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On a Multivalued Iterative Equation of Order n^*

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Because of no Lipschitz condition for upper semi-continuous (USC for short) multifunctions and some other technical difficulties, only the second order polynomial-like iterative equation with multifunctions was discussed but the general case of order n remains open. In this paper we consider the general case for a special class of multifunctions, called unblended multifunctions. We investigate the set of all jumps for iterates of those multifunctions and consider the piecewise Lipschitz condition. Then we prove the existence of USC multi-valued solutions for a modified form of this equation, which gives the existence of USC multi-valued solutions for this equation of general order n in the inclusion sense.

Keywords: Iteration, functional equation, multifunction, upper semi-continuity, unblended

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1. Introduction

As shown in [2, 9], the polynomial-like iterative equation

$$\lambda_1 f(x) + \lambda_2 f^2(x) + ... + \lambda_n f^n(x) = g(x), \tag{1}$$

is an interesting form of functional equations including iterates f^i $(i = 2, ..., n)$ of the unknown function f. Here the i-th iterate f^i is defined inductively by $f^i(x) = f(f^{i-1}(x))$ and $f^0(x) = x$. Since iteration is an important object in both many mathematical subjects such as dynamical systems and numerical computation and many fields of natural science, in recent years great attentions have been paid to the equation and its generalizations, see for example in [4, 5, 11, 12, 21, 26, 27, 29, 30, 31, 32]. On the other hand, being an important class of mappings, multifunction (called

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multi-valued function sometimes) have been extensively employed in control theory [7], stochastics [3], artificial intelligence [10] and economics [25]. Many nice results ([13, 16, 24, 28]) were given for functional equations with multifunctions. Since some efforts were made to iteration of multifunctions ([22]) and related problems ([6, 15, 17, 18, 19, 23]), it gets more interesting to study multi-valued solutions for equation (1), i.e., the equation

$$\lambda_1 F(x) + \lambda_2 F^2(x) + \cdots + \lambda_n F^n(x) = G(x), \quad \forall x \in I := [a, b], \tag{2}$$

where $n \geq 2$ is an integer, $\lambda_1, \lambda_2, ...\lambda_n$ are real constants, G is a given multifunction and F is an unknown multifunction. Here the i-th iterate F^i of the multifunction F is defined recursively as

$$F^i(x) := \cup\{F(y) : y \in F^{i-1}(x)\}$$

and $F^0(x) :\equiv x$ for all $x \in I$. It is worthy mentioning that it is a trivial question to find continuous multi-valued solutions for this equation with continuous multifunction G because that can be simplified to discussing the corresponding equation with the boundaries of G, which is well known as a single-value case ([30, 31]). In 2004 Nikodem and Zhang [14] discussed equation (2) for $n = 2$ with an increasing upper semi-continuous (abbreviated by USC) multifunction G on $I = [a, b]$. They proved the existence and uniqueness of USC solutions under the assumption that G has fixed points a and b and λ_1, λ_2 are both constants such that $\lambda_1 > \lambda_2 \geq 0$ and $\lambda_1 + \lambda_2 = 1$.

Actually, the generalization to USC multifunctions for equation (1) is rather difficult even if we only consider $n = 2$. Firstly, the upper semi-continuity for multifunctions is much weaker than the continuity for functions. The method used for continuous solutions and smooth solutions in [30, 31] may be used for continuous multi-valued solutions but does not work for USC multi-valued solutions. Secondly, according to [30, 31], the authors of [14] considered solutions in a class of USC and increasing multifunctions

$$\Phi(I) := \{F \in \mathcal{F}(I) : \ F \text{ is USC, increasing, } F(a) = \{a\}, \ F(b) = \{b\}\},$$

where $\mathcal{F}(I)$ is the set of all multifunctions $F : I \to cc(I)$ and $cc(I)$ denotes the family of all nonempty closed subintervals of I, but many tools used in [30, 31] cannot be applied because the class $\Phi(I)$ is not a Banach space. Fortunately, it was proved in Lemma 1 of [14] that this class, being a metric space equipped with the distance

$$D(F_1, F_2) := \sup\{h(F_1(x), F_2(x)) : \ x \in I\}, \quad \forall F_1, F_2 \in \Phi(I), \tag{3}$$

is complete.

It is of great interests to discuss equation (2) for $n \geq 3$. As pointed in the end of [14], the greatest difficulty comes from *Lipschitz condition*. For $n = 2$ the operator L defined in (3.10) of [14], i.e., the operator $L_F := \lambda_1 \mathrm{id} + \lambda_2 F + \cdots + \lambda_n F^{n-1}$ defined in (3.2) of [31], is independent of iterates F^i $(i \geq 2)$ and therefore the existence in [14] can be proved without Lipschitz condition of F and G. Obviously, it is not the case for $n \geq 3$ when we want to employ the idea of [30, 31] because we have to estimate

$h(F_1^2, F_2^2)$ with $h(F_1, F_2)$, where h is the Hausdorff distance for sets. One can show that a multifunction in $\Phi(I)$ with a Lipschitz condition has to be single-valued.

In this paper we discuss the iterative equation

$$\lambda_1 F(x) = G(x) - \lambda_2 F^2(x) - \cdots - \lambda_n F^n(x), \quad \forall x \in I, \tag{4}$$

a modified form of (2), for the general $n \geq 3$. We specially consider the class of unblended multifunctions and investigate the set of all jumps for iterates of those multifunctions. Then we introduce the concept of piecewise Lipschitz condition and use it to prove the existence of USC multi-valued solutions of equation (4) in the class of unblended multifunctions. As a corollary, our this result of existence for (4) implies the existence of a USC multifunction F satisfying

$$\lambda_1 F(x) + \lambda_2 F^2(x) + \cdots + \lambda_n F^n(x) \supset G(x), \quad \forall x \in I, \tag{5}$$

which can be regarded as a weak form of equation (2) *in inclusion sense*, as considered in [17, 19] for set-valued iterative roots. In the end of this paper we indicate that equation (4) is equivalent to equation (2) only in the case of single-valued F. It implies that the existence of USC set-valued solutions of the weak form (5) is the best result which we can obtain from (4).

2. Unblended Multifunctions

As mentioned in the Introduction, the family $cc(I)$ endowed with the Hausdorff metric h, defined by

$$h(A, B) = \max\{\sup\{d(x, B) : x \in A\}, \sup\{d(y, A) : y \in B\}\} \tag{6}$$

where $d(x, B) = \inf\{|x - y| : y \in B\}$, is a complete metric space (cf. [8, Corollary 4.3.12]). Some useful properties are summarized in the following lemma (cf. [20, 24]).

Lemma 2.1. *For $A, B, C, D \in cc(I)$ and for an arbitrary real λ, the following properties hold:*

(a) $h(A + C, B + C) = h(A, B)$,
(b) $h(\lambda A, \lambda B) = |\lambda| h(A, B)$,
(c) $h(A + C, B + D) \leq h(A, B) + h(C, D)$.

As defined in [1, Definition 3.5.1], a multifunction $F : I \to cc(I)$ is *increasing* (resp. *strictly increasing*) if $\max F(x_1) \leq \min F(x_2)$ (resp. $\max F(x_1) < \min F(x_2)$) for all $x_1, x_2 \in I$ with $x_1 < x_2$. A multifunction $F : I \to cc(I)$ is *upper semi-continuous* (abbreviated by *USC*) at a point $x_0 \in I$ if for every open set $V \subset \mathbb{R}$ with $F(x_0) \subset V$ there exists a neighborhood U_{x_0} of x_0 such that $F(x) \subset V$ for every $x \in U_{x_0}$. F is USC on I if it is USC at every point in I. For convenience, let

$$\mathbf{USI}(I) := \{F \in \mathcal{F}(I) : \ F \text{ is USC and strictly increasing}\}.$$

Lemma 2.2. *$F_1 \circ F_2 \in \mathbf{USI}(I)$ for $F_1, F_2 \in \mathbf{USI}(I)$.*

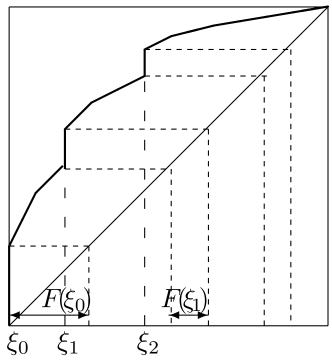

Fig. 2.1: F is blended

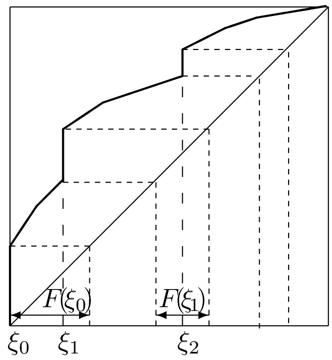

Fig. 2.2: F is unblended

Proof. Obviously, $F_1 \circ F_2 \in \mathcal{F}(I)$ and $F_1 \circ F_2$ is USC (cf. [8, Theorem 7.3.11]). Since F_1 and F_2 is strictly increasing, we see that $\max F_2(x_1) < \min F_2(x_2)$ and therefore $\max F_1(\max F_2(x_1)) < \min F_1(\min F_2(x_2))$ for all $x_1, x_2 \in I$ with $x_1 < x_2$. Clearly,

$$\max F_1(\max F_2(x_1)) = \max F_1 \circ F_2(x_1),$$
$$\min F_1(\min F_2(x_2)) = \min F_1 \circ F_2(x_2).$$

It follows that $\max F_1 \circ F_2(x_1) < \min F_1 \circ F_2(x_2)$ for all x_1, x_2 with $x_1 < x_2$, implying that $F_1 \circ F_2$ is strictly increasing. Thus $F_1 \circ F_2 \in \mathbf{USI}(I)$. □

If a function $F \in \mathbf{USI}(I)$ is not single-valued, there exists at least a point $\xi \in I$ such that the cardinal of the set $F(\xi)$ is more than 1, i.e., $\operatorname{card} F(\xi) \geq 2$. Actually, $F(\xi)$ is a nontrivial interval because $F(\xi) \in cc(I)$. Since F is strictly increasing, there exist two small open intervals $V_\delta^-(\xi) := \{x \in I \mid \xi - \delta < x < \xi\}$ and $V_\delta^+(\xi) := \{x \in I \mid \xi < x < \xi + \delta\}$ such that F is single-valued in both of them and satisfies

$$\min F(\xi) > F(x) \quad \forall x \in V_\delta^-(\xi) \quad \text{and} \quad \max F(\xi) < F(x) \quad \forall x \in V_\delta^+(\xi).$$

We call ξ a *jump-point* of F or a *jump* simply. For every $F \in \mathbf{USI}(I)$, let $J(F)$ denote the set of all jumps of F. We easily see that each $F \in \mathbf{USI}(I)$ has at most *countably infinite many* jumps, i.e., the cardinal $\operatorname{card} J(F) \leq \aleph_0$. In fact, for each $\xi \in J(F)$, the set $F(\xi)$ is a nontrivial compact subinterval of I. By the strict monotonicity, $\{F(\xi) : \xi \in J(F)\}$ is a set of disjoint nonempty compact subintervals of I. Choose a rational number $r(\xi) \in F(\xi)$ for each $\xi \in J(F)$. Then $\operatorname{card} J(F) \leq \operatorname{card} \mathbb{Q} = \aleph_0$. It is easy to construct an example in $\mathbf{USI}(I)$ which has infinitely many jumps. For example, the multifunction

$$F(x) = \begin{cases} 0, & x = 0, \\ x + \frac{1}{(n+1)^2}, & x \in (\frac{1}{n+1}, \frac{1}{n}),\ n = 2, 3, ..., \\ [\frac{1}{n} + \frac{1}{(n+1)^2}, \frac{1}{n} + \frac{1}{n^2}], & x = \frac{1}{n},\ n = 2, 3, ..., \\ \frac{1}{2}x + \frac{1}{2}, & x \in (\frac{1}{2}, 1], \end{cases}$$

is in the class $\mathbf{USI}([0, 1])$ and $J(F)$ is infinite.

The main purpose in this section is to find a "good" class of multifunctions for each of which either the number of jumps does not increase under iteration (i.e., $J(F^k) \subset J(F)$) or all jumps of iterates remain in a given set. A multifunction $F \in \mathbf{USI}(I)$ is said to be *unblended* on a sequence $S := \{\zeta_k\}_{k\geq 0}$ in I if $\zeta_{k+1} \in F(\zeta_k)$ (i.e., $\zeta_{k+1} \in [\min F(\zeta_k), \max F(\zeta_k)]$) for each k. Otherwise, $F \in \mathbf{USI}(I)$ is said to be *blended* on S. We show this in Figures 2.1 and 2.2.

Since functions in $\mathbf{USI}(I)$ are strictly increasing, it suffices to discuss multifunctions F in $\mathbf{USI}(I)$ which satisfy either $\min F(x) > x$ for all $x \in \operatorname{int} I$ or $\max F(x) < x$ for all $x \in \operatorname{int} I$. Let $\mathbf{USI}^*(I)$ and $\mathbf{USI}_*(I)$ denote the two classes of multifunctions respectively.

Lemma 2.3. *Suppose that $F \in \mathbf{USI}^*(I)$ (resp. $\in \mathbf{USI}_*(I)$) is unblended on the strictly increasing (resp. decreasing) sequence $S = \{\zeta_k\}_{k\geq 0}^{\infty}$. If $S \supset J(F)$ and satisfies that $\xi_0 = a$ and $\lim_{k\to\infty} \zeta_k = b$ (resp. $\xi_0 = b$ and $\lim_{k\to\infty} \zeta_k = a$), then for each integer $i \geq 1$,*

(i) $J(F^i) \subset S$, *and*

(ii) $F^i((\zeta_k, \zeta_{k+1})) \subset (\zeta_{k+i}, \zeta_{k+1+i})$ *(resp.* $F^i((\zeta_{k+1}, \zeta_k)) \subset (\zeta_{k+1+i}, \zeta_{k+i})$*)*, $\forall k \geq 0$.

Proof. We only prove this lemma for $F \in \mathbf{USI}^*(I)$. The other case is similar.

Suppose that there exists a point $\xi \in J(F^2) \setminus S$. Then $F(\xi) \in J(F)$. Since $J(F) \subset S$ and $F \in \mathbf{USI}^*(I)$, there exists a ζ_k with $k \geq 1$ such that $F(\xi) = \zeta_k$. Noting that F is unblended, we have $F(\xi) = \zeta_k \in F(\zeta_{k-1})$. Because F is strictly increasing in I, we infer that $\xi = \zeta_{k-1}$, a contradiction to the assumption of ξ. Hence $J(F^2) \subset S$. We can similarly prove (i) for general i.

For (ii), we note that F is single-valued on x and $F(x) \in (\max F(\zeta_k), \min F(\zeta_{k+1}))$ for every $x \in (\zeta_k, \zeta_{k+1})$ because $F \in \mathbf{USI}^*(I)$ and $J(F) \subset S$. Since F is unblended on S, we have $\zeta_{k+1} \leq \max F(\zeta_k)$ and $\zeta_{k+2} \geq \min F(\zeta_{k+1})$, implying that

$$F((\zeta_k, \zeta_{k+1})) = (\max F(\zeta_k), \min F(\zeta_{k+1})) \subset (\zeta_{k+1}, \zeta_{k+2}).$$

We can similarly prove (ii) for general i. □

For a strictly increasing sequence $S = \{\zeta_k\}_{k\geq 0}^{\infty}$ in I such that $\xi_0 = a$ and $\lim_{k\to\infty} \zeta_k = b$, define

$$\mathbf{USI}_u^*(I, S) := \{F \in \mathbf{USI}^*(I) : F \text{ is unblended on } S \text{ and } J(F) \subset S\}.$$

Similarly, for a strictly decreasing sequence $S = \{\zeta_k\}_{k\geq 0}^{\infty}$ in I such that $\xi_0 = b$ and $\lim_{k\to\infty} \zeta_k = a$, define

$$\mathbf{USI}_{u*}(I, S) := \{F \in \mathbf{USI}_*(I) : F \text{ is unblended on } S \text{ and } J(F) \subset S\}.$$

By Lemma 2.3, we assert that iteration of an unblended multifunction, which belongs to $\mathbf{USI}_u^*(I, S)$ or $\mathbf{USI}_{u*}(I, S)$, may increase the number of jumps but always keep all jumps in S. In particular, we can assert that the number of jumps does not increase under iteration when $S = J(F)$.

3. Piecewise Lipschitz Condition

Suppose that Λ is a strictly monotonic sequence in I such that $\mathrm{int}(I)\backslash\Lambda$ is a union of disjoint open intervals, i.e.,

$$\mathrm{int}(I)\backslash\Lambda = \bigcup_{k\geq 0}(\eta_k, \eta_{k+1}), \tag{7}$$

where each η_k is either an element of Λ or an endpoint of I.

A function $F \in \mathbf{USI}(I)$ is said to be *piecewise Lipschitzian* on I with the sequence Λ and constants $M > m > 0$ if for each $k \geq 0$,

(L_1) $J(F) \subset \Lambda$, i.e., the restriction of F to each interval (η_k, η_{k+1}) is single-valued, continuous and strictly increasing,

(L_2) $m(x_2 - x_1) \leq F(x_2) - F(x_1) \leq M(x_2 - x_1)$ $\forall x_1, x_2 \in (\eta_k, \eta_{k+1})$ with $x_1 < x_2$, and

(L_3) $\max F(\eta_{k+1}) - \min F(\eta_k) \leq M(\eta_{k+1} - \eta_k)$ $\forall \eta_k, \eta_{k+1} \in \Lambda$.

Let $\mathbf{USI}(I, \Lambda, m, M)$ be the set of all those piecewise Lipschitzian functions in $\mathbf{USI}(I)$ with the sequence Λ and constants $M > m > 0$.

Lemma 3.1. $\mathbf{USI}(I, \Lambda, m, M)$ *is a complete metric space equipped with the distance* D*, defined in* (3).

Proof. Let $\{F_j\}$ be a Cauchy sequence in $\mathbf{USI}(I, \Lambda, m, M)$. Similarly to Lemma 1 of [14], we can prove that the limit $F(x) := \lim_{j\to\infty} F_j(x)$ is well defined for every fixed $x \in I$ and that F is USC and increasing on I. Since $F_j \in \mathbf{USI}(I, \Lambda, m, M)$, we have $J(F) \subset \Lambda$ and

$$0 < m(x_2 - x_1) \leq F(x_2) - F(x_1) = \lim_{j\to\infty} \big(F_j(x_2) - F_j(x_1)\big) \leq M(x_2 - x_1)$$

for all $x_1, x_2 \in (\eta_k, \eta_{k+1})$ with $x_1 < x_2$. In the sequel we prove that F satisfies (L_3). For a reduction to absurdity, suppose that there exists an integer $k \geq 0$ such that $\max F(\eta_{k+1}) - \min F(\eta_k) > M(\eta_{k+1} - \eta_k)$. Put

$$\varepsilon := \max F(\eta_{k+1}) - \min F(\eta_k) - M(\eta_{k+1} - \eta_k).$$

Since $F_j(\eta_k) \to F(\eta_k)$ and $F_j(\eta_{k+1}) \to F(\eta_{k+1})$ as $j \to \infty$, we can find $j_0 \in \mathbb{N}$ such that $F(\eta_k) \subset F_{j_0}(\eta_k) + (-\varepsilon/2, \varepsilon/2)$ and $F(\eta_{k+1}) \subset F_{j_0}(\eta_{k+1}) + (-\varepsilon/2, \varepsilon/2)$. Hence

$$\min F(\eta_k) > \min F_{j_0}(\eta_k) - \frac{\varepsilon}{2}, \qquad \max F(\eta_{k+1}) < \max F_{j_0}(\eta_{k+1}) + \frac{\varepsilon}{2}.$$

It implies that

$$\begin{aligned} &\max F_{j_0}(\eta_{k+1}) - \min F_{j_0}(\eta_k) \\ >\ &\max F(\eta_{k+1}) - \varepsilon/2 - \big(\min F(\eta_k) + \varepsilon/2\big) \\ =\ &\max F(\eta_{k+1}) - \min F(\eta_k) - \varepsilon \\ =\ &M(\eta_{k+1} - \eta_k), \end{aligned}$$

a contradiction to the fact that F_{j_0} satisfies (L_3). □

Define

$$\begin{aligned}\mathbf{USI}^*(I,S,m,M) &:= \mathbf{USI}^*(I)\cap\mathbf{USI}(I,S,m,M)\\ \mathbf{USI}_*(I,S,m,M) &:= \mathbf{USI}_*(I)\cap\mathbf{USI}(I,S,m,M)\\ \mathbf{USI}_u^*(I,S,m,M) &:= \mathbf{USI}_u^*(I,S)\cap\mathbf{USI}(I,S,m,M),\\ \mathbf{USI}_{u*}(I,S,m,M) &:= \mathbf{USI}_{u*}(I,S)\cap\mathbf{USI}(I,S,m,M).\end{aligned}$$

Clearly, $\mathbf{USI}^*(I,S,m,M), \mathbf{USI}_*(I,S,m,M), \mathbf{USI}_u^*(I,S,m,M)$ and $\mathbf{USI}_{u*}(I,S,m,M)$ are all closed subsets of $\mathbf{USI}(I,S,m,M)$.

Lemma 3.2. *$F^i\in\mathbf{USI}^*(I,S,m^i,M^i)$ (resp. $\mathbf{USI}_*(I,S,m^i,M^i)$) if $F\in\mathbf{USI}_u^*(I,S,m,M)$ (resp. $\mathbf{USI}_{u*}(I,S,m,M)$).*

Proof. We only prove the case that $F\in\mathbf{USI}_u^*(I,S,m,M)$. In the other case the proof will be similar.

The assertion is trivial for $i=1$. Assume that $F^i\in\mathbf{USI}^*(I,S,m^i,M^i)$ holds for some i. Since $F\in\mathbf{USI}_u^*(I,S,m,M)$, we have $F\circ F^i=F^{i+1}\in\mathbf{USI}^*(I)$ by Lemma 2.2 and that $J(F^{i+1})\subset S$ and $F^i\big((\zeta_k,\zeta_{k+1})\big)\subset(\zeta_{k+i},\zeta_{k+1+i})$ by Lemma 2.3. Therefore, the fact that $F^i\in\mathbf{USI}^*(I,S,m^i,M^i)$ implies that

$$F(F^i(x_2))-F(F^i(x_1))\geq m(F^i(x_2)-F^i(x_1))\geq m^{i+1}(x_2-x_1), \tag{8}$$

$$F(F^i(x_2))-F(F^i(x_1))\leq M(F^i(x_2)-F^i(x_1))\leq M^{i+1}(x_2-x_1) \tag{9}$$

for all $x_1,x_2\in(\zeta_k,\zeta_{k+1})$ with $x_2>x_1$. Since F is unblended on S, we have $\zeta_{j+i}\in F^i(\zeta_j)$ for all $j\geq 0$, which implies that $\zeta_{j+i}\leq\max F^i(\zeta_j)<\min F^i(\zeta_{j+1})\leq\zeta_{j+1+i}$. It follows that

$$\begin{aligned}&\max F(F^i(\zeta_{k+1}))-\min F(F^i(\zeta_k))\\ =\ &\max F(\max F^i(\zeta_{k+1}))-\min F(\min F^i(\zeta_k))\\ =\ &\max F(\max F^i(\zeta_{k+1}))-\max F(\zeta_{k+1+i})+\max F(\zeta_{k+1+i})-\min F(\zeta_{k+i})\\ &\quad+\min F(\zeta_{k+i})-\min F(\min F^i(\zeta_k))\\ \leq\ &M(\max F^i(\zeta_{k+1})-\zeta_{k+1+i})+M(\zeta_{k+1+i}-\zeta_{k+i})+M(\zeta_{k+i}-\min F^i(\zeta_k))\\ =\ &M(\max F^i(\zeta_{k+1})-\min F^i(\zeta_k))\\ \leq\ &M^{i+1}(\zeta_{k+1}-\zeta_k).\end{aligned} \tag{10}$$

From those inequalities (8)–(10) we get that $F^{i+1}\in\mathbf{USI}^*(I,S,m^{i+1},M^{i+1})$ and therefore the lemma is proved by induction. □

Lemma 3.3. *If either $F_1,F_2\in\mathbf{USI}_u^*(I,S,m,M)$ or $F_1,F_2\in\mathbf{USI}_{u*}(I,S,m,M)$, then*

$$D(F_1^i,F_2^i)\leq\left(\sum_{j=0}^{i-1}M^j\right)D(F_1,F_2). \tag{11}$$

Proof. We only prove the case that $F_1, F_2 \in \mathbf{USI}_u^*(I, S, m, M)$. The other case can be proved similarly. The result is trivial for $i = 1$. Assume that the result holds for some integer $i \geq 1$. Since $F_p \in \mathbf{USI}_u^*(I, S, m, M)$ for $p = 1, 2$, we have $F_p^i((\zeta_k, \zeta_{k+1})) \subset (\zeta_{k+i}, \zeta_{k+1+i})$ by Lemma 2.3 and $F_p^i \in \mathbf{USI}^*(I, S, m^i, M^i)$ by Lemma 3.2. By the definitions of h and D, for every integer $k \geq 0$ and $x \in (\zeta_k, \zeta_{k+1})$,

$$\begin{aligned}
&h(F_1^{i+1}(x), F_2^{i+1}(x)) = h(F_1(F_1^i(x)), F_2(F_2^i(x))) \\
&\leq h(F_1(F_1^i(x)), F_1(F_2^i(x))) + h(F_1(F_2^i(x)), F_2(F_2^i(x))) \\
&\leq Mh(F_1^i(x), F_2^i(x)) + D(F_1, F_2) \\
&\leq MD(F_1^i, F_2^i) + D(F_1, F_2) \\
&\leq M\left(\sum_{j=0}^{i-1} M^j\right) D(F_1, F_2) + D(F_1, F_2) \\
&= \left(\sum_{j=0}^{i} M^j\right) D(F_1, F_2).
\end{aligned}$$

On the other hand, for each integer $k \geq 0$ and $x = \zeta_k$, there is no loss of generality in assuming that $\min F_1^i(\zeta_k) \leq \min F_2^i(\zeta_k)$. Since $F_1, F_2 \in \mathbf{USI}_u^*(I, S, m, M)$, we have

$$\zeta_{k-1+i} \leq \max F_1^i(\zeta_{k-1}) < \min F_1^i(\zeta_k) \leq \min F_2^i(\zeta_k) \leq \zeta_{k+i}.$$

It follows that

$$\begin{aligned}
&\min F_1(F_2^i(\zeta_k)) - \min F_1(F_1^i(\zeta_k)) \\
&= \min F_1(\min F_2^i(\zeta_k)) - \min F_1(\min F_1^i(\zeta_k)) \\
&\leq M(\min F_2^i(\zeta_k) - \min F_1^i(\zeta_k)).
\end{aligned}$$

Clearly, $\min F_1(F_2^i(\zeta_k)) - \min F_1(F_1^i(\zeta_k)) \geq 0$. Thus, for every ζ_k,

$$|\min F_1(F_2^i(\zeta_k)) - \min F_1(F_1^i(\zeta_k))| \leq M|\min F_2^i(\zeta_k) - \min F_1^i(\zeta_k))|. \tag{12}$$

Similarly, we have

$$|\max F_1(F_2^i(\zeta_k)) - \max F_1(F_1^i(\zeta_k))| \leq M|\max F_2^i(\zeta_k) - \max F_1^i(\zeta_k))|. \tag{13}$$

Therefore, by (12) and (13),

$$\begin{aligned}
&h(F_1^{i+1}(\zeta_k), F_2^{i+1}(\zeta_k)) = h(F_1(F_1^i(\zeta_k)), F_2(F_2^i(\zeta_k))) \\
&\leq h(F_1(F_1^i(\zeta_k)), F_1(F_2^i(\zeta_k))) + h(F_1(F_2^i(\zeta_k)), F_2(F_2^i(\zeta_k))) \\
&= \max\{|\min F_1(F_2^i(\zeta_k)) - \min F_1(F_1^i(\zeta_k))|, |\max F_1(F_2^i(\zeta_k)) - \max F_1(F_1^i(\zeta_k))|\} \\
&\qquad + h(F_1(F_2^i(\zeta_k)), F_2(F_2^i(\zeta_k))) \\
&\leq M\max\{|\min F_1^i(\zeta_k) - \min F_2^i(\zeta_k)|, |\max F_1^i(\zeta_k) - \max F_2^i(\zeta_k)|\} + D(F_1, F_2) \\
&= Mh(F_1^i(\zeta_k), F_2^i(\zeta_k)) + D(F_1, F_2) \\
&\leq MD(F_1^i, F_2^i) + D(F_1, F_2) \\
&\leq \left(\sum_{j=0}^{i} M^j\right) D(F_1, F_2).
\end{aligned}$$

This completes the proof of (11) by induction. □

4. Main Results

Theorem 4.1. *Suppose that* $\lambda_1 > 0, \lambda_i \le 0$ $(i = 2, .., n)$ *and* $\sum_{i=1}^n \lambda_i = 1$ *and that* $G \in \mathbf{USI}^*(I, S, m_0, M_0)$ *with* $M_0 > m_0 > 0$*, where the sequence* $S = \{\zeta_k\}_{k\ge 0}^\infty$ *in* I *is strictly increasing and satisfies* $\zeta_0 = a, \lim_{k\to\infty} \zeta_k = b$ *and*

$$\sum_{i=1}^{n} \lambda_i \zeta_{k+i} \in G(\zeta_k) \quad \forall k \ge 0. \tag{14}$$

Then, for arbitrary constants $M > m > 0$ *satisfying*

$$m \le \frac{m_0 + \sum_{i=2}^n |\lambda_i| m^i}{\lambda_1}, \qquad M \ge \frac{M_0 + \sum_{i=2}^n |\lambda_i| M^i}{\lambda_1}, \tag{15}$$

equation (4) *has a unique solution* $F \in \mathbf{USI}_u^*(I, S, m, M)$ *if*

$$d := \frac{1}{\lambda_1} \sum_{i=2}^{n} |\lambda_i| \sum_{j=0}^{i-1} M^j < 1. \tag{16}$$

Proof. Since the sequence $S = \{\zeta_k\}_{k\ge 0}^\infty$ is strictly increasing in I and satisfies that $\zeta_0 = a$ and $\lim_{k\to\infty} \zeta_k = b$, we have $\mathrm{int}(I)\backslash S = \bigcup_{k\ge 0}(\zeta_k, \zeta_{k+1})$. Define the mapping $\mathcal{T} : \mathbf{USI}_u^*(I, S, m, M) \to \mathcal{F}(I)$ by

$$\mathcal{T}F(x) = \frac{1}{\lambda_1}\Big(G(x) - \sum_{i=2}^{n} \lambda_i F^i(x)\Big), \quad \forall x \in I, \tag{17}$$

where $F \in \mathbf{USI}_u^*(I, S, m, M)$. Note that $\lambda_1 > 0$, $\lambda_i \le 0$ $(i = 2, .., n)$ and $G \in \mathbf{USI}^*(I, S, m_0, M_0)$. We can check that

$$\mathcal{T}F \in \mathbf{USI}\left(I, S, \frac{1}{\lambda_1}\Big(m_0 + \sum_{i=2}^{n} |\lambda_i| m^i\Big), \frac{1}{\lambda_1}\Big(M_0 + \sum_{i=2}^{n} |\lambda_i| M^i\Big)\right) \tag{18}$$

by Lemma 3.2. Obviously, $\mathcal{T}F \in \mathbf{USI}^*(I)$ because $\sum_{i=1}^n \lambda_i = 1$. Moreover, since $F \in \mathbf{USI}_u^*(I, S, m, M)$ and $\sum_{i=1}^n \lambda_i \zeta_{k+i} \in G(\zeta_k)$ for each k, we have

$$\zeta_{k+1} \in \frac{1}{\lambda_1}\Big(G(\zeta_k) - \sum_{i=2}^{n} \lambda_i \zeta_{k+i}\Big) \subset \frac{1}{\lambda_1}\Big(G(\zeta_k) - \sum_{i=2}^{n} \lambda_i F^i(\zeta_k)\Big) = \mathcal{T}F(\zeta_k), \quad \forall k \ge 0,$$

implying that $\mathcal{T}F \in \mathbf{USI}_u^*(I, S, m, M)$ by (15).

Furthermore, by (17) and Lemma 2.1, for every $F_1, F_2 \in \mathbf{USI}_u^*(I, S, m, M)$,

$$\begin{aligned} D(\mathcal{T}F_1, \mathcal{T}F_2) &= \sup_{x\in I} h\left(\frac{1}{\lambda_1}\Big(G(x) - \sum_{i=2}^{n} \lambda_i F_1^i(x)\Big), \frac{1}{\lambda_1}\Big(G(x) - \sum_{i=2}^{n} \lambda_i F_2^i(x)\Big)\right) \\ &\le \frac{1}{\lambda_1} \sum_{i=2}^{n} |\lambda_i| \sup_{x\in I} h(F_1^i(x), F_2^i(x)). \end{aligned} \tag{19}$$

By Lemma 3.3,

$$
\begin{aligned}
D(\mathcal{T}F_1, \mathcal{T}F_2) &\le \frac{1}{\lambda_1}\sum_{i=2}^{n}|\lambda_i| D(F_1^i, F_2^i) \\
&\le \frac{1}{\lambda_1}\sum_{i=2}^{n}|\lambda_i| \sum_{j=0}^{i-1} M^j D(F_1, F_2) \\
&\le dD(F_1, F_2). \qquad (20)
\end{aligned}
$$

Thus, under condition (16) the mapping $\mathcal{T}$ is a contraction. By Banach's fixed point principle, $\mathcal{T}$ has a unique fixed point F in $\mathbf{USI}_u^*(I, S, m, M)$, i.e.,

$$F(x) = \frac{1}{\lambda_1}\Big(G(x) - \sum_{i=2}^{n}\lambda_i F^i(x)\Big), \quad \forall x \in I. \qquad (21)$$

This completes the proof. □

In order to answer to equation (2), we note the fact that $A + B \supset C$ if the sets A, B, C satisfy $A = C - B$. The special choice that $A = [-1, 2], B = [0, 1]$ and $C = [0, 2]$ demonstrates that the equality $A + B = C$ does not hold. It implies that every solution F of equation (4) satisfies

$$\sum_{i=1}^{n}\lambda_i F^i(x) \supset G(x), \quad \forall x \in I,$$

i.e., we obtain the following:

Corollary 4.2. *Under the same conditions as in Theorem* 4.1*, there exists a multifunction* $F \in \mathbf{USI}_u^*(I, S, m, M)$ *such that* (5) *holds.*

For multifunctions in the other class $\mathbf{USI}_*(I, S, m_0, M_0)$ we have a similar result to Theorem 4.1. It can be proved similarly.

Theorem 4.3. *Suppose that* $\lambda_1 > 0, \lambda_i \le 0$ $(i = 2, .., n)$ *and* $\sum_{i=1}^{n}\lambda_i = 1$ *and that* $G \in \mathbf{USI}_*(I, S, m_0, M_0)$ *with* $M_0 > m_0 > 0$*, where the sequence* $S = \{\zeta_k\}_{k\ge 0}^{\infty}$ *in* I *is strictly decreasing and satisfies* $\zeta_0 = b, \lim_{k\to\infty}\zeta_k = a$ *and* (14)*. Then for arbitrary constants* $M > m > 0$ *satisfying* (15) *equation* (4) *has a unique solution* $F \in \mathbf{USI}_{u*}(I, S, m, M)$ *if condition* (16) *holds.*

Corollary 4.4. *Under the same conditions as in Theorem* 4.3*, there exists a multifunction* $F \in \mathbf{USI}_{u*}(I, S, m, M)$ *such that* (5) *holds.*

In general, for $G \in \mathbf{USI}(I)$, let $P(G) := \{x \in I : x \in G(x)\}$. Then $\mathrm{int}(I)\backslash P(G)$ is a union of disjoint open intervals, i.e.,

$$\mathrm{int}(I)\backslash P(G) = \bigcup_r (\alpha_r, \beta_r),$$

where α_r (or β_r) either belongs to $P(G)$ or is an endpoint of I. It is obvious that the restriction $G_r := G|_{I_r}$ to the subinterval $I_r := [\alpha_r, \beta_r]$ belongs to either $\mathbf{USI}^*(I_r)$ or $\mathbf{USI}_*(I_r)$. Thus we can also obtain the existence of solutions for equation (4) with $G \in \mathbf{USI}(I)$ by Theorems 4.1 and 4.3. Results corresponding to Corollaries 4.2 and 4.4 can also be given.

5. Some Remarks

In order to demonstrate conditions in our theorems, consider the equation

$$\frac{9}{8}F(x) = G(x) + \frac{1}{8}F^3(x), \quad x \in I := [0,1], \tag{22}$$

where $n = 3, \lambda_1 = \frac{9}{8}, \lambda_2 = 0, \lambda_3 = -\frac{1}{8}$ and

$$G(x) = \begin{cases} \left[\frac{53}{128}, \frac{29}{64}\right], & x = 0,, \\ \frac{11}{32}x + \frac{29}{64}, & x \in \left(0, \frac{1}{2}\right), \\ \left[\frac{5}{8}, \frac{93}{128}\right], & x = \frac{1}{2}, \\ \frac{35}{64}x + \frac{29}{64}, & x \in \left(\frac{1}{2}, 1\right). \end{cases}$$

Clearly, $G \in \mathbf{USI}^*(I, S, m_0, M_0)$, where

$$m_0 = \min\left\{\frac{11}{32}, \frac{35}{64}\right\} = \frac{11}{32}, \qquad M_0 = \max\left\{\frac{11}{32}, \frac{35}{64}, \frac{\frac{93}{128} - \frac{53}{128}}{\frac{1}{2} - 0}\right\} = \frac{5}{8},$$

and $S = \{\zeta_k\}_{k\geq 0}^{\infty}$ with $\zeta_k := 1 - 2^{-k}$ for each $k \geq 0$. Note that

$$\frac{9}{8}(1 - 2^{-1}) - \frac{1}{8}(1 - 2^{-3}) = \frac{29}{64} \in G(0)$$

and

$$\frac{9}{8}(1 - 2^{-(k+1)}) - \frac{1}{8}(1 - 2^{-(k+3)}) = \frac{29}{64} + \frac{35}{64}(1 - 2^{-k}) \in G(1 - 2^{-k})$$

for each $k \geq 1$. It implies that the condition (14) holds, i.e., $\sum_{i=1}^{n} \lambda_i \zeta_{k+i} \in G(\zeta_k)$ for every $k \geq 0$. Let $m = 11/36$ and $M = 1$. It is easy to check that both (15) and (16) hold. Thus, by Theorem 4.1, equation (22) has a unique solution $F \in \mathbf{USI}_u^*(I, S, m, M)$.

Remark that we have to discuss solutions in the class of *strictly* increasing USC multifunctions for $n \geq 3$ because *the second order iterate of a USC multifunction which is increasing but not strictly increasing may not be increasing.* For example, the multifunction

$$F(x) = \begin{cases} 2x, & x \in \left[0, \frac{1}{4}\right), \\ \frac{1}{2}, & x \in \left[\frac{1}{4}, \frac{1}{3}\right), \\ \left[\frac{1}{2}, \frac{2}{3}\right], & x = \frac{1}{3}, \\ \frac{2}{3}, & x \in (\frac{1}{3}, \frac{1}{2}), \\ \left[\frac{2}{3}, \frac{3}{4}\right], & x = \frac{1}{2}, \\ \frac{1}{2}x + \frac{1}{2}, & x \in (\frac{1}{2}, 1] \end{cases}$$

obviously lies in the class $\Phi([0,1])$ and $J(F)=\{\frac{1}{3},\frac{1}{2}\}$. Its second order iterate

$$F^2(x)=\begin{cases}4x, & x\in[0,\frac{1}{8}),\\ \frac{1}{2}, & x\in[\frac{1}{8},\frac{1}{6}),\\ [\frac{1}{2},\frac{2}{3}], & x=\frac{1}{6},\\ \frac{2}{3}, & x\in(\frac{1}{6},\frac{1}{4}),\\ [\frac{2}{3},\frac{3}{4}], & x\in[\frac{1}{4},\frac{1}{3}),\\ [\frac{2}{3},\frac{5}{6}], & x=\frac{1}{3},\\ \frac{5}{6}, & x\in(\frac{1}{3},\frac{1}{2}),\\ [\frac{5}{6},\frac{7}{8}], & x=\frac{1}{2},\\ \frac{1}{4}x+\frac{3}{4}, & x\in(\frac{1}{2},1],\end{cases}$$

does not increase because $F^2(x)=[\frac{2}{3},\frac{3}{4}]$ for every $x\in[\frac{1}{4},\frac{1}{3})$. Thus, unlike [14], we prefer discussing in $\mathbf{USI}(I)$ rather than in $\Phi(I)$. For this reason in this paper we cannot construct the contraction mapping $\mathcal{T}$ in the same form $\mathcal{T}F:=L_F^{-1}\circ G$ as that in [14] because such a $\mathcal{T}F$ may not be strictly increasing (and in $\mathbf{USI}(I)$) even if both F and G are chosen to be strictly increasing (and in $\mathbf{USI}(I)$).

In contrast, the above mentioned difficulty is overcome by the new construction (17) of contraction mapping $\mathcal{T}$ in this paper, but we only obtain the existence of multi-valued solutions for equation (4), a modified form of equation (2). This result of existence gives the existence of multi-valued solutions for equation (5), a weak version of (2) in inclusion sense as considered in [17, 19] for set-valued iterative roots. Actually, *equation* (4) *is equivalent to equation* (2) *only in the case of single-valued* F. In fact, if F is a solution of equation (4) with cardinal $\operatorname{card} F(x_1)\ge 2$, we see that $\operatorname{card}(\sum_{i=2}^n\lambda_iF^i(x_1))\ge 2$ and therefore

$$-\sum_{i=2}^n\lambda_iF^i(x_1)+\sum_{i=2}^n\lambda_iF^i(x_1)\supsetneq\{0\}, \tag{23}$$

which implies that

$$\lambda_1F(x_1)+\sum_{i=2}^n\lambda_iF^i(x_1)=G(x_1)-\sum_{i=2}^n\lambda_iF^i(x_1)+\sum_{i=2}^n\lambda_iF^i(x_1)\supsetneq G(x_1),$$

i.e., the equality of (2) does not hold. On the contrary, if F is a solution of equation (2) with cardinal $\operatorname{card} F(x_1)\ge 2$, we similarly see from (23) that

$$\lambda_1F(x_1)\subsetneq\lambda_1F(x_1)+\sum_{i=2}^n\lambda_iF^i(x_1)-\sum_{i=2}^n\lambda_iF^i(x_1)=G(x_1)-\sum_{i=2}^n\lambda_iF^i(x_1),$$

i.e., the equality of (4) does not hold.

The above assertion also implies that the setting of equation (4) is better than the original (2). Note that *if* $F,G\in\mathbf{USI}(I)$ *satisfy* (2) *with* $\lambda_1>0,\lambda_i\le 0$ *(*$i=2,..,n$*),* *then both F and G must be single-valued.* Indeed, by (2) we have

$$\lambda_1 F(x) \subset \lambda_1 F(x) + \sum_{i=2}^{n} \lambda_i F^i(x) - \sum_{i=2}^{n} \lambda_i F^i(x) = G(x) - \sum_{i=2}^{n} \lambda_i F^i(x).$$

Since both $\lambda_1 F$ and $G - \sum_{i=2}^{n} \lambda_i F^i$ belong to $\mathbf{USI}(I)$, by Lemma 2 in [14] we get

$$\lambda_1 F(x) = G(x) - \sum_{i=2}^{n} \lambda_i F^i(x),$$

i.e., (4) holds. However, as shown in the above paragraph, it holds if and only if F is single-valued. Then, G is also single-valued by (2). In spite of this, our theorems indicate that there exist multi-valued solutions of (4).

In addition, we remark that our theorems *do not require G to fix endpoints* of the interval I, which relaxes the restriction of the theorem in [14] at endpoints. Since we require that $\lambda_2 \le 0$ in this paper, our theorems do not cover the result in [14] for $n = 2$ even if we considered the result in [14] in the weak sense.

Similarly to [31], one can prove the stability of the solution under the same condition as in Theorem 4.1 or 4.3.

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