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Higher Integrability for Solutions to Variational Problems with Fast Growth

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We prove higher integrability properties of solutions to variational problems of minimizing

$$\int_\Omega \left[e^{f(\|\nabla u(x)\|)} + g(x,u(x))\right] dx \tag{1}$$

where f is a convex function satisfying some additional conditions.

1. Introduction

In this paper we consider the properties of a solution $\tilde{u}$ to the problem of minimizing

$$\int_\Omega \left[e^{f(\|\nabla u(x)\|)} + g(x,u(x))\right] dx. \tag{2}$$

In general, in order to establish the validity of the Euler Lagrange equation for the solution to this problem , i.e., in order to prove that, for every admissible variation η, the equation

$$\int_\Omega \left\{ e^{f(\|\nabla\tilde{u}(x)\|)} f'(\|\nabla\tilde{u}(x)\|) \left\langle \frac{\nabla\tilde{u}(x)}{\|\nabla\tilde{u}(x)\|}, \nabla\eta(x)\right\rangle + g_u(x,\tilde{u}(x))\eta(x)\right\} dx = 0 \tag{3}$$

holds, one has preliminarly to prove that the integrand is in L^1, in particular, that $e^{f(\|\nabla\tilde{u}(\cdot)\|)} f'(\|\nabla\tilde{u}(\cdot)\|) \in L^1_{loc}$. However, for Lagrangeans L growing faster than exponential, the integrability of a term like

$$\int_\Omega L(\|\nabla u(x)\|)\, dx$$

does not imply the integrability of

$$\int_\Omega \nabla L(\|\nabla u(x)\|)\, dx.$$

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In fact, consider $L(s) = e^{s^2}$, so that $L' = 2se^{s^2}$. For $n = 1$, the function $\xi(\cdot)$ whose derivative is

$$\xi'(t) = \sqrt{-\ln(|t|(|\ln |t||)^{\frac{3}{2}})}$$

is such that $e^{\xi'(t)^2} = \frac{1}{|t|\,|\ln|t||^{\frac{3}{2}}}$ is integrable on $(-\frac{1}{2}, \frac{1}{2})$; however, for $|t|$ small,

$$\begin{aligned}\xi'(t)e^{\xi'(t)^2} &= \frac{1}{|t|(|\ln |t||)^{\frac{3}{2}}}\sqrt{-\ln(|t|(|\ln |t||)^{\frac{3}{2}})}\\ &> \frac{1}{|t|(|\ln |t||)^{\frac{3}{2}}}\sqrt{\frac{1}{2}|\ln |t||} = \frac{1}{\sqrt{2}|t|\,|\ln |t||},\end{aligned}$$

hence $L'(\xi'(\cdot))$ is not locally integrable.

This problem does not occur when we are able to prove some additional regularity properties of the solution $\tilde{u}$. When $g = 0$, by using a barrier as in [7], one can prove that the gradient of the solution is in $L^\infty(\Omega)$; alternatively, taking advantage of the regularity properties of solutions to elliptic equations, as in [2] for the case $L(t) = e^{t^2}$, and in [4], [5] for the case $L(t) = e^{f(t)}$, under general assumptions on f, one proves that the gradient of the solution is in L^∞_{loc}. Both these methods demand additional smoothness assumptions: smoothness of the boundary and of the second derivative of f, in the case of a barrier; smoothness of the second derivative of f in the other case.

In the present paper we prove a higher integrability result for $\tilde{u}$: our result is weaker than the local boundedness of $\nabla\tilde{u}$, the result proved in [2], [4], [5]; however, it holds for a larger class of functionals, where, possibly, the stronger boundedness result might not hold. In fact, we do not assume further regularity on f besides its being convex and differentiable: in particular, we do not assume the existence of a second derivative of f, nor we assume its strict convexity. Moreover, we allow also a dependence on x and on u, assuming that g is a standard Carathéodory function. Our method of proof is based on a simple variation and on the properties of polarity.

2. Higher integrability

In what follows, Ω is a bounded open subset of $\mathbb{R}^N$. The function f^* is the *polar* or *conjugate* [6] of f, a possibly extended valued function. Moreover, since there is no assumption of strict convexity of f, the map f^* is convex but not necessarily differentiable: its subgradient will be denoted by ∂f^*: it is a maximal monotone map. In the Theorem that follows, we use the notation $\frac{1}{p\partial f^*(p)}$: we mean 0 when $p \notin \mathrm{Dom}(\partial f^*)$ and, when $p \in \mathrm{Dom}(\partial f^*)$, we mean any selection from the set-valued map $p \to \frac{1}{p\partial f^*(p)}$: since $\frac{1}{p\partial f^*(p)}$ is strictly decreasing, it is multi-valued at most on a countable set, and any two selections will differ only on a set of measure zero.

Theorem 2.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be convex, differentiable, symmetric, $f(0) = 0$ and assume that*

$$\int^\infty \frac{1}{p\partial f^*(p)}dp < \infty.$$

Let g be differentiable with respect to u, and let g and g_u be Carathéodory functions, and assume that for every U there exists $\alpha_U \in L^1_{loc}$ such that $|v| \le U$ implies

$|g_u(x,v)| \le \alpha_U(x)$. *Let* $\tilde u \in u^0 + W_0^{1,1}(\Omega)$ *be a locally bounded solution to the problem of minimizing*

$$\int_\Omega \left[e^{f(\|\nabla u(x)\|)} + g(x,u(x))\right] dx.$$

Then, for every function ξ *such that* $\int_\Omega e^{f(\xi(x))}\, dx < \infty$, *we have that*

$$e^{f(\|\nabla \tilde u(\cdot)\|)} f'(\|\nabla \tilde u(\cdot)\|)\xi(\cdot) \in L^1_{loc}(\Omega).$$

The result applies, in particular, to the function $\xi(x) = \|\nabla \tilde u(x)\|$, so that we have $e^{f(\|\nabla \tilde u(\cdot)\|)} f'(\|\nabla \tilde u(\cdot)\|)\|\nabla \tilde u(\cdot)\| \in L^1_{loc}(\Omega)$.

Examples. The map $f(s) = s - 2\sqrt{s+1} + 2$ is convex, differentiable and of linear growth. Its conjugate is the extended-valued function $f^*(p) = \frac{p^2}{1-|p|}$ for $|p| < 1$, $= \infty$ elsewhere. The conditions of the theorem are satisfied.

A map satisfying the assumption of the theorem is $f(s) = 2e^{s^{\frac12}}(s^{\frac12} - 1) - s$; then $f^{*\prime}(p) = (\ln(p+1))^2$ and $\int^\infty \frac{1}{p(\ln(p+1))^2} dp < \infty$.

A map f that does not satisfy the assumption of the theorem is $f(s) = \frac1e(e^s - s - 1)$; in this case, we have $f^{*\prime}(p) = 1 + \ln(p + \frac1e)$.

Remark 2.2. In Theorem 2.1 we assume the solution $\tilde u$ to be locally bounded. The validity of this assumption can be guaranteed:

i) when $g = 0$, assuming that the boundary datum u^0 is in L^∞, through a standard comparison result, noticing that, with the exception of the case $f \equiv 0$, $e^{f(\|v\|)}$ is a strictly convex function of z.

ii) in general, assuming that there exist $p \in \mathbb{R}^+$, $\alpha \in L^1(\Omega)$ and $\beta \in \mathbb{R}$ such that $u_0 \in W^{1,p}(\Omega)$ and

$$|g(x,u)| \le \alpha(x) + \beta|u|^p.$$

In fact, with the exception of the case $f \equiv 0$, there are A and $B > 0$ such that that $f(t) \ge A + Bt$; hence, fix N^* larger than $\sup\{N,p\}$. For suitable constants, we have

$$\begin{aligned}\infty > \int_\Omega \left[e^{f(\|\nabla \tilde u(x)\|)} + g(x,\tilde u(x))\right] dx &\ge \int_\Omega \left[e^{A+B\|\nabla \tilde u(x)\|} - |\alpha(x)| - |\beta||\tilde u(x)|^p\right] dx \\ &\ge A_1 + B_1\|\nabla \tilde u(x)\|^{N^*}_{L^{N^*}(\Omega)} - |\beta|\|\tilde u\|^p_{L^p(\Omega)} \\ &\ge A_1 + B_1\|\nabla \tilde u(x)\|^{N^*}_{L^{N^*}(\Omega)} - C_1\|u_o\|^p_{L^p(\Omega)} - C_1\|\tilde u - u_o\|^p_{L^p(\Omega)}.\end{aligned}$$

By Poincaré's inequality,

$$\infty > A_2 + B_1\|\nabla \tilde u(x)\|^{N^*}_{L^{N^*}(\Omega)} - C_2\|\nabla \tilde u - \nabla u_0\|^p_{L^p(\Omega)}.$$

By Holder's inequality,

$$\infty > A_2 + B_1\|\nabla \tilde u(x)\|^{N^*}_{L^{N^*}(\Omega)} - C_3\|\nabla u_0\|^p_{L^p(\Omega)} - D\|\nabla \tilde u\|^p_{L^{N^*}(\Omega)},$$

so that there are positive constants h and k such that

$$\infty > -h + k\|\nabla \tilde u(x)\|^{N^*}_{L^{N^*}(\Omega)}.$$

Hence, $\tilde{u}$ belongs to $C_B(\Omega)$ ([1]).

The proof of Theorem 2.1 relies on directly comparing the value of the functional on the solution $\tilde{u}$ and on a variation $\tilde{u} + \varepsilon v$. For it, we shall need the following Lemmas.

Lemma 2.3. *Let $G : \mathbb{R} \to 2^{\mathbb{R}}$ be upper semicontinuous, strictly increasing and such that $G(0) = \{0\}$. Assume that, for a selection g from G,*

$$\int^{\infty} g\left(\frac{1}{s}\right) ds < \infty. \tag{4}$$

Then, the implicit Cauchy problem

$$x(t) \in G(x'(t)), \qquad x(0) = 0$$

admits a solution $\tilde{x}$, positive on some interval $(0, \tau)$.

Notice that the condition expressed by (4) is independent on the selection g; in fact, G is multi-valued at most on countably many points, and the map $s \to \frac{1}{s}$ is strictly monotonic.

Proof. Set $\gamma = G^{-1}$: γ is single-valued, continuous and $\gamma(0) = 0$. We claim that for every $z > 0$

$$\int_{(0,z)} \frac{1}{\gamma(y)} dy = z \frac{1}{\gamma(z)} + \text{meas}(R),$$

where $R = \{(y, x) : 0 \le y \le z; \frac{1}{\gamma(z)} \le x \le \frac{1}{\gamma(y)}\}$. In fact, we have also that $R = \{(x, y) : 0 \le y \le \gamma^{-1}(\frac{1}{x}); \frac{1}{\gamma(z)} \le x < \infty\}$, so that $\text{meas}(R) = \int_{\frac{1}{\gamma(z)}}^{\infty} \gamma^{-1}(\frac{1}{y}) dy = \int_{\frac{1}{\gamma(z)}}^{\infty} g(\frac{1}{s}) ds$, that is finite by assumption.

Hence, the map $\Phi(x) = \int_0^x \frac{1}{\gamma(y)} dy$ is well defined, differentiable, positive for $x > 0$ and $\Phi(0) = 0$. Define $\tilde{x}(t)$ implicitely by

$$\Phi(\tilde{x}(t)) - t = 0;$$

then, $\tilde{x}$ is a differentiable function, $\tilde{x}(0) = 0$ and $x'(t) = \gamma(x(t))$. □

Let $O \subset\subset \Omega$, set $O_\delta = O + B(0, \delta)$ and let $\delta > 0$ be such that $\overline{O_\delta}$ is in Ω.

Lemma 2.4. *Let f be as in Theorem 2.1. Then, for every non-negative ξ in $L^1(O_\delta)$ and $U \in \mathbb{R}$, there exist $\eta \in C_c^1(O_\delta)$ and K such that*

$$f(\xi(1 - \varepsilon\eta) + \varepsilon\|\nabla\eta\|U) - f(\xi) \le \varepsilon K.$$

Proof. Consider the function

$$G(z) = z \frac{2U}{\partial f^* \left(\frac{1}{z}\right)}. \tag{5}$$

We claim that G satisfies the assumptions of Lemma 2.3. In fact, $G(0) = \{0\}$ and G is a strictly increasing multi-valued map (single-valued except on a countable set); we have

$$G\left(\frac{1}{x'}\right) = \frac{2U}{x' \partial f^*(x')}$$

so that, by the assumptions of Lemma 2.4, the condition of Lemma 2.3 is satisfied.

Consider $\tilde{x}$, the solution to $\tilde{x} \in G(\tilde{x}')$, provided by Lemma 2.3. Define η as follows. Let $d(x)$ be the distance from a point $x \in O_\delta$ to ∂O_δ and set

$$\eta(x) = \inf\left\{\frac{1}{\tilde{x}(\delta)}\tilde{x}(d(x)), 1\right\}$$

so that, in particular, $\eta = 1$ on O. Almost everywhere, d is differentiable with $\|\nabla d\| = 1$ and, at a point of differentiability, we have

$$\nabla\eta(x) = \begin{cases} 0 & \text{if } d(x) > \delta \\ \dfrac{1}{\tilde{x}(\delta)}\tilde{x}'(d(x))\nabla d(x) & \text{if } d(x) < \delta. \end{cases}$$

Hence, a.e., we have that $\|\nabla\eta\| \le \frac{1}{\tilde{x}(\delta)}\tilde{x}'(\delta)$ and that, either $\nabla\eta = 0$, or that

$$\eta = \frac{1}{\tilde{x}(\delta)}\tilde{x} = \frac{1}{\tilde{x}(\delta)}\tilde{x}'\frac{2U}{\partial f^*(\frac{1}{\tilde{x}'})} = \|\nabla\eta\| h(\tilde{x}(\delta)\|\nabla\eta\|)$$

with $h(z) = \frac{2U}{\partial f^*(\frac{1}{z})}$, an increasing function.

Set $F(\varepsilon,\xi) = f((1-\varepsilon\eta(x))\xi(x) + \varepsilon\|\nabla\eta(x)\|U)$. From the convexity of f, we obtain

$$\begin{aligned} & F(\varepsilon,\xi) - f(\xi) \\ & \le \begin{cases} \varepsilon f'(\xi(1-\varepsilon\eta) + \varepsilon\|\nabla\eta\|U)[-\eta\xi + \|\nabla\eta\|U] & \text{if } -\eta\xi + \|\nabla\eta\|U > 0 \\ \varepsilon f'(\xi)[-\eta\xi + \|\nabla\eta\|U] & \text{if } -\eta\xi + \|\nabla\eta\|U \le 0. \end{cases} \end{aligned} \tag{6}$$

In the second case, take K to be 0. In the first case, we cannot have $\nabla\eta = 0$, hence we have, a.e., $\eta = \|\nabla\eta\| h(\tilde{x}(\delta)\|\nabla\eta\|)$ and

$$\begin{aligned} & f(\xi(1-\varepsilon\eta) + \varepsilon\|\nabla\eta\|U) - f(\xi) \\ & \le \varepsilon\|\nabla\eta\| f'(\xi(1-\varepsilon\eta) + \varepsilon\eta U)[-h(\tilde{x}(\delta)\|\nabla\eta\|)\xi + U]. \end{aligned}$$

In addition, from $-\eta\xi + \|\nabla\eta\|U > 0$, we infer $\xi \le \frac{U}{h(\tilde{x}(\delta)\|\nabla\eta\|)}$, so that

$$\xi(1-\varepsilon\eta) + \varepsilon\|\nabla\eta\| \le \frac{U}{h(\tilde{x}(\delta)\|\nabla\eta\|)} + \varepsilon\|\nabla\eta\|U$$

and

$$\|\nabla\eta\| f'(\xi(1-\varepsilon\eta) + \varepsilon\eta U) \le \|\nabla\eta\| f'\left(\frac{U}{h(\tilde{x}(\delta)\|\nabla\eta\|)} + \varepsilon\|\nabla\eta\|U\right).$$

There exists σ such that, for $\|\nabla\eta\| < \sigma$, we have $\frac{U}{h(\tilde{x}(\delta)\|\nabla\eta\|)} + \varepsilon\|\nabla\eta\|U \le \frac{2U}{h(\tilde{x}(\delta)\|\nabla\eta\|)}$. For those x such that $\|\nabla\eta(x)\| < \sigma$, recalling (5),

$$\begin{aligned} & \|\nabla\eta\| f'\left(\frac{U}{h(\tilde{x}(\delta)\|\nabla\eta\|)} + \varepsilon\|\nabla\eta\|U\right) \le \|\nabla\eta\| f'\left(\frac{2U}{h(\tilde{x}(\delta)\|\nabla\eta\|)}\right) \\ & = \|\nabla\eta\| f'\left(\partial f^*\left(\frac{1}{\tilde{x}(\delta)\|\nabla\eta\|}\right)\right) = \frac{1}{\tilde{x}(\delta)} = K. \end{aligned}$$

It is left to consider the case $\|\nabla\eta\| \ge \sigma$: in this case, $\xi \le \frac{U}{h(\tilde{x}(\delta)\sigma)}$ and the result follows from the boundedness of $\|\nabla\eta\|$. □

Lemma 2.5. *Let ψ non negative and such that*

$$\int_O \psi e^{f(\psi)} f'(\psi) \le M.$$

Then, for any ξ such that $\int_O e^{f(\xi)}$ is bounded, we have that

$$\int_O \xi e^{f(\psi)} f'(\psi)$$

is bounded.

Proof. a) Consider the strictly increasing function $z(t) = f'(t)e^{f(t)}$ and call $t = i(z)$ its inverse, so that we have

$$z = e^{f(i(z))} f'(i(z)). \tag{7}$$

We have that $i(v) \to \infty$ as $v \to \infty$. Define the function ϕ as $\phi(z) = i(z)z$, hence, in terms of t,

$$\phi(f' e^{f(t)}) = t f' e^{f(t)}. \tag{8}$$

b) We wish to compute the polar g^* of the function $g(b) = e^{f(b)}$. Define b_z implicitely, setting

$$z = g'(b_z) = e^{f(b_z)} f'(b_z),$$

and notice that the previous equality defines b_z uniquely and we have $b_z = i(z)$, where i is defined in a). Then

$$g^*(z) = \sup_b bz - g(b) = b_z e^{f(b_z)} f'(b_z) - e^{f(b_z)} = b_z z - e^{f(b_z)} = i(z)z - e^{f(i(z))}$$

so that, by (8) and (7), $g^*(z) \le \phi(f'(b_z)e^{f(b_z)}) = \phi(f'(i(z))e^{f(i(z))}) = \phi(z)$.

For any t and b, we have

$$bf'(t)e^{f(t)} = bv(t) \le g^*(v(t)) + g(b) \le \phi(v(t)) + g(b).$$

Set, in the previous inequality, $t = \psi$ and $b = \xi$. From the definition of ϕ, we obtain

$$\begin{aligned} \xi f'(\psi)e^{f(\psi)} &\le \phi(f'(\psi)e^{f(\psi)}) + e^{f(\xi)} \\ &= \psi f'(\psi)e^{f(\psi)} + e^{f(\xi)}. \end{aligned}$$

From the assumptions of the Lemma, the proof is completed. □

Proof of Theorem 2.1. In the proof, we shall first prove the higher integrability result for the special case where $\xi(\cdot) = \|\nabla\tilde{u}(\cdot)\|$ and then extend this result to the general case.

a) Let O and O_δ as before. Set $U = \sup\{|\tilde{u}(x)| : x \in O_\delta\}$. Since $\tilde{u}$ is a minimum, for every variation v we have

$$\int_\Omega \left[e^{f(\|\nabla\tilde{u}(x)+\varepsilon\nabla v\|)} + g(x, \tilde{u}(x) + \varepsilon v(x))\right] dx \ge \int_\Omega \left[e^{f(\|\nabla\tilde{u}(x)\|)} + g(x, \tilde{u}(x))\right] dx.$$

set $v = -\eta\tilde{u}$, so that $\nabla v = -\tilde{u}\nabla\eta - \eta\nabla\tilde{u}$ and $|v| \le U$. For $\varepsilon > 0$ (and $\varepsilon < 1$), we obtain

$$\begin{aligned} &\int_\Omega \left(\frac{e^{f(\|\nabla\tilde{u}(x)(1-\varepsilon\eta)-\varepsilon\tilde{u}\nabla\eta\|)} - e^{f(\|\nabla\tilde{u}(x)\|)}}{\varepsilon} \right) dx \\ &\ge -\int_\Omega \frac{g(x,\tilde{u}(x)+\varepsilon v(x)) - g(x,\tilde{u}(x))}{\varepsilon}. \end{aligned} \tag{9}$$

b) By Lemma 2.3, η can be defined so that, for some $K \ge 0$, we have:

$$\begin{aligned} &f(\|\nabla\tilde{u}(1-\varepsilon\eta) - \varepsilon\tilde{u}\nabla\eta\|) - f(\|\nabla\tilde{u}\|) \\ &\le f(\|\nabla\tilde{u}\|(1-\varepsilon\eta) + \varepsilon U\|\nabla\eta\|) - f(\|\nabla\tilde{u}\|) \le \varepsilon K. \end{aligned} \tag{10}$$

Set $F(\varepsilon,\nabla\tilde{u}) = f((1-\varepsilon\eta(x))\|\nabla\tilde{u}(x)\| + \varepsilon\|\nabla\eta(x)\|U)$. From (9), we have

$$\begin{aligned} &-\int_\Omega \frac{g(x,\tilde{u}(x)+\varepsilon v(x)) - g(x,\tilde{u}(x))}{\varepsilon} \\ &\le \int_\Omega \left(\frac{e^{F(\varepsilon,\nabla\tilde{u})} - e^{f(\|\nabla\tilde{u}(x)\|)}}{\varepsilon} \right) dx \\ &= \int_\Omega \left(\frac{e^{F(\varepsilon,\nabla\tilde{u})-\varepsilon K+\varepsilon K} - e^{f(\|\nabla\tilde{u}(x)\|)}}{\varepsilon} \right) dx \\ &= \int_\Omega e^{F(\varepsilon,\nabla\tilde{u})-\varepsilon K} \left[\frac{e^{\varepsilon K} - e^{f(\|\nabla\tilde{u}(x)\|)-F(\varepsilon,\nabla\tilde{u})+\varepsilon K)}}{\varepsilon} \right] dx \\ &= \int_\Omega e^{F(\varepsilon,\nabla\tilde{u})-\varepsilon K} \left[\frac{e^{\varepsilon K} - 1 + 1 - e^{f(\|\nabla\tilde{u}(x)\|)-F(\varepsilon,\nabla\tilde{u})+\varepsilon K)}}{\varepsilon} \right] dx. \end{aligned}$$

The previous inequality can be written as

$$\begin{aligned} &\int_\Omega e^{F(\varepsilon,\nabla\tilde{u})-\varepsilon K} \left[\frac{e^{\varepsilon K} - 1}{\varepsilon} \right] dx + \int_\Omega \frac{g(x,\tilde{u}(x)+\varepsilon v(x)) - g(x,\tilde{u}(x))}{\varepsilon} \\ &\ge \int_\Omega e^{F(\varepsilon,\nabla\tilde{u})-\varepsilon K} \left[\frac{e^{f(\|\nabla\tilde{u}\|)-(F(\varepsilon,\nabla\tilde{u})-\varepsilon K)} - 1}{\varepsilon} \right] dx. \end{aligned} \tag{11}$$

c) From (10) we infer that $F(\varepsilon,\nabla\tilde{u}) - \varepsilon K \le f(\|\nabla\tilde{u}(x)\|)$; moreover, $\frac{e^{\varepsilon K}-1}{\varepsilon} \le Ke^K$. In addition,

$$\left| \frac{g(x,\tilde{u}(x)+\varepsilon v(x)) - g(x,\tilde{u}(x))}{\varepsilon} \right| = |g_u(x,u_{\varepsilon,x})\eta\tilde{u}(x)|$$

for some value $u_{\varepsilon,x}$ in the interval of extremes $\tilde{u}(x)$ and $\tilde{u}(x) - \varepsilon\eta(x)\tilde{u}(x)$, so that

$$|g_u(x,u_{\varepsilon,x})\eta(x)\tilde{u}(x)| \le [\alpha_U(x)]U.$$

Hence, the left hand side of (11) is bounded by some M, independent of ε.

d) Consider the right hand side. For some $t_{\varepsilon,x}$ in the interval of extremes $\|\nabla\tilde{u}\|$ and $(1-\varepsilon\eta)\|\nabla\tilde{u}\| + \varepsilon\|\nabla\eta\|U$, we have

$$f((1-\varepsilon\eta)\|\nabla\tilde{u}\| + \varepsilon\|\nabla\eta\|U) - f(\|\nabla\tilde{u}\|) = \varepsilon f'(t_{\varepsilon,x})(-\eta\|\nabla\tilde{u}\| + \|\nabla\eta\|U).$$

As $\varepsilon \to 0$, $t_{\varepsilon,x} \to \|\nabla\tilde{u}(x)\|$ pointwise, so that $f'(t_{\varepsilon,x})$ converges to $f'(\|\nabla\tilde{u}(x)\|)$; moreover, $f(\|\nabla\tilde{u}\|) - (F(\varepsilon, \nabla\tilde{u}) - \varepsilon K) = -\varepsilon f'(t_{\varepsilon,x})(-\eta\|\nabla\tilde{u}\| + \|\nabla\eta\|U) + \varepsilon K = -\varepsilon f'(\|\nabla\tilde{u}\|)(-\eta\|\nabla\tilde{u}\| + \|\nabla\eta\|U) + \varepsilon K + \varepsilon o(1)$, so that

$$\frac{e^{f(\|\nabla\tilde{u}\|)-(F(\varepsilon,\nabla\tilde{u})-\varepsilon K)} - 1}{\varepsilon}$$

converges pointwise to $K + f'(\|\nabla\tilde{u}\|)(\eta\|\nabla\tilde{u}\| - \|\nabla\eta\|U)$. In addition, by (10), $\varepsilon K - f((1-\varepsilon\eta)\|\nabla\tilde{u}\| + \varepsilon\|\nabla\eta\|U) + f(\|\nabla\tilde{u}\|) \geq 0$, so that the integrand at the right hand side is non negative. Finally, pointwise, $e^{F(\varepsilon,\nabla\tilde{u})-\varepsilon K} \to e^{f(\|\nabla\tilde{u}(x)\|)}$. Hence, applying Fatou's lemma, we obtain

$$\int_\Omega e^{f(\|\nabla\tilde{u}\|)} \left[K + f'(\|\nabla\tilde{u}\|)(\eta\|\nabla\tilde{u}\| - \|\nabla\eta\|U)\right] \leq M.$$

Since $K + f'(\|\nabla\tilde{u}\|)(\eta\|\nabla\tilde{u}\| - \|\nabla\eta\|U) \geq 0$, and $\nabla\eta = 0$ and $\eta = 1$ on O, we have obtained that

$$\int_O \|\nabla\tilde{u}\| e^{f(\|\nabla\tilde{u}\|)} f'(\|\nabla\tilde{u}\|) \leq M_1. \tag{12}$$

This proves the result for the case $\xi = \|\nabla\tilde{u}\|$. An application of Lemma 2.5 completes the proof. □

Notice that $\tilde{u}$ does not have to be a minimizer: a local minimizer would do.

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