since every time we mention the space of realization. In this case, the operator $\mathcal{A}: L^r(0,T;\mathbb{H}) \rightrightarrows L^q(0,T;\mathbb{H})$, $\frac{1}{r}+\frac{1}{q}=1$ is defined by $(x,g)\in\mathcal{A}$ if

- $\int_s^t \langle x(r)-z, g(r)-z^*\rangle\, dr \geq 0$, for all $0\leq s\leq t\leq T$ and for all $(z,z^*)\in A$,

or (clearly), equivalently

- $g(t)\in A(x(t))$, a.e. $t\in[0,T]$.

Arguing similar to the previous situation, we obtain that $\mathcal{A}$ is a maximal monotone operator.

1.2. Stochastic framework

Let $(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\}_{t\geq 0})$ be a stochastic basis i.e. $(\Omega,\mathcal{F},\mathbb{P})$ is a complete probability space and $\{\mathcal{F}_t\}_{t\geq 0}$ is a filtration satisfying the usual assumptions of right continuity and completeness:

$$\mathcal{N}_{\mathbb{P}}\subset\mathcal{F}_s\subset\mathcal{F}_t=\bigcap_{\varepsilon>0}\mathcal{F}_{t+\varepsilon}\ ,$$

for all $0\leq s\leq t$, where $\mathcal{N}_{\mathbb{P}}$ is the set of all $\mathbb{P}$-null sets.

Let $(\mathbb{H},|\cdot|_{\mathbb{H}})$ be a real separable Hilbert space; if F is a closed subset of $\mathbb{H}$, denote by $\mathcal{B}_F$ the σ-algebra generated by the closed subsets of F.

Denote by $S^p_{\mathbb{H}}[0,T]$, $p\geq 0$, the space of progressively measurable continuous stochastic processes $X:\Omega\times[0,T]\to\mathbb{H}$ (i.e. $t\longmapsto X(\omega,t)$ is continuous a.s. $\omega\in\Omega$, and $(\omega,s)\longmapsto X(\omega,s):\Omega\times[0,T]\to\mathbb{H}$ is $\left(\mathcal{F}_t\otimes\mathcal{B}_{[0,t]},\mathcal{B}_{\mathbb{H}}\right)$ measurable for all $t\in[0,T]$), such that

$$\|X\|_{S^p_{\mathbb{H}}[0,T]}=\begin{cases}(\mathbb{E}\ \|X\|^p_T)^{\frac{1}{p}\wedge 1}<\infty, & \text{if } p>0,\\ \mathbb{E}\left[1\wedge\|X\|_T\right], & \text{if } p=0,\end{cases}$$

where

$$\|X\|_T := \sup_{t\in[0,T]}|X_t|\,.$$

The space $(S^p_{\mathbb{H}}[0,T], \|\cdot\|_{S^p_{\mathbb{H}}[0,T]})$, $p\geq 1$, is a Banach space and $S^p_{\mathbb{H}}[0,T]$, $0\leq p<1$, is a complete metric space with the metric $\rho(Z_1,Z_2)=\|Z_1-Z_2\|_{S^p_d[0,T]}$ (when $p=0$ the metric convergence coincides with the probability convergence).

If $\mathbb{H}=\mathbb{R}^d$ we will denote $S^p_{\mathbb{H}}[0,T]$ by $S^p_d[0,T]$.

Let $\left(\mathbb{H}_0,|\cdot|_{\mathbb{H}_0}\right)$ be a real separable Hilbert space and

$$B=\{B_t(\varphi):(t,\varphi)\in[0,T]\times\mathbb{H}_0\}\subset L^0(\Omega,\mathcal{F},\mathbb{P})$$

a Gaussian family of real-valued random variables with zero mean and covariance function

$$\mathbb{E}\left[B_t(\varphi)B_s(\psi)\right] = (t \wedge s) \times \langle \varphi, \psi \rangle_{\mathbb{H}_0}, \quad \forall \varphi, \psi \in \mathbb{H}_0, \ \forall s, t \in [0,T],$$

where $t \wedge s = \min\{t, s\}$. We call $(B,\{\mathcal{F}_t\})$ a $\mathbb{H}_0$- *Wiener process* if, for all $t \in [0,T]$, we have

(i) $\mathcal{F}_t^B = \sigma\{B_s(\varphi);\ s \in [0,t], \varphi \in \mathbb{H}_0\} \vee \mathcal{N}_{\mathbb{P}} \subset \mathcal{F}_t$ and
(ii) $B_{t+h}(\varphi) - B_t(\varphi)$ is independent of $\mathcal{F}_t$, for all $h > 0$, $\varphi \in \mathbb{H}_0$.

Note that, given any orthonormal basis $\{e_i;\ i \in I \subseteq \mathbb{N}^*\}$ of $\mathbb{H}_0$, the sequence $\beta^i = \{\beta_t^i = B_t(e_i);\ t \in [0,T]\}$, $i \in I$, defines a family of independent real-valued standard Wiener processes (Brownian motions). Moreover, if $\mathbb{H}_0$ is of finite dimension, we have

$$B_t = \sum_{i \geq 1} \beta_t^i e_i, \quad t \in [0,T].$$

In the general case this series does not converge in $\mathbb{H}_0$, but rather in a larger space $\tilde{\mathbb{H}}_0, \mathbb{H}_0 \subset \tilde{\mathbb{H}}_0$ which is such that the injection of $\mathbb{H}_0$ into $\tilde{\mathbb{H}}_0$ is Hilbert-Schmidt. Moreover, $B \in \mathcal{M}^2(0,T;\tilde{\mathbb{H}}_0)$.

By $\mathcal{M}^p(0,T;\mathbb{H})$, $p \geq 1$, we denote the space of $\mathbb{H}$-valued continuous, p-integrable martingales M, that is, the space of all continuous stochastic processes $M : \Omega \times [0,T] \to \mathbb{H}$ satisfying, $\mathbb{P}$-$a.s$,

(m_1) $M_0 = 0$,
(m_2) $\mathbb{E}\,|M_t|^p < \infty, \quad \forall t \in [0,T]$,
(m_3) $\mathbb{E}\left[M_t|\mathcal{F}_s\right] = M_s$, for all $0 \leq s \leq t \leq T$.

$\mathcal{M}^p(0,T;\mathbb{H})$ is a Banach space with respect to the norm $\|X\|_{\mathcal{M}^p} = (\mathbb{E}\ |X_T|^p)^{1/p}$; in the case $p > 1$, $\mathcal{M}^p(0,T;\mathbb{H})$ is a closed linear subspace of $S_{\mathbb{H}}^p[0,T]$.

In order to define the stochastic integral with respect to the $\mathbb{H}_0$-Wiener process B, we introduce a class of processes with values in the separable Hilbert space $\mathcal{L}^2(\mathbb{H}_0;\mathbb{H})$ of Hilbert–Schmidt operators from $\mathbb{H}_0$ into $\mathbb{H}$, i.e. the space of linear operators $F : \mathbb{H}_0 \to \mathbb{H}$ satisfying

$$\|F\|_{HS}^2 = \sum_{i=1}^{\infty} |Fe_i|_{\mathbb{H}}^2 = \mathbf{Tr} F^* F = \mathbf{Tr} F F^* < \infty.$$

Denote $\Lambda_{\mathbb{H}\times\mathbb{H}_0}^p(0,T)$, $p \in [0,\infty[$, the space of progressively measurable processes $Z : \Omega \times]0,T[\to \mathcal{L}^2(\mathbb{H}_0;\mathbb{H})$ such that:

$$\|Z\|_{\Lambda^p} = \begin{cases} \left[\mathbb{E}\left(\int_0^T \|Z_s\|_{HS}^2 ds\right)^{\frac{p}{2}}\right]^{\frac{1}{p}\wedge 1}, & \text{if } p > 0, \\ \mathbb{E}\left[1 \wedge \left(\int_0^T \|Z_s\|_{HS}^2 ds\right)^{\frac{1}{2}}\right], & \text{if } p = 0. \end{cases}$$

The space $(\Lambda^p_{\mathbb{H}\times\mathbb{H}_0}(0,T), \|\cdot\|_{\Lambda^p})$, $p \geq 1$, is a Banach space and $\Lambda^p_{\mathbb{H}\times\mathbb{H}_0}(0,T)$, $0 \leq p < 1$, is a complete metric space with the metric $\rho(Z_1, Z_2) = \|Z_1 - Z_2\|_{\Lambda^p}$.

Consider $\{e_i;\ i \in I \subset \mathbb{N}^*\}$ an orthonormal basis of $\mathbb{H}_0$. Let $Z \in \Lambda^p_{\mathbb{H}\times\mathbb{H}_0}(0,T)$, with $p \geq 0$. The stochastic integral I is defined by $Z \overset{I}{\mapsto} I_\cdot(Z)$, where

$$I_t(Z) := \int_0^t Z_s dB_s = \sum_{i\in I}\int_0^t Z_s(e_i)dB_s(e_i), \quad t \in [0,T].$$

Note that it doesn't depend on the choice of the orthonormal basis of $\mathbb{H}_0$. The application

$$I : \Lambda^p_{\mathbb{H}\times\mathbb{H}_0}(0,T) \to S^p_{\mathbb{H}}[0,T]$$

is a linear continuous operator and it has the following properties:

(a) $\mathbb{E}I_t(Z) = 0$, if $p \geq 1$,

(b) $\mathbb{E}|I_T(Z)|^2 = \|Z\|^2_{\Lambda^2}$, if $p \geq 2$,

(c) $\frac{1}{c_p}\|Z\|^p_{\Lambda^p} \leq \mathbb{E}\sup_{t\in[0,T]}|I_t(Z)|^p \leq c_p\|Z\|^p_{\Lambda^p}$, if $p > 0$,

(Burkholder-Davis-Gundy inequality)

(d) $I(Z) \in \mathcal{M}^p(\Omega\times[0,T];\mathbb{H})$, $p \geq 1$.

The definition and the properties of the stochastic integral can be found in Pardoux & Răşcanu [14] or Da Prato & Zabczyk [15].

If $\mathbb{H}_0 = \mathbb{R}^k$ and $\mathbb{H} = \mathbb{R}^d$ then $\{B_t, t \geq 0\}$ is a k-dimensional Wiener process (Brownian motion); $\mathcal{L}^2(\mathbb{H}_0;\mathbb{H})$ is the space of real matrices $F = (f_{ij})_{d\times k}$ and $|F|^2 := \|F\|^2_{HS} = \sum_{i,j} f^2_{i,j}$. In this situation, the space $\Lambda^p_{\mathbb{H}\times\mathbb{H}_0}(0,T)$ will be denoted by $\Lambda^p_{d\times k}(0,T)$.

2. Fitzpatrick function approach

2.1. A Generalized Skorohod problem

Throughout this section $\mathbb{H}$ is a real separable Hilbert space with the norm $|\cdot|$ and the scalar product $\langle\cdot,\cdot\rangle$.

We study the multivalued monotone differential equation

$$\begin{cases} dx(t) + Ax(t)(dt) \ni dm(t), \\ x(0) = x_0, \quad t \geq 0, \end{cases} \qquad (GSP) \tag{7}$$

where we assume

$$(H_{GSP}) : \quad \begin{cases} \text{(i)} & A : \mathbb{H} \rightrightarrows \mathbb{H} \text{ is a maximal monotone operator,} \\ \text{(ii)} & x_0 \in \overline{\mathrm{Dom}(A)}, \\ \text{(iii)} & m : [0,\infty) \longrightarrow \mathbb{H} \text{ is continuous and } m(0) = 0. \end{cases}$$

Definition 2.1. A continuous function $x:[0,T]\rightarrow\mathbb{H}$ is a solution of Eq. (7) if $x(t)\in\overline{\mathrm{Dom}(A)}$ for all $0\leq t\leq T$, (T arbitrarily fixed) and there exists $k\in C([0,T];\mathbb{H})\bigcap BV_0([0,T];\mathbb{H})$ such that

$$x(t)+k(t)=x_0+m(t),\quad \forall 0\leq t\leq T$$

and

$$\int_s^t \langle x(r)-z,dk(r)-z^*dr\rangle\geq 0,\quad \forall(z,z^*)\in A,\ \forall 0\leq s\leq t\leq T. \tag{8}$$

(Without confusion, the uniqueness of k will permit us to call the pair (x,k) solution of the generalized Skorohod problem (GSP) and we write $(x,k)=\mathcal{GSP}(A;x_0,m)$.)

In virtue of this definition, the (classical) Skorohod problem (for more details, one can consult Cépa [8] or [9]) is obtained for $A=\partial I_E:\mathbb{R}^d\rightrightarrows\mathbb{R}^d$, where E is a closed convex subset of $\mathbb{R}^d$,

$$I_E(x)=\begin{cases}0, & \text{if } x\in E,\\ +\infty, & \text{if } x\in\mathbb{R}^d\setminus E\end{cases}$$

and

$$\partial I_E(x)=\begin{cases}0, & \text{if } x\in \mathrm{int}(E),\\ \left\{\nu\in\mathbb{R}^d:\langle\nu,y-x\rangle\leq 0,\ \text{for all } y\in E\right\}, & \text{if } x\in Bd(E),\\ \emptyset, & \text{if } x\notin E.\end{cases}$$

The definition of the solution can be given in a equivalent form as follows.

Definition 2.2. A continuous function $x:[0,T]\rightarrow\mathbb{R}^d\times\mathbb{R}^d$ is a solution of Skorohod problem in E if $x(t)\in E$ for all $0\leq t\leq T$ and there exists $k\in C\left([0,T];\mathbb{R}^d\right)\bigcap BV_0\left([0,T];\mathbb{R}^d\right)$ such that

$$\begin{cases}\text{(a)} & \updownarrow k\updownarrow_t=\displaystyle\int_0^t \mathbf{1}_{x(s)\in Bd(E)}d\updownarrow k\updownarrow_s,\\ \text{(b)} & k(t)=\displaystyle\int_0^t n_{x(s)}d\updownarrow k\updownarrow_s,\quad \text{where } n_{x(s)}\in N_E(x(s))\\ & \qquad\text{and } \left|n_{x(s)}\right|=1,\ d\updownarrow k\updownarrow_s\text{-a.e.}\end{cases}$$

and

$$x(t)+k(t)=x_0+m(t),\quad \forall t\in[0,T].$$

($N_E(x)$ denotes the outward normal cone to E at $x\in E$.)

Let $\mathcal{A}:C([0,T];\mathbb{H})\rightrightarrows BV_0([0,T];\mathbb{H})$ be the realization of the maximal monotone operator $A:\mathbb{H}\rightrightarrows\mathbb{H}$ and

$$\mathbb{X}=\{\mu\in C([0,T];\mathbb{H}):\mu(0)=0\}$$

the linear closed subspace of $C([0,T];\mathbb{H})$. For each $R>0$, we define

$$\mathbb{Y}_R=\{k\in C([0,T];\mathbb{H}):k(0)=0,\ \updownarrow k\updownarrow_T\leq R\};$$

$\mathbb{Y}_R$ is a closed subset of $C([0,T];\mathbb{H})$ and, consequently, it is a metric space with respect to the metric from $C([0,T];\mathbb{H})$. Remark that, by Helly-Foiaş theorem (see Barbu & Precupanu [3], Theorem 3.5 & Remark 3.2), it is also a bounded w^*-closed subset of $BV_0([0,T];\mathbb{H})$.

Let $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ a continuous function such that $\alpha(0) = 0$. Denote

$$C_\alpha = \{x \in \mathbb{X} : \mathbf{m}_x(\varepsilon) \le \alpha(\varepsilon) \text{ for all } \varepsilon \ge 0\}.$$

Here the function $\mathbf{m}_x : \mathbb{R}_+ \to \mathbb{R}_+$ represents the *modulus of continuity* of the continuous function $x : [0,T] \to \mathbb{H}$ and it is defined by

$$\mathbf{m}_x(\delta) = \mathbf{m}_{x,T}(\delta) = \sup\{|x(t) - x(s)| : |t-s| \le \delta,\ t,s \in [0,T]\}.$$

Clearly, C_α is a bounded closed convex subset of $\mathbb{X}$.

Consider, for each $(u,u^*) \in \mathcal{A}$ and $\nu \in \mathbb{X}$, the function $J_{(u,u^*,\nu)} : \mathbb{H} \times \mathbb{X} \times \mathbb{Y}_R \times \mathbb{X} \to \mathbb{R}$ given by

$$\begin{aligned} & J_{(u,u^*,\nu)}(a,x,k,\mu) \\ = \ & |a - x_0|^2 + \int_0^T [\langle u(t), dk(t)\rangle + \langle x(t), du^*(t)\rangle - \langle u(t), du^*(t)\rangle] \\ & \quad - \int_0^T \langle x(t), dk(t)\rangle + 2R\|\mu - m\|_T + \int_0^T \langle \mu(t) - \nu(t), dk(t)\rangle - R\|\nu - m\|_T \end{aligned}$$

and $\hat{J} : \mathbb{H} \times \mathbb{X} \times \mathbb{Y}_R \times \mathbb{X} \to]-\infty, +\infty]$, defined by

$$\begin{aligned} \hat{J}(a,x,k,\mu) &= \sup_{(u,u^*)\in\mathcal{A},\ \nu\in C_\alpha} J_{(u,u^*,\nu)}(a,x,k,\mu) \\ &= |a - x_0|^2 + \mathcal{H}(x,k) - \langle\!\langle x,k\rangle\!\rangle + 2R\|\mu - m\|_T \\ &\qquad + \sup_{\nu\in C_\alpha}\{\langle\!\langle \mu-\nu,k\rangle\!\rangle - R\|\nu - m\|_T\}, \end{aligned} \tag{9}$$

where $\mathcal{H} : C([0,T];\mathbb{H}) \times BV_0([0,T];\mathbb{H}) \to]-\infty,+\infty]$ is the Fitzpatrick function associated to the maximal monotone operator $\mathcal{A}$.

Remark 2.3. $\hat{J} : \mathbb{H} \times \mathbb{X} \times \mathbb{Y}_R \times \mathbb{X} \to]-\infty,+\infty]$ is a lower semicontinuous function as the supremum of the continuous functions $J_{(u,u^*,\nu)}$.

Remark also that, for $\mu \in C_\alpha$,

$$2R\|\mu - m\|_T + \sup_{\nu\in C_\alpha}\{\langle\!\langle \mu-\nu,k\rangle\!\rangle - R\|\nu - m\|_T\} \ge R\|\mu - m\|_T \ge 0.$$

Proposition 2.4. *Let $R > 0$ and $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ a continuous function such that $\alpha(0) = 0$. The function $\hat{J}$ has the following properties*

(*a*) $\hat{J}(a,x,k,\mu) \ge 0$, *for all* $(a,x,k,\mu) \in \mathbb{H} \times \mathbb{X} \times \mathbb{Y}_R \times C_\alpha$.

(*b*) *Let* $(\hat{a},\hat{x},\hat{k},\hat{\mu}) \in \mathbb{H} \times \mathbb{X} \times \mathbb{Y}_R \times C_\alpha$. *Then* $\hat{J}(\hat{a},\hat{x},\hat{k},\hat{\mu}) = 0$ *iff* $\hat{a} = x_0$, $\hat{\mu} = m$ *and* $\hat{k} \in \mathcal{A}(\hat{x})$.

(c) *The restriction of $\hat{J}$ to the closed convex set*

$$\mathbb{K} = \{(a, x, k, \mu) \in \mathbb{H} \times \mathbb{X} \times \mathbb{Y}_R \times C_\alpha : x + k = a + \mu\}$$

is a convex lower semicontinuous function; for $(\hat{a}, \hat{x}, \hat{k}, \hat{\mu}) \in \mathbb{K}$, we have

$$\hat{J}(\hat{a}, \hat{x}, \hat{k}, \hat{\mu}) = 0 \quad \textit{iff } \hat{a} = x_0, \ \hat{\mu} = m \textit{ and } (\hat{x}, \hat{k}) = \mathcal{GSP}\,(A; x_0, m)\,.$$

Proof. The points (a) and (b) clearly are consequences of the properties of the Fitzpatrick function $\mathcal{H}$. Let us prove (c). We have $(a, x, k, \mu) \in \mathbb{K}$ and

$$\begin{aligned}
&\hat{J}\,(a, x, k, \mu) \\
&= |a - x_0|^2 + \mathcal{H}\,(x, k) - \langle\!\langle x, k\rangle\!\rangle + 2R\,\|\mu - m\|_T \\
&\qquad + \sup_{\nu \in C_\alpha} \{\langle\!\langle \mu - \nu, k\rangle\!\rangle - R\,\|\nu - m\|_T\} \\
&= |a - x_0|^2 + \mathcal{H}\,(x, k) + \frac{1}{2}\,|x\,(T) - \mu\,(T)|^2 - \frac{1}{2}\,|a|^2 - \int_0^T \langle \mu\,(s)\,, dk\,(s)\rangle \\
&\qquad + 2R\,\|\mu - m\|_T + \sup_{\nu \in C_\alpha} \{\langle\!\langle \mu - \nu, k\rangle\!\rangle - R\,\|\nu - m\|_T\} \\
&= |x_0|^2 - 2\,\langle a, x_0\rangle + \frac{1}{2}\,|a|^2 + \mathcal{H}\,(x, k) + \frac{1}{2}\,|x\,(T) - \mu\,(T)|^2 \\
&\qquad + 2R\,\|\mu - m\|_T + \sup_{\nu \in C_\alpha} \{\langle\!\langle -\nu, k\rangle\!\rangle - R\,\|\nu - m\|_T\}
\end{aligned}$$

and the convexity of $\hat{J}$ follows. □

In the sequel we prove the existence and uniqueness of the solution of the multivalued monotone differential equation (7). Our proof is strongly connected with the one from Răşcanu [16]. First highlight some properties of a solution $(x, k) = \mathcal{GSP}\,(A; x_0, m)$.

Consider $\mathcal{M}$ a bounded and equicontinuous subset of $C\,([0, T]\,; \mathbb{H})$ and we denote

$$\|\mathcal{M}\|_T = \sup\{\|y\|_T : y \in \mathcal{M}\} \quad \text{and} \quad \mathbf{m}_{\mathcal{M},T}\,(\delta) = \sup\{\mathbf{m}_{y,T}\,(\delta) : y \in \mathcal{M}\}\,.$$

Proposition 2.5. *Fix $T > 0$. Let the assumption (H_{GSP}) be satisfied and*

$$\mathrm{int}\,(\mathrm{Dom}\,(A)) \neq \emptyset.$$

Then, there exists a positive constant $C_\mathcal{M}$ such that

(a) *If $m \in \mathcal{M}$ and $(x, k) = \mathcal{GSP}\,(A; x_0, m)$ then*

$$\|x\|_T^2 + \updownarrow k \updownarrow_T \leq C_\mathcal{M}(1 + |x_0|^2). \tag{10}$$

(b) *If $m, \hat{m} \in \mathcal{M}$, $(x, k) = \mathcal{GSP}\,(A; x_0, m)$ and $(\hat{x}, \hat{k}) = \mathcal{GSP}(A; \hat{x}_0, \hat{m})$ then*

$$\|x - \hat{x}\|_T \leq C_\mathcal{M}\,(1 + |x_0| + |\hat{x}_0|)\,(|x_0 - \hat{x}_0| + \|m - \hat{m}\|_T^{1/2}). \tag{11}$$

In particular, the uniqueness follows, that is, if $x_0 = \hat{x}_0$ and $m = \hat{m}$ then $(x, k) = (\hat{x}, \hat{k})$.

Proof. (a) In the sequel we fix arbitrary $u_0 \in \mathbb{H}$ and $0 < r_0 \leq 1$ such that

$$\bar{B}\left(u_0, r_0\right) \subset \operatorname{Dom}\left(A\right)$$

and

$$A^{\#}_{u_0,r_0} := \sup\left\{|\hat{u}| : \hat{u} \in A\left(u_0 + r_0 v\right),\ |v| \leq 1\right\} < \infty.$$

If in (8) we consider $z = u_0 + r_0 v$, $|v| \leq 1$ and $z^* \in A\left(z\right)$, then $|z^*| \leq A^{\#}_{u_0,r_0}$ and we infer

$$r_0 d \updownarrow k \updownarrow_t \leq \left\langle x\left(t\right) - u_0, dk\left(t\right)\right\rangle + A^{\#}_{u_0,r_0}\left[r_0 + \left|x\left(t\right) - u_0\right|\right] dt. \tag{12}$$

Let $\delta_0 = \delta_{0,\mathcal{M}} > 0$ be defined by

$$\delta_0 + \mathbf{m}_{\mathcal{M},T}\left(\delta_0\right) = \frac{r_0}{4}.$$

By Energy Equality

$$\left|x\left(t\right) - m\left(t\right) - u_0\right|^2 + 2\int_0^t \left\langle x\left(r\right) - u_0, dk\left(r\right)\right\rangle = \left|x_0 - u_0\right|^2 + 2\int_0^t \left\langle m\left(r\right), dk\left(r\right)\right\rangle$$

and, using (12), we obtain

$$\begin{aligned} & \left|x\left(t\right) - m\left(t\right) - u_0\right|^2 + 2r_0 \updownarrow k \updownarrow_t \\ \leq\ & \left|x_0 - u_0\right|^2 + 2\int_0^t \left\langle m\left(r\right), dk\left(r\right)\right\rangle + 2A^{\#}_{u_0,r_0}\int_0^t \left[r_0 + \left|x\left(r\right) - u_0\right|\right] dr. \end{aligned}$$

Let $n_0 = \left\lceil \frac{T}{\delta_0} \right\rceil$ and consider the partition $0 = t_0 < t_1 < ... < t_{n_0} = t,\ t_{i+1} - t_i = \frac{t}{n\,0} \leq \delta_0,\ i = \overline{0, n_0 - 1}$ ($\lceil a \rceil$ is the smallest integer greater or equal to $a \in \mathbb{R}$). Then

$$\begin{aligned} & \int_0^t \left\langle m\left(r\right), dk\left(r\right)\right\rangle \\ = & \sum_{i=0}^{n_0-1} \int_{t_i}^{t_{i+1}} \left\langle m\left(r\right) - m\left(t_i\right), dk\left(r\right)\right\rangle + \sum_{i=0}^{n_0-1} \left\langle m\left(t_i\right), k\left(t_{i+1}\right) - k\left(t_i\right)\right\rangle \\ \leq & \mathbf{m}_{\mathcal{M},T}(\delta_0) \updownarrow k \updownarrow_t + \sum_{i=0}^{n_0-1} \left\langle m\left(t_i\right), m\left(t_{i+1}\right) - x\left(t_{i+1}\right) + u_0 - m\left(t_i\right) + x\left(t_i\right) - u_0\right\rangle \\ \leq & \frac{r_0}{4} \updownarrow k \updownarrow_t + 2(n_0 + 1) \left\|m\right\|_t \left\|x - u_0 - m\right\|_t\ . \end{aligned}$$

Hence

$$\begin{aligned} & \left|x\left(t\right) - m\left(t\right) - u_0\right|^2 + \frac{3r_0}{2} \updownarrow k \updownarrow_t \\ \leq\ & \left|x_0 - u_0\right|^2 + \left[4\left(n_0 + 1\right)\left\|m\right\|_t + 2tA^{\#}_{u_0,r_0}\right] \left\|x - u_0 - m\right\|_t \\ & + 2\left(t + t\left\|m\right\|_t\right) A^{\#}_{u_0,r_0}, \end{aligned}$$

which implies (10), where $C_{\mathcal{M}} = C(T, u_0, r_0, A^{\#}_{u_0,r_0}, \delta_0, \left\|\mathcal{M}\right\|_T)$.

(b) By ordinary differential calculus and (10) we infer

$$\begin{aligned}
&|x(t) - m(t) - \hat{x}(t) + \hat{m}(t)|^2 + 2\int_0^t \langle x(r) - \hat{x}(r), dk(r) - d\hat{k}(r)\rangle \\
&= |x_0 - \hat{x}_0|^2 + 2\int_0^t \langle m(r) - \hat{m}(r), dk(r) - d\hat{k}(r)\rangle \\
&\leq |x_0 - \hat{x}_0|^2 + 2\|m - \hat{m}\|_T[\updownarrow k \updownarrow_T + \updownarrow \hat{k} \updownarrow_T] \\
&\leq |x_0 - \hat{x}_0|^2 + 4C_{\mathcal{M}}\|m - \hat{m}\|_T(1 + |x_0|^2 + |\hat{x}_0|^2).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
|x(t) - m(t) - \hat{x}(t) + \hat{m}(t)|^2 &\geq \frac{1}{2}|x(t) - \hat{x}(t)|^2 - \|m - \hat{m}\|_T^2 \\
&\geq \frac{1}{2}|x(t) - \hat{x}(t)|^2 - 2\|\mathcal{M}\|_T\|m - \hat{m}\|_T
\end{aligned}$$

Combining these last two inequalities with (6), we deduce

$$\begin{aligned}
&|x(t) - \hat{x}(t)|^2 \\
&\leq 2|x_0 - \hat{x}_0|^2 + 4\|\mathcal{M}\|_T\|m - \hat{m}\|_T + 8C_{\mathcal{M}}\|m - \hat{m}\|_T(1 + |x_0|^2 + |\hat{x}_0|^2)
\end{aligned}$$

and (11) easily follows, with a constant $\hat{C}_{\mathcal{M}}$; the two relations (10) and (11) can be written with a common constant $C_{\mathcal{M}} := \max\{C_{\mathcal{M}}, \hat{C}_{\mathcal{M}}\}$. □

Theorem 2.6. *Under the assumptions* (H_{GSP})*, if we have also* $\mathrm{int}(\mathrm{Dom}(A)) \neq \emptyset$*, then the generalized convex Skorohod problem* (7) *has a unique solution* (x, k) *and estimates* (10) *and* (11) *hold.*

Proof. The uniqueness and estimates (10) and (11) have been obtained in the above result. It suffices to prove the existence on an arbitrary fixed interval $[0, T]$.

Let $x_{0,n} \in \mathrm{Dom}(A)$ and $m_n \in C^\infty([0, T]; \mathbb{H})$ be such that

$$x_{0,n} \to x_0 \ \text{ in } \mathbb{H} \quad \text{and} \quad m_n \to m \ \text{ in } C([0, T]; \mathbb{H}).$$

Notice that $\mathcal{M} = \{m, m_1, m_2, \ldots\}$ is a bounded equicontinuous subset of $C([0, T]; \mathbb{H})$. We set $\alpha(\varepsilon) = \mathbf{m}_{\mathcal{M},T}(\varepsilon)$ and let $\hat{J}$ (resp. $\hat{J}_n$): $\mathbb{H} \times \mathbb{X} \times \mathbb{Y}_R \times \mathbb{X} \to]-\infty, +\infty]$ be the functions defined by (9) associated to (x_0, m, A) (and resp. $(x_{0,n}, m_n, A)$). Then

$$\begin{aligned}
&\hat{J}(a, x, k, \mu) \\
&= \hat{J}_n(a, x, k, \mu) - |a - x_{0,n}|^2 - 2R\|\mu - m_n\|_T + |a - x_0|^2 \\
&\qquad + \sup_{\nu \in C_\alpha}\{\langle\!\langle \mu - \nu, k\rangle\!\rangle - R\|\nu - m\|_T\} - \sup_{\nu \in C_\alpha}\{\langle\!\langle \mu - \nu, k\rangle\!\rangle - R\|\nu - m_n\|_T\} \\
&\leq \hat{J}_n(a, x, k, \mu) - |a - x_{0,n}|^2 - 2R\|\mu - m_n\|_T + |a - x_0|^2 \\
&\qquad + R\sup_{\nu \in C_\alpha}\{\|\nu - m_n\|_T - \|\nu - m\|_T\} \\
&\leq \hat{J}_n(a, x, k, \mu) - |a - x_{0,n}|^2 - 2R\|\mu - m_n\|_T + |a - x_0|^2 + R\|m - m_n\|_T \ .
\end{aligned}$$

In particular,

$$\hat{J}\left(x_{0,n},x,k,m_n\right) \leq \hat{J}_n\left(x_{0,n},x,k,m_n\right) + \left|x_{0,n}-x_0\right|^2 + R\left\|m-m_n\right\|_T \ . \tag{13}$$

By a classical result (see Barbu [2], Theorem 2.2) there exist $x_n \in C\left([0,T];\mathbb{H}\right)$ and $h_n \in L^1\left(0,T;\mathbb{H}\right)$, $h_n\left(t\right) \in Ax_n(t)$, a.e. $t \in [0,T]$, such that

$$x_n\left(t\right) + \int_0^t h_n\left(s\right)ds = x_{0,n} + m_n\left(t\right). \tag{14}$$

If we denote $k_n\left(t\right) = \int_0^t h_n\left(s\right)ds$, then $\left(x_n,k_n\right) \in \mathcal{A}$ and therefore, by Fitzpatrick's Theorem, $\mathcal{H}\left(x_n,k_n\right) = \langle\!\langle x_n,k_n\rangle\!\rangle$.

Then, using Proposition 2.5, there exists a positive constant $\mathcal{C}$, not depending on n, such that, for all $n,j \in \mathbb{N}^*$,

$$\|x_n\|_T^2 + \updownarrow k_n \updownarrow_T \leq \mathcal{C} \quad \text{and}$$
$$\|x_n - x_j\|_T \leq \mathcal{C}(|x_{0,n}-x_{0,j}| + \|m_n - m_j\|_T^{1/2}).$$

Hence, there exists $x \in C\left([0,T];\mathbb{H}\right)$ such that, as $n \to \infty$,

$$x_n \to x \ \text{ in } C([0,T];\overline{\mathrm{Dom}(A)}).$$

Let

$$k\left(t\right) = x_0 + m\left(t\right) - x\left(t\right).$$

We deduce that

$$k_n = x_{0,n} + m_n - x_n \ \longrightarrow \ k \ \text{ in } C\left([0,T];\mathbb{H}\right)$$

and clearly follows

$$k \in BV\left([0,T];\mathbb{H}\right), \qquad \updownarrow k \updownarrow_T \leq \mathcal{C}.$$

Setting $R = \mathcal{C}$, the quantities $\hat{J}\left(x_{0,n},x_n,k_n,m_n\right)$ and $\hat{J}_n\left(x_{0,n},x_n,k_n,m_n\right)$ are well defined. Moreover, by Proposition 2.4, $\hat{J}_n\left(x_{0,n},x_n,k_n,m_n\right) = 0$. Passing to $\liminf_{n\to+\infty}$ in (13), the lower-semicontinuity of $\hat{J}$ implies

$$0 \leq \hat{J}\left(x_0,x,k,m\right) \leq \liminf_{n\to+\infty} \hat{J}\left(x_{0,n},x_n,k_n,m_n\right) = 0,$$

that is, there exists a minimum point for which $\hat{J}$ is zero. By Proposition 2.4(c) we infer that the generalized convex Skorohod problem (7) has a solution. □

Remark 2.7. We highlight that the existence problem is reduced to the minimization of a specific l.s.c. convex function on a bounded closed convex subset of $\mathbb{H}\times\mathbb{X}\times BV\left([0,T];\mathbb{H}\right)\times\mathbb{X}$. Indeed, via Proposition 2.4(c), the minimization of $\hat{J}$ is on the set $\mathbb{H}_{\rho_0}\times\mathbb{X}_R\times\mathbb{Y}_R\times C_\alpha$, where

$$\mathbb{H}_{\rho_0} = \left\{h \in \mathbb{H} : |h| \leq \rho_0 := \sup\{|x_0|,|x_{0,n}| : n \in \mathbb{N}^*\}\right\},$$

$\mathbb{X}_R = \{x \in \mathbb{X} : \|x\|_T \leq R\}$ and $R = \mathcal{C}$. Classical results (see Zeidler [18], Theorem 38.A) establish sufficient conditions for a functional defined on a subset of a reflexive Banach space to attain its minimum.

We note that, in the framework of Hilbert spaces, the assumption $\text{int}(\text{Dom}(A)) \neq \emptyset$ from the above results is fairly restrictive. One can renounce at this condition, but we have to consider a stronger assumption on m and, moreover, to weaken the notion of solution for the generalized Skorohod problem (7). Therefore, along $\mathbb{H}$, we consider $(\mathbb{V}, \|\cdot\|_{\mathbb{V}})$ a real separable Banach space with separable dual $(\mathbb{V}^*, \|\cdot\|_{\mathbb{V}^*})$ such that

$$\mathbb{V} \subset \mathbb{H} \cong \mathbb{H}^* \subset \mathbb{V}^*,$$

where the embeddings are continuous, with dense range (the duality paring $(\mathbb{V}^*, \mathbb{V})$ is denoted also by $\langle \cdot, \cdot \rangle$, and, for $k : [0, \infty) \longrightarrow \mathbb{V}^*$, $k(0) = 0$, we use the adequate notation $\updownarrow k \updownarrow_{*T} = \|k\|_{BV([0,T];\mathbb{V}^*)}$.

Reconsider the multivalued monotone differential equation (7) under the assumptions

$$\bar{H}_{GSP} : \quad \begin{cases} H_{GSP} : (i) \quad \text{and} \quad (ii), \\ (iii') \quad m : [0, \infty) \longrightarrow \mathbb{V} \text{ is continuous and } m(0) = 0. \end{cases}$$

Definition 2.8. A continuous function $x : [0, \infty) \to \mathbb{H}$ is a solution of Eq. (7) if

(i) there exist the sequences $\{x_{0,n}\} \subset \text{Dom}(A)$ and $m_n : [0, \infty) \longrightarrow \mathbb{V}$, $m_n(0) = 0$ of C^1-continuous functions satisfying, for all $T > 0$,

$$|x_{0,n} - x_0| + \|m_n - m\|_{C([0,T];\mathbb{V})} \to 0, \quad \text{as } n \to \infty,$$

(ii) there exist $x_n \in C([0, \infty); \overline{\text{Dom}(A)})$, $k_n \in C([0, \infty); \mathbb{H}) \cap BV_{0,loc}(\mathbb{R}_+; \mathbb{V}^*)$, $k_n(0) = 0$, and a function k such that

$$x_n(t) + k_n(t) = x_{0,n} + m_n(t), \quad \forall t \geq 0$$

and, for all $T > 0$,

(a) $\|x_n - x\|_T + \|k_n - k\|_T \to 0, \quad \text{as } n \to \infty,$

(b) $\sup_{n \in \mathbb{N}^*} \updownarrow k_n \updownarrow_{*T} < \infty,$

(c) $\int_s^t \langle x_n(r) - z, dk_n(r) - z^* dr \rangle \geq 0, \quad \forall (z, z^*) \in A, \ \forall 0 \leq s \leq t \leq T.$

(Without confusion, the uniqueness of k will permit us to call the pair (x, k) solution of the generalized Skorohod problem (7) and we write $(x, k) = \mathcal{GSP}(A; x_0, m)$.)

Remark 2.9. If $(x, k) = \mathcal{GSP}(A; x_0, m)$ then we clearly have

(iii) $x(t) \in \overline{\text{Dom}(A)}$, for all $t \geq 0$,

(iv) $k \in C([0, \infty); \mathbb{H}) \cap BV_{0,loc}(\mathbb{R}_+; \mathbb{V}^*)$, $k(0) = 0$ and

(v) $x(t) + k(t) = x_0 + m(t), \ \forall t \geq 0$.

Replacing now the condition $\text{int}(\text{Dom}(A)) \neq \emptyset$ we obtain (see, for example, Răşcanu [16], Theorem 2.3) the following result of existence and uniqueness of a solution for the generalized Skorohod problem (7).

Theorem 2.10. *Under the hypothesis* $(\bar{H}_{GSP})$*, if there exist* $h_0 \in \mathbb{H}$ *and* $r_0, a_1, a_2 > 0$ *such that*

$$r_0 \|z^*\|_{\mathbb{V}^*} \leq \langle z^*, z - h_0 \rangle + a_1 |z|^2 + a_2, \quad \forall (z, z^*) \in A \tag{15}$$

then the differential equation (7) *has a unique solution* (x, k) *in the sense of Definition* 2.8. *Moreover, for all* $T > 0$,

(a) *if* $(x, k) = \mathcal{GSP}(A; x_0, m)$ *and* $(\hat{x}, \hat{k}) = \mathcal{GSP}(A; \hat{x}_0, \hat{m})$*, then there exists a positive constant* C *such that*

$$\|x - \hat{x}\|_T^2 \leq C \left[|x_0 - \hat{x}_0|^2 + \|m - \hat{m}\|_T^2 + \|m - \hat{m}\|_{C([0,T];\mathbb{V})} \updownarrow k - \hat{k} \updownarrow_{*T} \right] \text{ and}$$

(b) *for every equiuniform continuous subset* $\mathcal{M} \subset C([0, T]; \mathbb{V})$, $m \in \mathcal{M}$, *there exists* $C_0 = C_0(r_0, h_0, a_1, a_2, T, \mathcal{N}_\mathcal{M}) > 0$ *for which*

$$\|x\|_T^2 + \updownarrow k \updownarrow_{*T} \leq C_0 \left[1 + |x_0|^2 + \|m\|_T^2 \right].$$

(Here $\mathcal{N}_\mathcal{M}$ *is the constant of equiuniform continuity given by* $\sup\{\|f(t) - f(s)\|_\mathbb{V} : |t - s| \leq T/\mathcal{N}_\mathcal{M}\} \leq r_0/4$, $\forall f \in \mathcal{M}$.*)*

From Răşcanu [16] we mention three situations when the relation (15) is satisfied:

(a) $A = A_0 + \partial\varphi$, where $A_0 : \mathbb{H} \to \mathbb{H}$ is a continuous monotone operator on $\mathbb{H}$ and $\varphi : \mathbb{H} \to]-\infty, +\infty]$ is a proper convex l.s.c. function for which there exist $h_0 \in \mathbb{H}$, $R_0 > 0$, $a_0 > 0$ such that

$$\varphi(h_0 + x) \leq a_0, \quad \forall x \in \mathbb{V}, \ \|x\|_\mathbb{V} \leq R_0.$$

(b)
- There exists a separable Banach space $\mathbb{U}$ such that $\mathbb{U} \subset \mathbb{H} \subset \mathbb{U}^*$ densely and continuously and $\mathbb{U} \cap \mathbb{V}$ is dense in $\mathbb{V}$,
- $A : \mathbb{H} \rightrightarrows \mathbb{H}$ is a maximal monotone operator with $\mathrm{Dom}(A) \subset \mathbb{U}$,
- $\exists a, \lambda \in \mathbb{R}$, $a > 0$, such that for all $(x_1, y_1), (x_2, y_2) \in A$

$$(y_1 - y_2, x_1 - x_2) + \lambda |x_1 - x_2|^2 \geq a \|x_1 - x_2\|_\mathbb{V}^2,$$

- $\exists h_0 \in \mathbb{U}$, $\exists r_0, a_0 > 0$ such that

$$h_0 + r_0 e \in \mathrm{Dom}(A) \quad \text{and} \quad \left\| A^0(h_0 + r_0 e) \right\|_{\mathbb{U}^*} \leq r_0,$$

for all $e \in \mathbb{U} \cap \mathbb{V}$, $\|e\|_\mathbb{V} = 1$, where $A^0 x := \mathrm{Pr}_{Ax} 0$.

(c) A is a maximal monotone with $\mathrm{int}(\mathrm{Dom}(A)) \neq \emptyset$ and $\mathbb{V} = \mathbb{H}$.

2.2. Maximal monotone SDE with additive noise

Consider now the following stochastic differential equation (for short SDE), where by B we denote the $\mathbb{H}_0$-Wiener process defined in Section 1.2,

$$\begin{cases} dX_t + AX_t(dt) \ni G_t dB_t, \\ X_0 = \xi, \quad t \in [0, T], \end{cases} \tag{16}$$

where

$$(H_{MSDE}): \quad \begin{cases} (i) & A:\mathbb{H}\rightrightarrows\mathbb{H} \text{ is a maximal monotone operator,} \\ (ii) & \xi\in L^0(\Omega,\mathcal{F}_0,\mathbb{P};\overline{\mathrm{Dom}(A)}), \\ (iii) & G\in\Lambda^2_{\mathbb{H}\times\mathbb{H}_0}\ . \end{cases}$$

Setting $\mathbb{X}=L^2\left(\Omega;C\left([0,T];\mathbb{H}\right)\right)$, the space $L^2\left(\Omega;BV_0\left([0,T];\mathbb{H}\right)\right)$ is a linear subspace of the dual of $\mathbb{X}$ and, the natural duality

$$(X,K)\mapsto\mathbb{E}\int_0^T\langle X_t,dK_t\rangle$$

between these two suggests to use the notation $\mathbb{X}^*$ for $L^2\left(\Omega;BV_0\left([0,T];\mathbb{H}\right)\right)$, even it is not the entire dual space. On $\mathbb{X}$ we shall consider the strong topology and on $\mathbb{X}^*$ the w^*-topology. Let $\mathcal{A}$ the realization of A on $\mathbb{X}\times\mathbb{X}^*$.

Definition 2.11. By a solution of Eq. (16) we understand a pair of stochastic processes

$$(X,K)\in L^0\left(\Omega;C\left([0,T];\mathbb{H}\right)\right)\times\left[L^0\left(\Omega;C\left([0,T];\mathbb{H}\right)\right)\cap L^0\left(\Omega;BV_0\left([0,T];\mathbb{H}\right)\right)\right],$$

satisfying, $\mathbb{P}$-a.s. $\omega\in\Omega$, for all $0\le s\le t\le T$,

$$\begin{aligned} &(c_1) && X_t\in\overline{\mathrm{Dom}(A)}, \\ &(c_2) && X_t+K_t=\xi+\int_0^t G_s dB_s \text{ and} \\ &(c_3) && \int_s^t\langle X_r-u,dK_r-vdr\rangle\ge 0,\ \forall(u,v)\in A. \end{aligned}$$

Clearly,

$$(X(\omega,\cdot),K(\omega,\cdot))=\mathcal{GSP}\left(A;\xi(\omega),M(\omega,\cdot)\right),\quad \mathbb{P}\text{-a.s. }\omega\in\Omega,$$

where $M_t=\int_0^t G_s dB_s\in\mathcal{M}^2(0,T;\mathbb{H})$. Consequently, under the hypothesis (H_{MSDE}), if $\mathrm{int}(\mathrm{Dom}(A))\neq\emptyset$ then by Theorem 2.6 there exists a unique solution (X,K) (in the sense of Definition 2.11) for Eq. (16). Moreover, if

$$\mathbb{E}\left|\xi\right|^4+\mathbb{E}\left(\int_0^T\|G_t\|^2_{HS}\,dt\right)^2<+\infty$$

then $X\in L^4\left(\Omega;C\left([0,T];\mathbb{H}\right)\right)\subset\mathbb{X}$ and $K\in\mathbb{X}\cap\mathbb{X}^*$ (see for example Pardoux & Răşcanu [14], Proposition 4.22).

In the sequel we define a convex functional whose minimum point coincide with the solution of Eq. (16).

Let

$$\mathbb{S}=L^2\left(\Omega,\mathcal{F}_0,\mathbb{P};\mathbb{H}\right)\times\mathbb{X}\times\mathbb{X}^*\times\Lambda^2_{\mathbb{H}\times\mathbb{H}_0}\,.$$

Define, for each $(U,U^*)\in\mathcal{A}$,

$$J_{(U,U^*)}:\mathbb{S}\to\mathbb{R}$$

by

$$J_{(U,U^*)}(\eta,X,K,g)=\frac{1}{2}\mathbb{E}\left|\eta-\xi\right|^2+\frac{1}{2}\mathbb{E}\int_0^T\left\|g_t-G_t\right\|_{HS}^2dt$$
$$+\mathbb{E}\int_0^T\left[\langle U_t,dK_t\rangle+\langle X_t,dU_t^*\rangle-\langle U_t,dU_t^*\rangle-\langle X_t,dK_t\rangle\right]$$

and $\hat{J}:\mathbb{S}\to]-\infty,+\infty]$

$$\begin{aligned}\hat{J}(\eta,X,K,g)&=\sup_{(U,U^*)\in\mathcal{A}}J_{(U,U^*)}(\eta,X,K,g)\\&=\frac{1}{2}\mathbb{E}\left|\eta-\xi\right|^2+\mathcal{H}(X,K)-\langle\langle X,K\rangle\rangle+\frac{1}{2}\mathbb{E}\int_0^T\left\|g_t-G_t\right\|_{HS}^2dt,\end{aligned}$$

where $\mathcal{H}:\mathbb{X}\times\mathbb{X}^*\to]-\infty,+\infty]$ is the Fitzpatrick function associated to the maximal monotone operator $\mathcal{A}$. It is clear that

Remark 2.12. $\hat{J}:\mathbb{S}\to]-\infty,+\infty]$ is a lower semicontinuous function as supremum of continuous functions.

Since $\mathcal{H}(X,K)\geq\langle\langle X,K\rangle\rangle$, then we easily deduce

Proposition 2.13. *$\hat{J}$ has the following properties:*

(a) $\hat{J}(\eta,X,K,g)\geq 0$, *for all* $(\eta,X,K,g)\in\mathbb{S}$.
(b) $\hat{J}(\eta,X,K,g)=0$ *iff* $\eta=\xi$, $g=G$ *and* $K\in\mathcal{A}(X)$.
(c) *Let* $R>0$. *The restriction of* $\hat{J}$ *to the bounded closed convex set*

$$\mathcal{L}=\left\{(\eta,X,K,g)\in\mathbb{S}:X_t+K_t=\eta+\int_0^t g_s dB_s,\ \forall t\in[0,T],\right.$$
$$\left.\mathbb{E}\left|\eta\right|^2+\mathbb{E}\left\|X\right\|_{\mathbb{X}}^2+\mathbb{E}\updownarrow K\updownarrow_{\mathbb{X}^*}+\mathbb{E}\int_0^T\left\|g_s\right\|_{HS}^2ds\leq R\right\}$$

is a convex l.s.c. function and $\hat{J}(\eta,X,K,g)=0$ *iff* $\eta=\xi$, $g=G$ *and* (X,K) *is the solution of the SDE* (16).

Proof. The points (a) and (b) clearly are consequences of the properties of the Fitzpatrick function $\mathcal{H}$. Let us prove (c). Since, by Energy Equality

$$\frac{1}{2}\mathbb{E}\left|X_T\right|^2+\mathbb{E}\int_0^T\langle X_t,dK_t\rangle=\frac{1}{2}\mathbb{E}\left|\eta\right|^2+\frac{1}{2}\mathbb{E}\int_0^T\left\|g_t\right\|_{HS}^2dt$$

then

$$\begin{aligned}\hat{J}(\eta,X,K,g)&=\frac{1}{2}\mathbb{E}\left|\eta-\xi\right|^2+\mathcal{H}(X,K)-\langle\langle X,K\rangle\rangle+\frac{1}{2}\mathbb{E}\int_0^T\left\|g_t-G_t\right\|_{HS}^2dt\\&=\frac{1}{2}\mathbb{E}\left|\xi\right|^2-\mathbb{E}\langle\eta,\xi\rangle+\mathcal{H}(X,K)+\frac{1}{2}\mathbb{E}\left|X_T\right|^2\\&\quad-\mathbb{E}\int_0^T\langle g_t,G_t\rangle dt+\frac{1}{2}\mathbb{E}\int_0^T\left\|G_t\right\|_{HS}^2dt\end{aligned}$$

and the convexity of $\hat{J}$ on the set $\mathcal{L}$ follows. □

To complete this section, we will situate in the extended framework introduced in the final part of Subsection 2.1. We will consider once again the spaces $\mathbb{H}$ and $\mathbb{V}$ and we assume that $\mathbb{V} \subset \mathbb{H} \cong \mathbb{H}^* \subset \mathbb{V}^*$, where the embeddings are continuous with dense range. Concerning the SDE (16), the hypothesis (H_{MSDE}) will be replaced by

$$(\bar{H}_{MSDE}): \quad \begin{cases} \text{(i)} & \left| \begin{array}{l} A : \mathbb{H} \rightrightarrows \mathbb{H} \text{ is a maximal monotone operator and} \\ \text{there exist } h_0 \in \mathbb{H} \text{ and } r_0, a_1, a_2 > 0 \text{ such that} \\ r_0 \|z^*\|_{\mathbb{V}^*} \leq \langle z^*, z - h_0 \rangle + a_1 |z|^2 + a_2, \ \forall (z, z^*) \in A \end{array} \right. \\ \text{(ii)} & \xi \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \overline{\mathrm{Dom}(A)}), \\ \text{(iii)} & G \in \Lambda^2_{\mathbb{H}\times\mathbb{H}_0}(0, T; \mathcal{L}^2(\mathbb{H}_0, \mathbb{H})). \end{cases}$$

Definition 2.14. Let $M_t := \int_0^t G_s dB_s$. A *stochastic process* $X \in L^0_{ad}(\Omega; C([0,T];\mathbb{H}))$ that satisfies, $\mathbb{P}$-a.s., $X_0 = \xi$ and $X_t \in \overline{\mathrm{Dom}(A)}$, $\forall t \in [0,T]$ *is a (generalized) solution of multivalued SDE* (16) if there exist

$$K \in L^0_{ad}(\Omega; C([0,T];\mathbb{H})) \cap L^0(\Omega; BV(0,T;\mathbb{V}^*)), \quad K_0 = 0 \ \mathbb{P}\text{-a.s.}$$

and a sequence of stochastic processes $\{M^n\}_{n\in\mathbb{N}^*}$ satisfying

$$\begin{cases} M^n \in L^2_{ad}(\Omega; C([0,T];\mathbb{V})) \cap \mathcal{M}^2(0,T;\mathbb{H}), \\ M^n \longrightarrow M \ \text{ in } \mathcal{M}^2(0,T;\mathbb{H}) \end{cases} \tag{17}$$

such that, denoting for a.s. $\omega \in \Omega$,

$$(X^n(\omega,\cdot), K^n(\omega,\cdot)) = \mathcal{GSP}(A; \xi(\omega), M^n(\omega,\cdot)),$$

we have $X^n \to X$, $K^n \to K$ in $L^0_{ad}(\Omega, C([0,T];\mathbb{H}))$ as $n \to \infty$ and $\sup_n \mathbb{E}\updownarrow K^n \updownarrow_{*T} < +\infty$.

(Without confusion, the uniqueness of K permits us to call the pair (X,K) a generalized solution *of the multivalued SDE* (16).)

Recall, from Răşcanu [16], the following existence result which is a consequence of the corresponding deterministic case here above.

Theorem 2.15. *Under the assumption* $(\bar{H}_{MSDE})$ *the problem* (16) *has a unique generalized solution* (X,K). *Moreover the solution satisfies*

$$\mathbb{E} \sup_{t\in[0,T]} |X_t|^2 + \mathbb{E} \sup_{t\in[0,T]} |K_t|^2 + \mathbb{E} \updownarrow K \updownarrow_{*T} \leq C_0 \left[1 + \mathbb{E}|\xi|^2 + \mathbb{E}\int_0^T \|G_t\|^2_{HS}\, dt\right], \tag{18}$$

where $C_0 = C_0(T, r_0, h_0, a_1, a_2) > 0$. *If* (X,K) *and* $(\tilde{X}, \tilde{K})$ *are two solutions of* (16) *corresponding to* (ξ, G) *and, respectively,* $(\tilde{\xi}, \tilde{G})$ *then*

$$\mathbb{E} \sup_{t\in[0,T]} |X_t - \tilde{X}_t|^2 \leq C(T)\left[\mathbb{E}|\xi - \tilde{\xi}|^2 + \mathbb{E}\int_0^T ||G_t - \tilde{G}_t||^2_{HS}\, dt\right]. \tag{19}$$

Proof. Since the process M does not have $\mathbb{V}$-valued continuous trajectories, we use the deterministic result approximating the stochastic integral by the sequence

$$M_t^n := \sum_{i=1}^{n} \langle M_t, e_i \rangle \, e_i,$$

where $\{e_i;\ i \in \mathbb{N}^*\} \subset \mathbb{V}$ is an orthonormal basis in $\mathbb{H}$. By Theorem 2.10, there exists $(X^n(\omega), K^n(\omega)) = \mathcal{GSP}(A; \xi(\omega), M^n(\omega))$, $\mathbb{P}$-a.s. $\omega \in \Omega$. It is not difficult to prove that the following inequalities hold

$$\mathbb{E} \sup_{t\in[0,T]} |X_t^n|^2 + \mathbb{E} \sup_{t\in[0,T]} |K_t^n|^2 + \mathbb{E} \updownarrow\!\updownarrow K^n \updownarrow\!\updownarrow_{*T} \leq C_0 \left[1 + \mathbb{E}|\xi|^2 + \mathbb{E}|M_T^n|^2\right]$$

and, if $(\tilde{X}^n(\omega), \tilde{K}^n(\omega)) = \mathcal{GSP}(A; \tilde{\xi}(\omega), \tilde{M}^n(\omega))$, then

$$\mathbb{E} \sup_{t\in[0,T]} |X_t^n - \tilde{X}_t^n|^2 + \mathbb{E} \sup_{t\in[0,T]} |K_t^n - \tilde{K}_t^n|^2 \leq C(T) \left[\mathbb{E}|\xi - \tilde{\xi}|^2 + \mathbb{E}|M_T^n - \tilde{M}_T^n|^2\right].$$

So (replacing $\tilde{M}^n$ by $\tilde{M}^{n'}$), there exist $X, K \in L^2_{ad}(\Omega; C([0,T];\mathbb{H}))$ such that $X^n \to X$ and $K^n \to K$ in $L^2_{ad}(\Omega; C([0,T];\mathbb{H}))$ as $n \to \infty$. The inequalities (18) and (19) are immediate consequences and, as a by-product, (X, K) is a solution of Eq. (16). For more details, we invite the interested reader to consult Răşcanu [16]. □

2.3. Backward stochastic $\mathcal{A}$−representation

Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ be a stochastic basis, where $\{\mathcal{F}_t\}_{t\geq 0}$ is the standard filtration associated to a $\mathbb{H}_0$-Wiener process $\{B_t\}_{t\geq 0}$.

By the representation theorem, for $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{H})$ there exists a unique $Z \in \Lambda^2_{\mathbb{H}\times\mathbb{H}_0}(0,T)$ such that

$$\xi = \mathbb{E}\xi + \int_0^T Z_s dB_s$$

and, for each $(\xi, H) \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{H}) \times \Lambda^2_{\mathbb{H}}(0,T)$, there exists a unique pair

$$(Y, Z) \in S^2_{\mathbb{H}}[0,T] \times \Lambda^2_{\mathbb{H}\times\mathbb{H}_0}(0,T)$$

such that

$$Y_t + \int_t^T H_s ds = \xi - \int_t^T Z_s dB_s$$

and the mapping $(\xi, H) \mapsto (Y, Z) : L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{H}) \times \Lambda^2_{\mathbb{H}}(0,T) \to S^2_{\mathbb{H}}[0,T] \times \Lambda^2_{\mathbb{H}\times\mathbb{H}_0}(0,T)$ is linear and continuous. (Y, Z) is defined as

$$Y_t = \mathbb{E}\left(\xi - \int_t^T H_s ds \,\middle|\, \mathcal{F}_t\right) \quad \text{and} \quad \xi - \int_0^T H_s ds = \mathbb{E}\left(\xi - \int_0^T H_s ds\right) + \int_0^T Z_s dB_s.$$

Denote

$$Y_t = C_t(\xi, H) \quad \text{and} \quad Z_t = D_t(\xi, H).$$

Remark that, by the Energy Equality, we have

$$\mathbb{E}\left|Y_t\right|^2+\mathbb{E}\int_t^T\|Z_s\|_{HS}^2 ds=\mathbb{E}\left|\xi\right|^2+2\mathbb{E}\int_t^T\left\langle Y_s,F_s\right\rangle ds. \tag{20}$$

If $A:\mathbb{H}\rightrightarrows\mathbb{H}$ is a maximal monotone operator then the realization of A on $\Lambda_{\mathbb{H}}^2(0,T)$ is the maximal monotone operator $\mathcal{A}:\Lambda_{\mathbb{H}}^2(0,T)\rightrightarrows\Lambda_{\mathbb{H}}^2(0,T)$ defined by $H\in\mathcal{A}(Y)$ iff $H_t(\omega)\in A(Y_t(\omega))$, $d\mathbb{P}\otimes dt$-a.e. $(\omega,t)\in\Omega\times]0,T[$. The inner product in $\Lambda_{\mathbb{H}}^2(0,T)$ is given by $\langle\!\langle U,V\rangle\!\rangle=\mathbb{E}\int_0^T\langle U_t,V_t\rangle\,dt$.

Consider the backward stochastic differential equation

$$\begin{cases}-dY_t+A(Y_t)\,dt\ni -Z_t dB_t,\ t\in[0,T],\\ Y_T=\xi,\end{cases} \tag{21}$$

where

$$\begin{cases}\text{(i)}\quad A:\mathbb{H}\rightrightarrows\mathbb{H}\text{ is a maximal monotone operator and}\\ \text{(ii)}\quad \xi\in L^2(\Omega,\mathcal{F}_T,\mathbb{P};\overline{\mathrm{Dom}(A)}).\end{cases}$$

Definition 2.16. $Y\in S_{\mathbb{H}}^2[0,T]$ is a solution of Eq. (21) if there exist $H\in\Lambda_{\mathbb{H}}^2(0,T)$ and $Z\in\Lambda_{\mathbb{H}\times\mathbb{H}_0}^2(0,T)$ such that

$$Y_t+\int_t^T H_s ds=\xi-\int_t^T Z_s dB_s$$

and $H\in\mathcal{A}(Y)$ (that is, $H_t(\omega)\in A(Y_t(\omega))$, $d\mathbb{P}\otimes dt$-a.e. $(\omega,t)\in\Omega\times]0,T[$).

Let $R>0$ and the ball $\mathbb{F}_R=\left\{\eta\in L^2(\Omega,\mathcal{F}_T,\mathbb{P};\mathbb{H}):\mathbb{E}\left|\eta\right|^2\le R\right\}$.

For $(U,U^*)\in\mathcal{A}$ and $\zeta\in\mathbb{F}_R$ define

$$J_{(\zeta,U,U^*)}:L^2(\Omega,\mathcal{F}_T,\mathbb{P};\mathbb{H})\times\Lambda_{\mathbb{H}}^2(0,T)\times\Lambda_{\mathbb{H}}^2(0,T)\to\mathbb{R}$$

by

$$\begin{aligned}J_{(\zeta,U,U^*)}(\eta,Y,H)=\frac{1}{2}\mathbb{E}\left|\eta-\xi\right|^2+\mathbb{E}\int_0^T\left[\langle U_t,H_t\rangle+\langle Y_t,U_t^*\rangle-\langle U_t,U_t^*\rangle-\langle Y_t,H_t\rangle\right]dt\\ +\frac{1}{2}\left[\mathbb{E}\left|\zeta-\eta\right|^2-\mathbb{E}\left|\zeta-\xi\right|^2\right]\end{aligned}$$

and $\hat{J}:L^2(\Omega,\mathcal{F}_T,\mathbb{P};\mathbb{H})\times\Lambda_{\mathbb{H}}^2(0,T)\times\Lambda_{\mathbb{H}}^2(0,T)\to]-\infty,+\infty]$,

$$\begin{aligned}\hat{J}(\eta,Y,H)&=\sup\left\{J_{(\zeta,U,U^*)}(\eta,Y,H):(U,U^*)\in\mathcal{A},\ \zeta\in\mathbb{F}_R\right\}\\ &=\frac{1}{2}\mathbb{E}\left|\eta-\xi\right|^2+\mathcal{H}(Y,H)-\langle\!\langle Y,H\rangle\!\rangle+\frac{1}{2}\sup_{\zeta\in\mathbb{F}_R}\left[\mathbb{E}\left|\zeta-\eta\right|^2-\mathbb{E}\left|\zeta-\xi\right|^2\right],\end{aligned} \tag{22}$$

where $\mathcal{H}:\Lambda_{\mathbb{H}}^2(0,T)\times\Lambda_{\mathbb{H}}^2(0,T)\to]-\infty,+\infty]$ is the Fitzpatrick function associated to the maximal monotone operator $\mathcal{A}$.

Remark 2.17. $\hat{J} : L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{H}) \times \Lambda^2_{\mathbb{H}}(0,T) \times \Lambda^2_{\mathbb{H}}(0,T) \to]-\infty, +\infty]$ is a l.s.c. function as the supremum of the continuous functions $J_{(\zeta,U,U^*)}(\eta, Y, H)$.

If $\xi \in \mathbb{F}_R$ then

$$2R^2 + 2\mathbb{E}|\eta|^2 \geq \sup_{\zeta \in \mathbb{F}_R} \left(\mathbb{E}|\zeta - \eta|^2 - \mathbb{E}|\zeta - \xi|^2\right) \geq \mathbb{E}|\eta - \xi|^2$$

and clearly follows

Proposition 2.18. *Let $R > 0$ and $\xi \in \mathbb{F}_R$. $\hat{J}$ has the following properties:*

(a) $\hat{J}(\eta, Y, H) \geq \mathcal{H}(Y, H) - \langle\langle Y, H \rangle\rangle \geq 0$, *for all* $(\eta, Y, H) \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{H}) \times \Lambda^2_{\mathbb{H}}(0,T) \times \Lambda^2_{\mathbb{H}}(0,T)$.

(b) *Let* $(\hat{\eta}, \hat{Y}, \hat{H}) \in \mathbb{F}_R \times \Lambda^2_{\mathbb{H}}(0,T) \times \Lambda^2_{\mathbb{H}}(0,T)$. *Then* $\hat{J}(\hat{\eta}, \hat{Y}, \hat{H}) = 0$ *iff* $\hat{\eta} = \xi$, $\hat{H} \in \mathcal{A}(\hat{Y})$.

(c) *The restriction of $\hat{J}$ to the closed convex set*

$$\mathbb{K} = \left\{(\eta, Y, H) \in \mathbb{F}_R \times \Lambda^2_{\mathbb{H}}(0,T) \times \Lambda^2_{\mathbb{H}}(0,T) : \ Y_t = C_t(\eta, H), \ \forall t \in [0,T]\right\}$$

is a convex lower semicontinuous function and for $(\hat{\eta}, \hat{Y}, \hat{H}) \in \mathbb{K}$ the following assertions are equivalent:

(c_1) $\inf_{(\eta,Y,H) \in \mathbb{F}_R \times \Lambda^2_{\mathbb{H}}(0,T) \times \Lambda^2_{\mathbb{H}}(0,T)} \hat{J}(\eta, Y, H) = \hat{J}(\hat{\eta}, \hat{Y}, \hat{H}) = 0.$

(c_2) $\hat{\eta} = \xi$ *and* $(\hat{Y}, \hat{H}, \hat{Z})$, *with* $\hat{Z}_s = D_s(\xi, \hat{H})$, *is the solution of the BSDE* (21).

Proof. (Sketch) Since the points (a) and (b) are obvious, we focus on (c). The convexity of $\hat{J}$ on $\mathbb{K}$ is obtained as follows. By Energy Equality we have

$$\frac{1}{2}|C_0(\eta, H) - C_0(\zeta, 0)|^2 + \mathbb{E}\int_0^T \langle Y_s - C_s(\zeta, 0), H_s \rangle ds$$
$$+ \frac{1}{2}\mathbb{E}\int_0^T |D_s(\eta, H) - D_s(\zeta, 0)|^2 ds = \frac{1}{2}\mathbb{E}|\eta - \zeta|^2.$$

Then

$$\begin{aligned}
&J_{(\zeta,U,\tilde{U})}(\eta, Y, H) \\
&= \frac{1}{2}\mathbb{E}|\eta - \xi|^2 + \mathbb{E}\int_0^T [\langle U_t, H_t \rangle + \langle Y_t, U_t^* \rangle - \langle U_t, U_t^* \rangle - \langle Y_t, H_t \rangle] dt \\
&\qquad + \frac{1}{2}\left[\mathbb{E}|\zeta - \eta|^2 - \mathbb{E}|\zeta - \xi|^2\right] \\
&= \frac{1}{2}\mathbb{E}|\eta - \xi|^2 + [\langle\langle U, H \rangle\rangle + \langle\langle Y, U^* \rangle\rangle - \langle\langle U, U^* \rangle\rangle] + \frac{1}{2}|C_0(\eta, H) - C_0(\zeta, 0)|^2 \\
&\qquad + \langle\langle C(\zeta, 0), H \rangle\rangle + \frac{1}{2}\|D(\eta, H) - D(\zeta, 0)\|^2 - \mathbb{E}|\zeta - \xi|^2
\end{aligned}$$

Hence

$$\begin{aligned}(\eta, Y, H) \quad \longmapsto \quad & \hat{J}(\eta, Y, H) \\ & = \frac{1}{2}\mathbb{E}\,|\eta - \xi|^2 + \mathcal{H}(Y, H) + \sup_{\zeta}\Big\{\frac{1}{2}\,|C_0(\eta, H) - C_0(\zeta, 0)|^2 \\ & \qquad + \langle\!\langle C(\zeta, 0), H\rangle\!\rangle + \frac{1}{2}\,\|D(\eta, H) - D(\zeta, 0)\|^2 - \mathbb{E}\,|\zeta - \xi|^2\Big\}\end{aligned}$$

is, clearly, a convex lower semicontinuous function. Then, the equivalence between (c_1) and (c_2) easily follows. □

Proving the existence of a solution for the backward stochastic differential equation (21) is therefore equivalent to solving a problem on convex analysis. More precisely, it is sufficient to show that the functional defined by the formula (22) attains a minimum and its value in that point is zero. Unfortunately, this is still an open problem, but we estimate that the perspective and the tools introduced along this paper will lead us to the desired result.

3. Fitzpatrick type method for SVI and BSVI

In the following sections we will consider the finite dimensional case $\mathbb{H} = \mathbb{R}^d$ and $\mathbb{H}_0 = \mathbb{R}^k$. Let $\{B_t, t \geq 0\}$ be a k-dimensional Brownian motion with respect to a given complete stochastic basis $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$.

3.1. Stochastic variational inequality

3.1.1. Known results

Let

$$F : \Omega \times [0, +\infty[\, \times \mathbb{R}^d \to \mathbb{R}^d, \qquad G : \Omega \times [0, +\infty[\, \times \mathbb{R}^d \to \mathbb{R}^{d\times k}.$$

Consider the stochastic variational inequality (for short SVI)

$$\begin{cases} dX_t + \partial\varphi(X_t)(dt) \ni F(t, X_t)dt + G(t, X_t)dB_t, \quad t \geq 0, \\ X_0 = \xi, \end{cases} \tag{23}$$

where will assume

$$(\mathbf{H}_0): \quad \xi \in L^0(\Omega, \mathcal{F}_0, P; \overline{\mathrm{Dom}(\varphi)}) \tag{24}$$

and

$$(\mathbf{H}_\varphi): \quad \begin{cases} \text{(i)} \quad \varphi : \mathbb{R}^d \to]-\infty, +\infty] \text{ is a convex l.s.c. function,} \\ \text{(ii)} \quad \mathrm{int}(\mathrm{Dom}(\varphi)) \neq \emptyset. \end{cases} \tag{25}$$

Definition 3.1. A pair $(X,K) \in S_d^0 \times S_d^0$, $K_0 = 0$, is a solution of the stochastic variational inequality (23) if the following conditions are satisfied, $\mathbb{P}$-*a.s.*:

$$\begin{cases} (d_1) & X_t \in \mathrm{Dom}(\varphi),\ \text{a.e. } t>0 \text{ and } \varphi(X) \in L^1_{loc}(0,\infty), \\ (d_2) & \updownarrow K \updownarrow_T < \infty,\ \forall T>0, \\ (d_3) & X_t + K_t = \xi + \int_0^t F(s,X_s)ds + \int_0^t G(s,X_s)dB_s,\ \forall t \geq 0, \\ (d_4) & \int_s^t \langle y(r) - X_r, dK_r\rangle + \int_s^t \varphi(X_r)dr \leq \int_s^t \varphi(y(r))dr, \\ & \quad \forall y : \mathbb{R}_+ \to \mathbb{R}^d \text{ continuous function and } \forall 0 \leq s \leq t. \end{cases} \tag{26}$$

Notation 3.2. The notation $dK_t \in \partial\varphi(X_t)(dt)$ will be used to say that (X,K) satisfy $(d_1), (d_2)$ and (d_4). The SDE (23) will be written, also, in the form

$$\begin{cases} X_t + K_t = \xi + \int_0^t F(s,X_s)ds + \int_0^t G(s,X_s)dB_s,\ \forall t \geq 0, \\ dK_t \in \partial\varphi(X_t)(dt). \end{cases}$$

Remark (see Asiminoaei & Răşcanu [1]) that the condition (d_4) from Definition 3.1 is equivalent to each of the following conditions, for any fixed $T > 0$,

$$\begin{aligned} &(a_1) \quad \int_s^t \langle z - X_r, dK_r\rangle + \int_s^t \varphi(X_r)dr \leq (t-s)\varphi(z), \quad \forall z \in \mathbb{R}^d, \quad \forall 0 \leq s \leq t \leq T, \\ &(a_2) \quad \int_s^t \langle X_r - z, dK_r - z^* dr\rangle \geq 0,\ \forall\ (z,z^*) \in \partial\varphi, \quad \forall 0 \leq s \leq t \leq T, \\ &(a_3) \quad \int_0^T \langle y(r) - X_r, dK_r\rangle + \int_0^T \varphi(X_r)dr \leq \int_0^T \varphi(y(r))dr, \quad \forall y \in C([0,T],\mathbb{R}^d). \end{aligned}$$

Hence, the condition (d_4) means that $(X_\cdot(\omega), K_\cdot(\omega)) \in \partial\tilde{\varphi}$, $\mathbb{P}$-a.s., where $\tilde{\varphi}$ is the realization of φ on $C([0,T];\mathbb{R}^d)$, that is $\tilde{\varphi} : C([0,T];\mathbb{R}^d) \to]-\infty,+\infty]$,

$$\tilde{\varphi}(x) = \begin{cases} \int_0^T \varphi(x(t))dt, & \text{if } \varphi(x) \in L^1(0,T), \\ +\infty, & \text{otherwise.} \end{cases} \tag{27}$$

Notation 3.3. We introduce the notation:

$$F_R^{\#}(t) := \operatorname{ess\,sup}\{|F(t,x)| : |x| \leq R\}.$$

We recall the basic assumptions on F and G under which we will study the multivalued stochastic equation (23):

- ○ the functions $F(\cdot,\cdot,x) : \Omega \times [0,+\infty[\to \mathbb{R}^d$ and $G(\cdot,\cdot,x) : \Omega \times [0,+\infty[\to \mathbb{R}^{d\times k}$ are progressively measurable stochastic processes for every $x \in \mathbb{R}^d$,

- there exist $\mu \in L^1_{loc}(0,\infty)$ and $\ell \in L^2_{loc}(0,\infty;\mathbb{R}_+)$, such that $d\mathbb{P}\otimes dt$-a.e.:

$$(\mathbf{H}_F):\begin{cases} & \text{Continuity:} \\ (\mathbf{C}_F): & x \mapsto F(t,x):\mathbb{R}^d \to \mathbb{R}^d \text{ is continuous,} \\ & \text{Monotonicity condition:} \\ (\mathbf{M}_F): & \langle x-y, F(t,x)-F(t,y)\rangle \le \mu(t)\,|x-y|^2,\ \forall x,y\in\mathbb{R}^d, \\ & \text{Boundedness condition:} \\ (\mathbf{B}_F): & \int_0^T F_R^{\#}(s)\,ds < \infty,\ \text{ for all } R,T\ge 0. \end{cases} \tag{28}$$

and

$$(\mathbf{H}_G)\ :\begin{cases} & \text{Lipschitz condition:} \\ (\mathbf{L}_G): & |G(t,x)-G(t,y)| \le \ell(t)\,|x-y|,\ \forall x,y\in\mathbb{R}^d, \\ & \text{Boundedness condition:} \\ (\mathbf{B}_g): & \int_0^T |G(t,0)|^2 dt < \infty. \end{cases} \tag{29}$$

Clearly $(\mathbf{H}_F)$ and $(\mathbf{H}_G)$ yield $F(\cdot,\cdot,X.) \in L^1_{loc}\left(\mathbb{R}_+;\mathbb{R}^d\right)$ and $G(\cdot,\cdot,X.)\in\Lambda^0_{d\times k}$ for all $X\in S^0_d$.

Theorem 3.4. *If the assumptions* (24), (25), (28) *and* (29) *are satisfied, then the SDE* (23) *has a unique solution* $(X,K)\in S^0_d\times S^0_d$ *(in the sense of Definition* 3.1*). Moreover, if there exist* $p\ge 2$ *and* $u_0\in \operatorname{int}(\operatorname{Dom}(\varphi))$ *such that, for all* $T\ge 0$,

$$\mathbb{E}\,|\xi|^p + \mathbb{E}\left(\int_0^T |F(t,u_0)|\,dt\right)^p + \mathbb{E}\left(\int_0^T |G(t,u_0)|^2 dt\right)^{p/2} < +\infty, \tag{30}$$

then

$$\mathbb{E}(\|X\|_T^p + \|K\|_T^{p/2} + \updownarrow K \updownarrow_T^{p/2}) + \mathbb{E}\left(\int_0^T |\varphi(X_r)|\,dr\right)^{p/2} < \infty.$$

(For the proof see Pardoux & Răşcanu [14], Theorem 4.14.)

3.1.2. Fitzpatrick approach

In this subsection, assumptions $(\mathbf{H}_F)$ and $(\mathbf{H}_G)$ are replaced by

(i) the functions $F(\cdot,\cdot,x):\Omega\times[0,+\infty[\,\to\mathbb{R}^d$ and $G(\cdot,\cdot,x):\Omega\times[0,+\infty[\,\to\mathbb{R}^{d\times k}$ are progressively measurable stochastic processes for every $x\in\mathbb{R}^d$ and, $d\mathbb{P}\otimes dt$-a.e.,

(ii) $x\mapsto F(t,x):\mathbb{R}^d\to\mathbb{R}^d$ and $x\mapsto G(t,x):\mathbb{R}^d\to\mathbb{R}^{d\times k}$ are continuous,

(iii) for all $x,y\in\mathbb{R}^d$

$$2\langle x-y, F(t,x)-F(t,y)\rangle + |G(t,x)-G(t,y)|^2 \le 0 \quad \text{and} \tag{31}$$

(iv) there exists $b > 0$ such that, for all $x \in \mathbb{R}^d$,

$$|F(t,x)| + |G(t,x)| \leq b\,(1 + |x|)\,. \tag{32}$$

Remark 3.5. If $\mu(t) + \frac{1}{2}\ell^2(t) \leq 0$, for every $t \geq 0$, then the assumptions (28-$\mathbf{M}_F$) and (29-$\mathbf{L}_G$) implies that (31) holds.

Denote

$$\mathbb{S}_{BV}[0,T] = \left\{K \in S_d^0[0,T] : K_0 = 0,\ \mathbb{E}\updownarrow K\updownarrow_T^2 < \infty\right\},$$

with the w^*-topology, that means $K^n \to K$ if $\lim_{n\to\infty} \mathbb{E}\int_0^T \langle X_t, dK_t^n\rangle = \mathbb{E}\int_0^T \langle X_t, dK_t\rangle$, for all $X \in L^2(\Omega; C([0,T];\mathbb{R}^d))$.

Let $\Phi : S_d^2[0,T] \to]-\infty, +\infty]$ defined by

$$\Phi(X) = \begin{cases} \mathbb{E}\displaystyle\int_0^T \varphi(X_t)dt, & \text{if } \varphi(X) \in L^1(\Omega\times]0,T[), \\ +\infty, & \text{otherwise.} \end{cases} \tag{33}$$

Since $\varphi : \mathbb{R}^d \to]-\infty, +\infty]$ is a proper convex l.s.c. function then Φ is also a proper convex l.s.c. function.

Let

$$\mathbb{S} := L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \overline{\mathrm{Dom}(\varphi)}) \times S_d^2[0,T] \times \mathbb{S}_{BV}[0,T] \times \Lambda_{d\times k}^2(0,T)$$

and, for each $U \in \mathrm{Dom}(\Phi) = \{X \in S_d^2[0,T] : \Phi(X) < \infty\}$, we consider the mapping $J_U : \mathbb{S} \to]-\infty, +\infty]$, defined by

$$\begin{aligned} &J_U(\eta, X, L, g) \\ &= \frac{1}{2}\mathbb{E}|\eta - \xi|^2 + \mathbb{E}\int_0^T \left[\langle U_s - X_s, F(s,U_s)\rangle + \frac{1}{2}|g_s - G(s,U_s)|^2\right] ds \\ &\quad + \mathbb{E}\int_0^T \langle U_s - X_s, dL_s\rangle + \Phi(X) - \Phi(U) \end{aligned} \tag{34}$$

and $\hat{J} : \mathbb{S} \longrightarrow]-\infty, +\infty]$

$$\hat{J}(\eta, X, L, g) := \sup_{U \in \mathrm{Dom}(\Phi)} J_U(\eta, X, L, g)\,.$$

Remark 3.6. $\hat{J} : \mathbb{S} \to]-\infty, +\infty]$ is a lower semicontinuous function as supremum of lower semicontinuous functions.

We now have

Proposition 3.7. *$\hat{J}$ has the following properties:*

(*a*) *$\hat{J}(\eta, X, L, g) \geq 0$, for all $(\eta, X, L, g) \in \mathbb{S}$ and $\hat{J}$ is not identically $+\infty$.*

(*b*) *Let $(\hat{\eta}, \hat{X}, \hat{L}, \hat{g}) \in \mathbb{S}$. Then*

$$\hat{J}(\hat{\eta}, \hat{X}, \hat{L}, \hat{g}) = 0 \quad \textit{iff } \hat{\eta} = \xi,\ \hat{g}_\cdot = G(\cdot, \hat{X}_\cdot),\ \hat{L} + \int_0^\cdot F(s, \hat{X}_s)ds \in \partial\Phi(\hat{X}).$$

(c) *The restriction of $\hat{J}$ to the closed convex set*

$$\mathbb{L} = \left\{ (\eta, X, L, g) \in \mathbb{S} : X_t + L_t = \eta + \int_0^t g_s dB_s, \ \forall t \in [0, T] \right\}$$

is a convex l.s.c. function. If $(\hat{\eta}, \hat{X}, \hat{L}, \hat{g}) \in \mathbb{L}$, then $\hat{J}(\hat{\eta}, \hat{X}, \hat{L}, \hat{g}) = 0$ iff

$\hat{\eta} = \xi$, $\hat{g}_\cdot = G(\cdot, \hat{X}_\cdot)$ and $(\hat{X}, \hat{L} + \int_0^\cdot F(s, \hat{X}_s) ds)$ is a solution of the SVI (23).

Proof. (a) If $X \notin \mathrm{Dom}\,(\Phi)$ then $\hat{J}\,(\eta, X, L, g) = +\infty$. If $X \in \mathrm{Dom}\,(\Phi)$ then

$$\begin{aligned} \hat{J}\,(\eta, X, L, g) &= \sup_{U \in \mathrm{Dom}(\Phi)} J_U\,(\eta, X, L, g) \\ &\geq J_X\,(\eta, X, L, g) \\ &= \frac{1}{2}\mathbb{E}\,|\eta - \xi|^2 + \frac{1}{2}\mathbb{E}\int_0^T |g_s - G\,(s, X_s)|^2\, ds \geq 0. \end{aligned}$$

$\hat{J}$ is a proper function since, for $v_0 \in \partial\varphi\,(u_0)$ and $\eta^0 = \xi$, $X_t^0 = u_0$, $L_t^0 = v_0 t - \int_0^t F\,(s, u_0)\, ds$, $g_s^0 = G\,(s, u_0)$, we have (using the assumption (31)) that

$$J_U\left(\eta^0, X^0, L^0, g^0\right) \leq 0, \ \text{ for all } U \in \mathrm{Dom}\,(\Phi)\,.$$

(b) If $\hat{J}(\hat{\eta}, \hat{X}, \hat{L}, \hat{g}) = 0$, then $\hat{X} \in \mathrm{Dom}\,(\Phi)$ and by the calculus from the proof of (a) we infer $\hat{\eta} = \xi$, $\hat{g} = G(\cdot, \hat{X}_\cdot)$ and

$$J_U(\hat{\eta}, \hat{X}, \hat{L}, \hat{g}) \leq 0, \quad \text{for all } U \in \mathrm{Dom}\,(\Phi)\,.$$

Hence

$$\mathbb{E}\int_0^T \left\langle U_s - \hat{X}_s, F\,(s, U_s)\, ds + d\hat{L}_s \right\rangle + \Phi(\hat{X}) \leq \Phi\,(U)\,, \ \text{ for all } U \in \mathrm{Dom}\,(\Phi)\,.$$

Let $V \in \mathrm{Dom}\,(\Phi)$ and $\lambda \in]0, 1[$ be arbitrary. Since $\mathrm{Dom}\,(\Phi)$ is a convex set, we can replace U by $(1 - \lambda)\,\hat{X} + \lambda V$. It follows

$$\begin{aligned} &\lambda\mathbb{E}\int_0^T \left\langle V_s - \hat{X}_s, F(s, \hat{X}_s + \lambda(V_s - \hat{X}_s))ds + d\hat{L}_s \right\rangle + \Phi(\hat{X}) \\ &\leq \Phi((1 - \lambda)\,\hat{X} + \lambda V) \leq (1 - \lambda)\,\Phi(\hat{X}) + \lambda\Phi\,(V)\,, \end{aligned}$$

which is equivalent to

$$\mathbb{E}\int_0^T \left\langle V_s - \hat{X}_s, F(s, \hat{X}_s + \lambda(V_s - \hat{X}_s))ds + d\hat{L}_s \right\rangle + \Phi(\hat{X}) \leq \Phi\,(V)\,,$$

for all $V \in \mathrm{Dom}\,(\Phi)\,.$ By the continuity of $x \mapsto F\,(t, x)$ and assumption (32) we can pass to limit under the last integral, and it follows that $\hat{L} + \int_0^\cdot F(s, \hat{X}_s)ds \in \partial\Phi(\hat{X})$.

Conversely, using (31), we have

$$\begin{aligned}&J_U(\xi,\hat{X},\hat{L},G(\cdot,\hat{X}_\cdot))\\&=\frac{1}{2}\mathbb{E}\int_0^T|G(s,\hat{X}_s)-G\left(s,U_s\right)|^2ds+\mathbb{E}\int_0^T\left\langle U_s-\hat{X}_s,F\left(s,U_s\right)-F(s,\hat{X}_s)ds\right\rangle\\&\quad+\mathbb{E}\int_0^T\left\langle U_s-\hat{X}_s,F(s,\hat{X}_s)ds+d\hat{L}_s\right\rangle+\Phi(\hat{X})-\Phi\left(U\right)\leq 0\end{aligned}$$

and, consequently, $\hat{J}(\xi,\hat{X},\hat{L},G(\cdot,\hat{X}_\cdot))=0$.

(c) The second part of this point is easy to observe, and, therefore, $(\hat{X},\hat{L}+\int_0^\cdot F(s,\hat{X}_s)ds)$ is a solution of the SVI (23).

It remains to prove the convexity of $\hat{J}$ on $\mathbb{L}$. By the Energy Equality we have

$$\frac{1}{2}\mathbb{E}\left|X_T\right|^2+\mathbb{E}\int_0^T\left\langle X_s,dL_s\right\rangle=\frac{1}{2}\mathbb{E}\left|\eta\right|^2+\frac{1}{2}\mathbb{E}\int_0^T\left|g_s\right|^2ds$$

and, using it in the formula (34), the functional $J_U\left(\eta,X,L,g\right)$ becomes

$$\begin{aligned}&J_U\left(\eta,X,L,g\right)\\&=\frac{1}{2}\mathbb{E}\left|\eta-\xi\right|^2+\mathbb{E}\int_0^T\left\langle U_s-X_s,F\left(s,U_s\right)ds\right\rangle+\frac{1}{2}\mathbb{E}\int_0^T\left|g_s-G\left(s,U_s\right)\right|^2ds\\&\quad+\left[\mathbb{E}\int_0^T\left\langle U_s,dL_s\right\rangle-\frac{1}{2}\mathbb{E}\left|\eta\right|^2-\frac{1}{2}\mathbb{E}\int_0^T\left|g_s\right|^2ds+\frac{1}{2}\mathbb{E}\left|X_T\right|^2\right]+\Phi\left(X\right)-\Phi\left(U\right)\\&=-\mathbb{E}\left\langle\eta,\xi\right\rangle+\frac{1}{2}\mathbb{E}\left|\xi\right|^2+\mathbb{E}\int_0^T\left\langle U_s-X_s,F\left(s,U_s\right)ds\right\rangle+\mathbb{E}\int_0^T\left\langle U_s,dL_s\right\rangle\\&\quad+\frac{1}{2}\mathbb{E}\left|X_T\right|^2+\frac{1}{2}\mathbb{E}\int_0^T\left|G\left(s,U_s\right)\right|^2ds-\mathbb{E}\int_0^T\left\langle g_s,G\left(s,U_s\right)\right\rangle ds+\Phi\left(X\right)-\Phi\left(U\right).\end{aligned}$$

It clearly follows that J_U is convex and lower semicontinuous for $\forall U\in\mathrm{Dom}\left(\Phi\right)$. Consequently, the mapping $(\eta,X,L,g)\longmapsto\hat{J}\left(\eta,X,L,g\right)=\sup_{U\in\mathrm{Dom}(\Phi)}J_U\left(\eta,X,L,g\right)$ has the same properties.

The proof is now complete. □

3.2. Backward stochastic variational inequality

In this section we suppose that the filtration $\{\mathcal{F}_t:\,t\geq 0\}$ is the natural filtration of the k–dimensional Brownian motion $\{B_t\,:\,t\geq 0\}$, i.e., for all $t\geq 0$,

$$\mathcal{F}_t=\mathcal{F}_t^B:=\sigma\left(\{B_s:0\leq s\leq t\}\right)\vee\mathcal{N}_{\mathbb{P}}.$$

3.2.1. Known results

Consider the backward stochastic variational inequality (for short BSVI)

$$\begin{cases}-dY_t+\partial\varphi\left(Y_t\right)dt\ni F\left(t,Y_t,Z_t\right)dt-Z_tdB_t,\;0\leq t<T,\\Y_T=\xi,\end{cases}\tag{35}$$

or, equivalently,

$$\begin{cases} Y_t + \displaystyle\int_t^T H_s ds = \xi + \int_t^T F(s, Y_s, Z_s)\, ds - \textstyle\int_t^T Z_s dB_s, \; t \in [0,T], \quad \mathbb{P}\text{-a.s.}, \\ H_t(\omega) \in \partial\varphi(Y_t(\omega)), \quad d\mathbb{P} \otimes dt\text{-a.e.} \end{cases}$$

We assume

($\mathbf{H}_\xi$): $\xi : \Omega \to \mathbb{R}^d$ is *a* F_T-measurable random vector,

($\mathbf{H}_\varphi$): $\partial\varphi$ is the subdifferential of the proper convex l.s.c. function $\varphi : \mathbb{R}^d \to]-\infty, +\infty]$,

($\mathbf{H}_F$): $F : \Omega \times [0,\infty[\times \mathbb{R}^d \times \mathbb{R}^{d\times k} \to \mathbb{R}^d$ satisfies

- the function $F(\cdot,\cdot,y,z) : \Omega \times [0,T] \to \mathbb{R}^d$ is a progressively measurable stochastic process for every $(y,z) \in \mathbb{R}^d \times \mathbb{R}^{d\times k}$,
- there exist some deterministic functions $\mu \in L^1(0,T;\mathbb{R})$ and $\ell \in L^2(0,T;\mathbb{R})$, such that,

$$\begin{cases} \text{(i)} \quad \text{for all } y, y' \in \mathbb{R}^d,\; z, z' \in \mathbb{R}^{d\times k},\; d\mathbb{P} \otimes dt\text{-a.e.:} \\ \qquad\qquad \text{Continuity:} \\ \qquad (\mathbf{C}_y): \quad y \longrightarrow F(t,y,z) : \mathbb{R}^d \to \mathbb{R}^d \text{ is continuous,} \\ \qquad\qquad \text{Monotonicity condition:} \\ \qquad (\mathbf{M}_y): \quad \langle y' - y, F(t,y',z) - F(t,y,z)\rangle \le \mu(t)\,|y'-y|^2, \\ \qquad\qquad \text{Lipschitz condition:} \\ \qquad (\mathbf{L}_z): \quad |F(t,y,z') - F(t,y,z)| \le \ell(t)\,|z'-z|, \\ \text{(ii)} \quad \text{Boundedness condition:} \\ \qquad (\mathbf{B}_F) \quad \displaystyle\int_0^T F_R^{\#}(t)\, dt < \infty, \quad \mathbb{P}\text{-a.s.}, \forall R \ge 0, \end{cases} \tag{36}$$

where

$$F_R^{\#}(t) = \sup\{|F(t,y,0)| : |y| \le R\}.$$

Definition 3.8. A pair $(Y,Z) \in S_d^0[0,T] \times \Lambda_{d\times k}^0(0,T)$ of stochastic processes is a solution of the backward stochastic variational inequality (35) if there exists a progressively measurable stochastic process H such that, $\mathbb{P}$-*a.s.*,

$$\begin{aligned} &\text{(a)} \quad \int_0^T |H_t|\, dt + \int_0^T |F(t,Y_t,Z_t)|\, dt < \infty, \\ &\text{(b)} \quad (Y_t(\omega), H_t(\omega)) \in \partial\varphi, \quad \text{a.e. } t \in [0,T] \end{aligned}$$

and, for all $t \in [0,T]$,

$$Y_t + \int_t^T H_s ds = \eta + \int_t^T F(s,Y_s,Z_s)\, ds - \int_t^T Z_s dB_s. \tag{37}$$

(Without confusion, *the uniqueness of the stochastic process* H will permit to call the triplet (Y,Z,H) a solution of Eq. (35).)

We introduce now a supplementary assumption

$(\mathbf{A})$: There exist $p \geq 2$, a positive stochastic process $\beta \in L^1(\Omega \times]0,T[)$, a positive function $b \in L^1(0,T)$ and a real number $\kappa \geq 0$, such that for all $(u,\hat{u}) \in \partial\varphi$ and $z \in R^{d\times k}$

$$\langle \hat{u}, F(t,u,z)\rangle \leq \frac{1}{2}|\hat{u}|^2 + \beta_t + b(t)|u|^p + \kappa |z|^2, \quad d\mathbb{P}\otimes dt\text{-a.e.}$$

Theorem 3.9. *Let assumptions $(\mathbf{H}_\xi)$, $(\mathbf{H}_\varphi)$, $(\mathbf{H}_F)$ and $(\mathbf{A})$ be satisfied. If there exists $u_0 \in \mathrm{Dom}(\partial\varphi)$ such that*

$$\mathbb{E}|\xi|^p + \mathbb{E}|\varphi(\xi)| + \mathbb{E}\left(\int_0^T |F(s,u_0,0)|\,ds\right)^p < \infty, \tag{38}$$

then the BSVI (35) *has a unique solution $(Y,Z) \in S_d^p[0,T] \times \Lambda_{d\times k}^p(0,T)$. Moreover, uniqueness holds in $S_d^{1+}[0,T] \times \Lambda_{d\times k}^0(0,T)$, where*

$$S_d^{1+}[0,T] := \bigcup_{p>1} S_d^p[0,T].$$

(For the proof see Pardoux & Răşcanu [14], Theorem 5.13.)

3.2.2. Fitzpatrick approach

In this subsection the assumptions $(\mathbf{H}_F)$ are replaced by

(i) the function $F(\cdot,\cdot,y,z) : \Omega\times[0,+\infty[\to \mathbb{R}^d$ is a progressively measurable stochastic processes for every $(y,z) \in \mathbb{R}^d \times \mathbb{R}^{d\times k}$,

(ii) $(y,z) \mapsto F(t,y,z) : \mathbb{R}^d \times \mathbb{R}^{d\times k} \to \mathbb{R}^d$ is continuous $d\mathbb{P}\otimes dt$-a.e.,

(iii) for all $y,y' \in \mathbb{R}^d$ and $z,z' \in \mathbb{R}^{d\times k}$

$$\langle y-y', F(t,y,z) - F(t,y',z')\rangle \leq \frac{1}{2}|z-z'|, \quad d\mathbb{P}\otimes dt\text{-a.e.}, \tag{39}$$

(iv) there exists $b > 0$ such that, for all $y \in \mathbb{R}^d$,

$$|F(t,y,z)| \leq b(1+|y|+|z|), \quad d\mathbb{P}\otimes dt\text{-a.e.}$$

Remark that, if

$$\mu(t) + \frac{1}{2}\ell^2(t) \leq 0, \quad \text{a.e. } t \geq 0,$$

then the assumptions $(\mathbf{H}_F)$ implies (i)–(iii).

Denote by $\Phi : S_d^2[0,T] \to]-\infty,+\infty]$ the proper convex lower semicontinuous function defined by

$$\Phi(X) := \begin{cases} \mathbb{E}\displaystyle\int_0^T \varphi(X_t)dt, & \text{if } \varphi(X) \in L^1(\Omega\times]0,T[), \\ +\infty, & \text{otherwise} \end{cases}$$

For each

$$(U, V) \in \mathbb{D} := \text{Dom}\,(\Phi) \times L^2\left(\Omega \times [0, T]\,; \mathbb{R}^d\right)$$

we introduce the function

$$J_{(U,V)} : \mathbb{S} := L^2(\Omega, \mathcal{F}_T, \mathbb{P}, \mathbb{R}^d) \times \Lambda^2_d\,(0, T) \times S^2_d\,(0, T) \times \Lambda^2_{d\times k}\,(0, T) \to \mathbb{R}$$

by

$$J_{(U,V)}(\eta, G, Y, Z) := \frac{1}{2}\mathbb{E}\,|\eta - \xi|^2 + \mathbb{E}\int_0^T \langle U_t - Y_t, F(t, U_t, V_t) - G_t\rangle\, dt \\ - \frac{1}{2}\mathbb{E}\int_0^T |Z_t - V_t|^2\, dt + \Phi\,(Y) - \Phi\,(U)$$

and consider the functional $\hat{J} : \mathbb{S} \to] - \infty, +\infty]$,

$$\hat{J}(\eta, G, Y, Z) := \sup_{(U,V)\in\mathbb{D}} J_{(U,V)}(\eta, G, Y, Z).$$

Remark 3.10. $\hat{J} : \mathbb{S} \to] - \infty, +\infty]$ is a lower semicontinuous function as supremum of lower semicontinuous functions.

We now have

Proposition 3.11. *The mapping $\hat{J}$ has the following properties:*

(a) $\hat{J}(\eta, G, Y, Z) \geq 0,\ \forall(\eta, G, Y, Z) \in \mathbb{S}$ *and* $\hat{J}$ *is not identical* $+\infty$.

(b) *Let* $(\hat{\eta}, \hat{G}, \hat{Y}, \hat{Z}) \in \mathbb{S}$. *Then*

$$\hat{J}(\hat{\eta}, \hat{G}, \hat{Y}, \hat{Z}) = 0 \quad \textit{iff}\ \hat{\eta} = \xi,\ F(\hat{Y}, \hat{Z}) - \hat{G} \in \partial\Phi(\hat{Y}).$$

(c) *The restriction of* $\hat{J}$ *to the closed convex set*

$$\mathbb{K} = \left\{(\eta, G, Y, Z) \in \mathbb{S} :\ Y_t = \eta + \int_t^T G_s ds - \int_t^T Z_s dB_s,\ \forall t \in [0, T]\right\}$$

is a convex lower semicontinuous function. If $(\hat{\eta}, \hat{G}, \hat{Y}, \hat{Z}) \in \mathbb{K}$, *then*

$$\hat{J}(\hat{\eta}, \hat{G}, \hat{Y}, \hat{Z}) = 0 \quad \textit{iff}\ \hat{\eta} = \xi\ \textit{and}\ (\hat{Y}, \hat{Z}, \hat{H}),\ \textit{with}\ \hat{H} = F(\hat{Y}, \hat{Z}) - \hat{G}$$

is a solution of the BSVI (35).

Proof. (a) If $Y \notin \text{Dom}\,(\Phi)$ then $J_{(U,V)}(\eta, G, Y, Z) = +\infty$ and if $Y \in \text{Dom}\,(\Phi)$, we have $\hat{J}(\eta, G, Y, Z) \geq J_{(Y,Z)}(\eta, G, Y, Z) \geq 0$. Moreover, $\hat{J}$ is a proper function since for $v_0 \in \partial\varphi\,(u_0)$ and $\eta^0 = \xi$, $Y^0_t = u_0$, $Z^0_t = 0$, $G^0_t = F\,(t, u_0, 0) - v_0$ we have (using the assumption (39)) that

$$\hat{J}_{(U,V)}\left(\eta^0, G^0, Y^0, Z^0\right) \leq 0, \quad \text{for all}\ \ (U, V) \in \mathbb{D}.$$

(b) If $\hat{J}(\hat{\eta},\hat{G},\hat{Y},\hat{Z})=0$ then

$$J_{(U,V)}(\hat{\eta},\hat{G},\hat{Y},\hat{Z})\leq 0,\quad \forall U\in \mathrm{Dom}\,(\Phi)\,,\ \forall V\in L^2\left(\Omega\times[0,T]\,;\mathbb{R}^d\right).$$

So, for all $(U,V)\in\mathbb{D}$,

$$\begin{aligned}\frac{1}{2}\mathbb{E}\,|\hat{\eta}-\xi|^2+\mathbb{E}\int_0^T\left\langle U_t-\hat{Y}_t,F(U_t,V_t)-\hat{G}_t\right\rangle dt\\ -\frac{1}{2}\mathbb{E}\int_0^T|\hat{Z}_t-V_t|^2dt+\Phi(\hat{Y})-\Phi\,(U)\leq 0,\end{aligned}$$

which yields $\hat{Y}\in\mathrm{Dom}\,(\Phi)$; taking in particular $U=\hat{Y}$ and $V=\hat{Z}$, we infer

$$\hat{\eta}=\xi,\quad \mathbb{P}\text{-a.s.}$$

Hence, for all $(U,V)\in\mathbb{D}$,

$$\mathbb{E}\int_0^T\left\langle U_t-\hat{Y}_t,F(U_t,V_t)-\hat{G}_t\right\rangle dt+\Phi(\hat{Y})\leq\frac{1}{2}\mathbb{E}\int_0^T|\hat{Z}_t-V_t|^2dt+\Phi\,(U)\,. \tag{40}$$

Since $\mathbb{D}$ is a convex set, we can replace (U,V) by $((1-\lambda)\,\hat{Y}+\lambda U,(1-\lambda)\,\hat{Z}+\lambda V)$, where $\lambda\in(0,1)$. The convexity of Φ leads to the following inequality

$$\begin{aligned}&\mathbb{E}\int_0^T\left\langle U_t-\hat{Y}_t,F((1-\lambda)\,\hat{Y}_t+\lambda U_t,(1-\lambda)\,\hat{Z}_t+\lambda V_t)-\hat{G}_t\right\rangle dt\\ &\leq\frac{\lambda}{2}\mathbb{E}\int_0^T|\hat{Z}_t-V_t|^2dt+\Phi\,(U)-\Phi(\hat{Y}).\end{aligned}$$

Passing to $\liminf_{\lambda\to 0}$, we deduce

$$\mathbb{E}\int_0^T\left\langle U_t-\hat{Y}_t,F(\hat{Y}_t,\hat{Z}_t)-\hat{G}_t\right\rangle dt+\Phi(\hat{Y})\leq\Phi\,(U)\,,\quad\forall U\in\mathrm{Dom}\,(\Phi)\,,$$

that is

$$F(\hat{Y},\hat{Z})-\hat{G}\in\partial\Phi(\hat{Y}).$$

Conversely, using assumption (39) we have

$$\begin{aligned}&J_{(U,V)}(\xi,\hat{G},\hat{Y},\hat{Z})\\ &=\mathbb{E}\int_0^T\left\langle U_t-\hat{Y}_t,F(U_t,V_t)-\hat{G}_t\right\rangle dt-\frac{1}{2}\mathbb{E}\int_0^T|\hat{Z}_t-V_t|^2dt+\Phi(\hat{Y})-\Phi\,(U)\\ &\leq\mathbb{E}\int_0^T\left\langle U_t-\hat{Y}_t,F(U_t,V_t)-F(\hat{Y}_t,\hat{Z}_t)\right\rangle dt-\frac{1}{2}\mathbb{E}\int_0^T|\hat{Z}_t-V_t|^2dt\\ &\quad+\mathbb{E}\int_0^T\left\langle U_t-\hat{Y}_t,F(\hat{Y}_t,\hat{Z}_t)-\hat{G}_t\right\rangle dt+\Phi(\hat{Y})-\Phi\,(U)\leq 0\end{aligned}$$

and, consequently, $\hat{J}(\hat{\eta},\hat{G},\hat{Y},\hat{Z})=0.$

(c) If, moreover, $(\hat{\eta},\hat{G},\hat{Y},\hat{Z})\in\mathbb{K}$, then

$$Y_t+\int_t^T(F(\hat{Y}_s,\hat{Z}_s)-\hat{G}_s)ds=\hat{\eta}+\int_t^T F(\hat{Y}_s,\hat{Z}_s)ds-\int_t^T Z_s dB_s$$

and

$$F(\hat{Y},\hat{Z})-\hat{G}\in\partial\Phi(\hat{Y}),$$

that is, $(\hat{Y},\hat{Z},F(\hat{Y},\hat{Z})-\hat{G})$ is solution of the SVI (35).

The convexity of $\hat{J}$ on $\mathbb{K}$ is obtained as follows: by the Energy Equality we have

$$|Y_0|^2+\mathbb{E}\int_0^T|Z_s|^2\,ds=\mathbb{E}\,|\eta|^2+2\mathbb{E}\int_0^T\langle Y_s,G_s\rangle\,ds$$

and $J_{(U,V)}(\eta,G,Y,Z)$ becomes

$$\begin{aligned}
&J_{(U,V)}(\eta,G,Y,Z)\\
&=\frac{1}{2}\mathbb{E}\,|\eta-\xi|^2+\mathbb{E}\int_0^T\langle U_t-Y_t,F(U_t,V_t)\rangle\,dt-\mathbb{E}\int_0^T\langle U_t,G_t\rangle\,dt\\
&\quad+\mathbb{E}\int_0^T\langle Y_t,G_t\rangle\,dt-\frac{1}{2}\mathbb{E}\int_0^T|Z_t-V_t|^2\,dt+\Phi(Y)-\Phi(U)\\
&=\frac{1}{2}\mathbb{E}\,|\xi|^2-\mathbb{E}\,\langle\eta,\xi\rangle+\mathbb{E}\int_0^T\langle U_t-Y_t,F(U_t,V_t)\rangle\,dt-\mathbb{E}\int_0^T\langle U_t,G_t\rangle\,dt\\
&\quad+\mathbb{E}\int_0^T\langle Z_t,V_t\rangle\,dt-\frac{1}{2}\mathbb{E}\int_0^T|V_t|^2\,dt+\frac{1}{2}\mathbb{E}\,|Y_0|^2+\Phi(Y)-\Phi(U).
\end{aligned}$$

Hence $\hat{J}$ is a convex l.s.c. function as supremum of convex l.s.c. functions.

The proof is now complete. □

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