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A Remark on a Free Interface Problem with Volume Constraint

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We study the optimal design problem of finding the minimal energy configuration for a mixture of two conducting materials with a perimeter penalization of the unknown domain, when a volume constraint is added. We prove that minimizers of the constrained problem also minimize an unconstrained problem with a volume penalization for which regularity results are available in the literature.

Keywords: Perimeter penalization, volume constraint, regularity

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1. Introduction and statements

In this paper we study minimal energy configurations of a mixture of two conductive materials in a bounded, connected open set $\Omega \subset \mathbb{R}^n$, where the energy includes penalization of the perimeter of the interface between the materials.

The energy we consider here is given by

$$\mathcal{F}(u,E) = \int_\Omega \sigma_E(x)|\nabla u|^2\,dx + P(E,\Omega)\,, \tag{1}$$

where $E \subset \Omega$ is a measurable set, $\sigma_E(x) = \alpha\chi_E(x) + \beta\chi_{\Omega\setminus E}(x)$, for some $0 < \beta < \alpha$, $u \in W^{1,2}(\Omega)$ and $P(E,\Omega)$ denotes the perimeter of E in Ω. We are interested in the following constrained problem

$$\min\left\{\mathcal{F}(u,E) :\ u = u_0 \text{ on } \partial\Omega,\ |E| = d\right\}\,, \tag{2}$$

where $u_0 \in W^{1,2}(\Omega)$ is a given boundary datum and $0 < d < |\Omega|$ is prescribed.

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Energies with surface terms competing with a volume term often appear in materials science as a model for optimal design [1], phase transiction [5], liquid crystals [6] or continuum mechanics [4].

A similar functional was studied by L. Ambrosio and G. Buttazzo in [1] where they consider an energy of the type

$$\int_\Omega \left[\sigma_E(x)|\nabla u|^2 + \chi_E(x)g_1(x,u) + \chi_{\Omega\setminus E}(x)g_2(x,u)\right]\,dx + P(E,\Omega)\,,$$

with $u = 0$ on $\partial\Omega$, and prove that if (u, E) is a minimal configuration then u is locally Hölder continuous in Ω and E is (equivalent to) an open set such that $P(E,\Omega) = \mathcal{H}^{n-1}(\partial E \cap \Omega)$.

A stronger regularity result is proved by F. H. Lin in [7] who shows that if (u, E) is a minimizer of functional (1) in the class of all configurations such that u and ∂E are prescribed on $\partial\Omega$, then $u \in C^{1/2}(\Omega)$ and $\partial^* E$ is a $C^{1,\alpha}$-hypersurface away from a closed singular set Σ of $\mathcal{H}^{n-1}$ measure zero (by $\partial^* E$ we denote the reduced boundary of E in the sense of De Giorgi).

A more detailed analysis of the minimal configurations of (1) is carried on in the two dimensional case by C. J. Larsen in [6]. However also in this case only a partial regularity of ∂E is obtained.

All minimum problems considered in these papers are unconstrained. Indeed, from the point of view of regularity the constraint $|E| = d$ introduces extra difficulties, since one can work only with variations which keep the volume constant.

The main result in this paper deals precisely with this issue, stating that every minimizer of the constrained problem (2) is also a minimizer of a suitable energy functional with a volume penalization, but no constraints.

Theorem 1.1. *There exists $\lambda_0 > 0$ such that if (u, E) is a minimizer of the constrained problem (2), then for all $\lambda \geq \lambda_0$, (u, E) is also a minimizer of the functional*

$$\mathcal{F}_\lambda(v,F) = \int_\Omega \sigma_F(x)|\nabla v|^2\,dx + P(F,\Omega) + \lambda\big||F| - d\big| \tag{3}$$

among all configurations (v, F) such that $v = u_0$ on $\partial\Omega$.

Though the functional in (3) is slightly different from the ones considered in [1], [7] or [6], from the point of view of the regularity the extra term $\lambda\big||F|-d\big|$ is a higher order, negligible perturbation, in the sense that if $x_0 \in \partial^* E \cap \partial\Omega$ then $|E \cap B_\varrho(x_0)|$ decays like ϱ^n as $\varrho \to 0^+$ while the leading term $\int_{B_\varrho(x_0)} \sigma_E|\nabla u|^2 + P(E, B_\varrho(x_0))$ decays like ϱ^{n-1}. Therefore, as an immediate consequence of Theorem 1.1, we have that the regularity results proved in above mentioned papers still hold for minimal configurations of the constrained problem (2).

However, the full regularity of the boundary of the optimal sets E remains an open problem, even in low dimensions. Next result gives a partial answer to this question.

Theorem 1.2. *There exists $\gamma_n > 1$ such that if $\frac{\alpha}{\beta} < \gamma_n$ and (u, E) is a minimizer of* *problem* (2)*, then* $u \in C^{0,1/2+\varepsilon}(\Omega)$*, for some* $0 < \varepsilon < 1/2$ *depending on* n, α, β*,* $\partial^* E$ *is a* $C^{1,\varepsilon}$*-hypersurface and* $\mathcal{H}^s(\partial E \setminus \partial^* E) = 0$ *for all* $s > n - 8$.

We conclude with a few words on the proofs. Theorem 1.1 is obtained by a contradiction argument. We show that if λ is sufficiently large and (u, E) is a minimizer of (3) with $|E| \neq d$, then one can construct a suitable bi-Lipschitz map $\Phi : \Omega \to \Omega$ such that, setting $\tilde{u} = u \circ \Phi^{-1}$ and $\widetilde{E} = \Phi(E)$, one has $\mathcal{F}_\lambda(\tilde{u}, \widetilde{E}) < \mathcal{F}_\lambda(u, E)$. The construction of this diffeomorphism is quite delicate and uses in an essential way the structure theorem of the reduced boundary $\partial^* E$ due to De Giorgi.

Theorem 1.2 follows instead by proving that if α/β is not too large (and the critical ratio γ_n can be explicitly estimated), then $\int_{B_\varrho(x_0)} |\nabla u|^2$ decays like $\varrho^{n-1+2\varepsilon}$. As already observed in [7] this estimate easily implies that E is an area almost minimizer in the sense of F. Almgren and the regularity of $\partial^* E$ follows from the results on minimal surfaces proved in [3] and [8].

2. Proof of Theorem 1.1

This section is entirely devoted to the proof of Theorem 1.1. We argue by contradiction assuming that there exist a sequence $\{\lambda_h\}_{h\in\mathbb{N}}$ such that $\lambda_h \to \infty$ as $h \to \infty$ and a sequence of configurations (u_h, E_h) minimizing $\mathcal{F}_{\lambda_h}$ under the boundary condition $u_h = u_0$ for all h such that $|E_h| \neq d$ for all h. Notice that

$$\mathcal{F}_{\lambda_h}(u_h, E_h) \leq \mathcal{F}(u_0, E_0) =: \Lambda \,, \tag{4}$$

where $E_0 \subset \Omega$ is a fixed set of finite perimeter such that $|E_0| = d$. Therefore from (4) it follows that the sequence u_h is bounded in $W^{1,2}(\Omega)$, the perimeters of the sets E_h are bounded and $|E_h| \to d$. Therefore, with no loss of generality, we may assume, passing possibly to a subsequence, that there exist a configuration (u, E) such that $u_h \to u$ weakly in $W^{1,2}(\Omega)$ and $\chi_{E_h} \to \chi_E$ a.e. in Ω. Thus E is set of finite perimeter in Ω with $|E| = d$. Finally we shall assume that $|E_h| < d$ for all h, since if $|E_h| > d$ the proof is similar. Our aim is to show that for h sufficiently large there exists a configuration $(\tilde{u}_h, \widetilde{E}_h)$ such that $\mathcal{F}_{\lambda_h}(\tilde{u}_h, \widetilde{E}_h) < \mathcal{F}_{\lambda_h}(u_h, E_h)$, thus proving the result by contradiction.

Proof. *Step 1.* We begin by taking a point $x \in \partial^* E \cap \Omega$; such a point exists since E has finite perimeter in Ω, $0 < |E| < |\Omega|$, and Ω is connected. By De Giorgi structure theorem for sets of finite perimeter (see [2, Theorem 3.59]) the sets $E_r = (E - x)/r$ converge locally in measure to the half space $H = \{z \cdot \nu_E(x) > 0\}$, i.e., $\chi_{E_r} \to \chi_H$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, where $\nu_E(x)$ is the generalized inner normal to E at x (see [2, Definition 3.54]). Let $y \in B_1(0) \setminus H$ be the point $y = -\nu_E(x)/2$. Given ε (to be chosen at the end of the proof), since $\chi_{E_r} \to \chi_H$ in $L^1(B_1(0))$ there exists $r > 0$ such that

$$|E_r \cap B_{1/2}(y)| < \varepsilon, \qquad |E_r \cap B_1(y)| \geq |E_r \cap B_{1/2}(0)| > \frac{\omega_n}{2^{n+2}} \,,$$

where ω_n denotes the measure of the unit ball of $\mathbb{R}^n$ and $B_r(x_r) \subset \Omega$, where $x_r = x + ry$. Therefore we have

$$|E \cap B_{r/2}(x_r)| < \varepsilon r^n, \qquad |E \cap B_r(x_r)| > \frac{\omega_n r^n}{2^{n+2}} \,.$$

Let us assume, without loss of generality, that $x_r = 0$ and from now on let us denote the balls centered at the origin by B_r. From the convergence of E_h to E we have that for all h sufficiently large

$$|E_h \cap B_{r/2}| < \varepsilon r^n, \qquad |E_h \cap B_r| > \frac{\omega_n r^n}{2^{n+2}}\,. \tag{5}$$

Let us now define the following bi-Lipschitz map which takes B_r into itself

$$\Phi(x) = \begin{cases} (1-\sigma(2^n-1))\,x & \text{if } |x| < \dfrac{r}{2}, \\ x + \sigma\left(1 - \dfrac{r^n}{|x|^n}\right)x & \text{if } \dfrac{r}{2} \le |x| < r, \\ x & \text{if } |x| \ge r\,, \end{cases} \tag{6}$$

for some $0 < \sigma < 1/2^n$ such that, setting

$$\widetilde{E}_h = \Phi(E_h), \qquad \tilde{u}_h = u_h \circ \Phi^{-1}\,,$$

we have $|\widetilde{E}_h| < d$. Let us now estimate

$$\begin{aligned} \mathcal{F}_{\lambda_h}(u_h, E_h) - \mathcal{F}_{\lambda_h}(\tilde{u}_h, \widetilde{E}_h) = {} & \left[\int_{B_r} \sigma_{E_h}(x)|\nabla u_h|^2\,dx - \int_{B_r} \sigma_{\widetilde{E}_h}(y)|\nabla \tilde{u}_h|^2\,dy\right] \\ & + \left(P(E_h, \overline{B}_r) - P(\widetilde{E}_h, \overline{B}_r)\right) + \lambda_h\left(|\widetilde{E}_h| - |E_h|\right) \\ = {} & I_{1,h} + I_{2,h} + I_{3,h}\,. \end{aligned} \tag{7}$$

Step 2. In order to estimate $I_{1,h}$ we need to estimate preliminarily the gradient and the jacobian of Φ in the annulus $B_r \setminus B_{r/2}$. If $r/2 < |x| < r$, we have

$$\frac{\partial \Phi_i}{\partial x_j}(x) = \left(1 + \sigma - \frac{\sigma r^n}{|x|^n}\right)\delta_{ij} + n\sigma r^n \frac{x_i x_j}{|x|^{n+2}} \quad \text{for all } i, j = 1, \dots n$$

and thus, if $\eta \in \mathbb{R}^n$, we get

$$(\nabla\Phi \circ \eta)\cdot\eta = \left(1 + \sigma - \frac{\sigma r^n}{|x|^n}\right)|\eta|^2 + n\sigma r^n \frac{(x\cdot\eta)^2}{|x|^{n+2}}$$

from which it follows that

$$|\nabla\Phi \circ \eta| \ge \left(1 + \sigma - \frac{\sigma r^n}{|x|^n}\right)|\eta|\,.$$

From this inequality we easily deduce an estimate on the norm of $\nabla\Phi^{-1}$

$$\begin{aligned} \left\|\nabla\Phi^{-1}(\Phi(x))\right\| = {} & \max_{|\eta|=1}\left|\nabla\Phi^{-1} \circ \left(\frac{\nabla\Phi\circ\eta}{|\nabla\Phi\circ\eta|}\right)\right| = \max_{|\eta|=1} \frac{1}{|\nabla\Phi\circ\eta|} \\ \le {} & \left(1 + \sigma - \frac{\sigma r^n}{|x|^n}\right)^{-1} \le (1-(2^n-1)\sigma)^{-1} \quad \text{for all } x \in B_r \setminus B_{r/2}\,. \end{aligned} \tag{8}$$

Concerning the jacobian, it is convenient to write, for $x \in B_r \setminus B_{r/2}$,

$$\Phi(x) = \varphi(|x|)\frac{x}{|x|}\,, \tag{9}$$

where

$$\varphi(t) = t\left(1+\sigma-\frac{\sigma r^n}{t^n}\right), \quad \text{for all } t \in [r/2, r]\,.$$

Let I denote the identity map in $\mathbb{R}^n$. Recalling that if $A = I + a \otimes a$ for some vector $a \in \mathbb{R}^n$ then $\det A = 1 + |a|^2$, a straightforward calculation gives for all $x \in B_r \setminus B_{r/2}$

$$\begin{aligned} J\Phi(x) &= \varphi'(|x|)\left(\frac{\varphi(|x|)}{|x|}\right)^{n-1} \\ &= \left(1+\sigma+\frac{(n-1)\sigma r^n}{|x|^n}\right)\left(1+\sigma-\frac{\sigma r^n}{|x|^n}\right)^{n-1} \geq 1 + C_1(n)\sigma\,, \end{aligned} \tag{10}$$

for some positive constant $C_1(n)$ depending only on n. On the other hand, from (10) we get also

$$J\Phi(x) \leq 1 + 2^n n\sigma\,. \tag{11}$$

Let us now turn to the estimate of $I_{1,h}$. Performing the change of variable $y = \Phi(x)$ in the second integral defining $I_{1,h}$, and observing that $\sigma_{\widetilde{E}_h}(\Phi(x)) = \sigma_{E_h}(x)$, we get

$$I_{1,h} = \int_{B_r} \sigma_{E_h}(x)\left[|\nabla u_h(x)|^2 - \left|\nabla u_h(x) \circ \nabla\Phi^{-1}(\Phi(x))\right|^2 J\Phi(x)\right] dx = A_{1,h} + A_{2,h}\,,$$

where $A_{1,h}$ stands for the above integral evaluated on $B_{r/2}$ and $A_{2,h}$ for the same integral evaluated on $B_r \setminus B_{r/2}$. Recalling the definition of Φ in (6) we get

$$\begin{aligned} A_{1,h} &= \int_{B_{r/2}} \sigma_{E_h}(x)\left[|\nabla u_h(x)|^2 - \left|\nabla u_h(x) \circ (1-\sigma(2^n-1))^{-1} I\right|^2 (1-\sigma(2^n-1))^n\right] dx \\ &= \int_{B_{r/2}} \sigma_{E_h}(x)|\nabla u_h(x)|^2 \left[1-(1-\sigma(2^n-1))^{n-2}\right] dx \geq 0\,. \end{aligned}$$

Recalling (8), (11) and (4) we have

$$\begin{aligned} A_{2,h} &= \int_{B_r\setminus B_{r/2}} \sigma_{E_h}(x)\left[|\nabla u_h(x)|^2 - \left|\nabla u_h(x) \circ \nabla\Phi^{-1}(\Phi(x))\right|^2 J\Phi(x)\right] dx \\ &\geq \int_{B_r\setminus B_{r/2}} \sigma_{E_h}(x)|\nabla u_h(x)|^2 \left[1-(1-(2^n-1)\sigma)^{-2}(1+2^n n\sigma)\right] dx \\ &\geq -C_2(n)\sigma \int_{B_r\setminus B_{r/2}} \sigma_{E_h}(x)|\nabla u_h(x)|^2\, dx \geq -C_2(n)\Lambda\sigma\,, \end{aligned}$$

for some $C_2(n) > 0$, depending only on n. Thus, from the above estimates we may conclude that

$$I_{1,h} \geq -C_2(n)\Lambda\sigma\,. \tag{12}$$

Step 3. To estimate $I_{2,h}$ we shall make use of the area formula for maps between rectifiable sets. To this aim let us denote for all $x \in \partial^* E_h$ by $T_{h,x} : \pi_{h,x} \to \mathbb{R}^n$ the tangential differential at x of Φ along the approximate tangent space $\pi_{h,x}$ to $\partial^* E_h$, which is defined by $T_{h,x}(\tau) = \nabla\Phi(x) \circ \tau$ for all $\tau \in \pi_{h,x}$. We recall (see [2, Definition 2.68]) that the $(n-1)$-dimensional jacobian of $T_{h,x}$ is given by

$$J_{n-1}T_{h,x} = \sqrt{\det\left(T^*_{h,x} \circ T_{h,x}\right)}\,,$$

where $T^*_{h,x}$ is the adjoint of the map $T_{h,x}$. To estimate $J_{n-1}T_{h,x}$ let us fix $x \in \partial^* E_h \cap (B_r \setminus B_{r/2})$. Let us denote by $\{\tau_1, \dots, \tau_{n-1}\}$ an orthonormal base for $\pi_{h,x}$ and by L the $n \times (n-1)$ matrix representing $T_{h,x}$ with respect to the fixed base in $\pi_{h,x}$ and the standard base $\{e_1, \dots, e_n\}$ in $\mathbb{R}^n$. From (9) we have

$$\begin{aligned} L_{ij} &= \nabla\Phi_i \cdot \tau_j \\ &= \frac{\varphi(|x|)}{|x|} e_i \cdot \tau_j + \left(\varphi'(|x|) - \frac{\varphi(|x|)}{|x|}\right) \frac{x_i}{|x|}\frac{x \cdot \tau_j}{|x|}, \quad i = 1, \dots, n,\ j = 1, \dots, n-1\,. \end{aligned}$$

Thus, we have that for $j, l = 1, \dots, n-1$

$$(L^* \circ L)_{jl} = \frac{\varphi^2(|x|)}{|x|^2} \sum_{i=1}^{n} (e_i \cdot \tau_j)(e_i \cdot \tau_l) + \left(\varphi'^2(|x|) - \frac{\varphi^2(|x|)}{|x|^2}\right) \frac{(x \cdot \tau_j)(x \cdot \tau_l)}{|x|^2}\,.$$

Since $J_{n-1}T_{h,x}$ is invariant by rotation, in order to evaluate $\det(L^* \circ L)$ we may assume without loss of generality that $\tau_j = e_j$, for all $j = 1, \dots, n-1$. Then, from the equality above we have

$$L^* \circ L = \frac{\varphi^2(|x|)}{|x|^2} I^{(n-1)} + \left(\varphi'^2(|x|) - \frac{\varphi^2(|x|)}{|x|^2}\right) \frac{x' \otimes x'}{|x|^2}\,,$$

where $I^{(n-1)}$ denotes the identity map on $\mathbb{R}^{n-1}$ and $x' = (x_1, \dots, x_{n-1})$. With a calculation similar to the one performed to obtain (10), from the equality above we easily get that

$$\det(L^* \circ L) = \left(\frac{\varphi^2(|x|)}{|x|^2}\right)^{n-1} \left[1 + \frac{|x|^2}{\varphi^2(|x|)} \left(\varphi'^2(|x|) - \frac{\varphi^2(|x|)}{|x|^2}\right) \frac{|x'|^2}{|x|^2}\right]$$

and thus, using (9) we can estimate for $x \in \partial^* E_h \cap (B_r \setminus B_{r/2})$

$$\begin{aligned} J_{n-1}T_{h,x} &= \sqrt{\det(L^* \circ L)} \\ &= \left(\frac{\varphi(|x|)}{|x|}\right)^{n-1} \sqrt{1 + \frac{|x|^2}{\varphi^2(|x|)} \left(\varphi'^2(|x|) - \frac{\varphi^2(|x|)}{|x|^2}\right) \frac{|x'|^2}{|x|^2}} \qquad (13) \\ &\le \left(\frac{\varphi(|x|)}{|x|}\right)^{n-2} \varphi'(|x|) \le \varphi'(|x|) \le 1 + \sigma + 2^n(n-1)\sigma\,. \end{aligned}$$

To estimate $I_{2,h}$, we use the area formula for maps between rectifiable sets ([2, Theorem 2.91]), thus getting

$$\begin{aligned} I_{2,h} = P(E_h, \overline{B}_r) - P(\widetilde{E}_h, \overline{B}_r) &= \int_{\partial^* E_h \cap \overline{B}_r} d\mathcal{H}^{n-1} - \int_{\partial^* E_h \cap \overline{B}_r} J_{n-1} T_{h,x} \, d\mathcal{H}^{n-1} \\ &= \int_{\partial^* E_h \cap \overline{B}_r \setminus B_{r/2}} (1 - J_{n-1} T_{h,x}) \; d\mathcal{H}^{n-1} + \int_{\partial^* E_h \cap B_{r/2}} (1 - J_{n-1} T_{h,x}) \; d\mathcal{H}^{n-1} . \end{aligned}$$

Notice that the last integral in the above formula is nonnegative since Φ is a contraction in $B_{r/2}$, hence $J_{n-1}T_{h,x} < 1$ in $B_{r/2}$, while from (13) and (4) we have

$$\int_{\partial^* E_h \cap \overline{B}_r \setminus B_{r/2}} (1 - J_{n-1} T_{h,x}) \; d\mathcal{H}^{n-1} \geq -2^n n P(E_h, \overline{B}_r) \sigma \geq -2^n n \Lambda \sigma ,$$

thus concluding that

$$I_{2,h} \geq -2^n n \Lambda \sigma . \tag{14}$$

Step 4. To estimate $I_{3,h}$ we recall (5), (6), (10), thus getting

$$\begin{aligned} I_{3,h} &= \lambda_h \int_{E_h \cap B_r \setminus B_{r/2}} (J\Phi(x) - 1) \; dx + \lambda_h \int_{E_h \cap B_{r/2}} (J\Phi(x) - 1) \; dx \\ &\geq \lambda_h C_1(n) \left(\frac{\omega_n}{2^{n+2}} - \varepsilon \right) \sigma r^n - \lambda_h \left[1 - (1 - (2^n - 1)\sigma)^n \right] \varepsilon r^n \\ &\geq \lambda_h \sigma r^n \left[C_1(n) \frac{\omega_n}{2^{n+2}} - C_1(n)\varepsilon - (2^n - 1) n \varepsilon \right] . \end{aligned}$$

Therefore, if we choose $0 < \varepsilon < \varepsilon(n)$, with $\varepsilon(n)$ depending only on the dimension, we have that

$$I_{3,h} \geq \lambda_h C_3(n) \sigma r^n ,$$

for some positive $C_3(n)$. From this inequality, recalling (7), (12) and (14) we obtain

$$\mathcal{F}_{\lambda_h}(u_h, E_h) - \mathcal{F}_{\lambda_h}(\tilde{u}_h, \widetilde{E}_h) \geq \sigma \left(\lambda_h C_3 r^n - \Lambda (C_2 + 2^n n) \right) > 0$$

if λ_h is sufficiently large. This contradicts the minimality of (u_h, E_h), thus concluding the proof. □

3. Proof of Theorem 1.2

In this section we are going to prove Theorem 1.2. To this aim, given a set $E \subset \Omega$ of finite perimeter in Ω for every ball $B_r(x) \subset\subset \Omega$ we measure how far E is from being an area minimizer in the ball by setting

$$\psi(E, B_r(x)) = P(E, B_r(x)) - \min \left\{ P(F, B_r(x)) : \; F \Delta E \subset\subset B_r(x) \right\} .$$

The following regularity result due to I. Tamanini ([8]), shows that if $\psi(E, B_r(x))$ decays fast enough when $r \to 0$, then E has essentially the same regularity properties of an area minimizing set.

Theorem 3.1. *Let Ω be an open subset of $\mathbb{R}^n$ and let E be a set of finite perimeter satisfying for $\delta \in (0,1/2)$*

$$\psi(E, B_r(x)) \le c r^{n-1+2\delta}$$

for every $x \in \Omega$ and every $r \in (0, r_0)$, with $c, r_0 > 0$. Then $\partial^* E$ is a $C^{1,\delta}$-hypersurface in Ω and $\mathcal{H}^s((\partial E \setminus \partial^* E) \cap \Omega)) = 0$ for all $s > n-8$.

Let us now prove Theorem 1.2.

Proof. *Step 1.* Let us first prove that $u \in C^{0,1/2+\varepsilon}$ for some ε if the the ratio $\alpha/\beta < \gamma_n$ for some $\gamma_n > 1$ depending only on the dimension. Let us fix a ball $B_r(x) \subset\subset \Omega$ and assume, with no loss of generality that $x = 0$. Let us denote by v the harmonic function in B_r coinciding with u on ∂B_r. We then have

$$\int_{B_r} \sigma_E(x) \nabla u \cdot \nabla \varphi \, dx = 0, \qquad \int_{B_r} \nabla v \cdot \nabla \varphi \, dx = 0\,, \tag{15}$$

for all $\varphi \in W_0^{1,2}(B_r)$. Multiplying the second equation in (15) by α and subtracting from the first equation, we obtain

$$\alpha \int_{B_r} (\nabla u - \nabla v) \cdot \nabla \varphi \, dx + (\beta - \alpha) \int_{B_r \setminus E} \nabla u \cdot \nabla \varphi \, dx = 0\,. \tag{16}$$

Similarly, if we multiply the second equation in (15) by β and subtract from the first equation, we obtain

$$\beta \int_{B_r} (\nabla u - \nabla v) \cdot \nabla \varphi \, dx + (\alpha - \beta) \int_{B_r \cap E} \nabla u \cdot \nabla \varphi \, dx = 0\,.$$

Choosing in this equation and in (16) $\varphi = u - v$ and adding the resulting equalities, we easily get

$$(\alpha + \beta) \int_{B_r} |\nabla u - \nabla v|^2 \, dx \le (\alpha - \beta) \left(\int_{B_r} |\nabla u|^2 \, dx \right)^{1/2} \left(\int_{B_r} |\nabla u - \nabla v|^2 \, dx \right)^{1/2},$$

hence

$$\int_{B_r} |\nabla u - \nabla v|^2 \, dx \le \frac{(\alpha - \beta)^2}{(\alpha + \beta)^2} \int_{B_r} |\nabla u|^2 \, dx\,. \tag{17}$$

Since v is harmonic, we have for $0 < \varrho < r$

$$\int_{B_\varrho} |\nabla v|^2 \, dx \le \left(\frac{\varrho}{r} \right)^n \int_{B_r} |\nabla v|^2 \, dx\,.$$

From this inequality and (17) we easily get

$$\begin{aligned} \left(\int_{B_\varrho} |\nabla u|^2 \, dx \right)^{1/2} &\le \left(\int_{B_\varrho} |\nabla u - \nabla v|^2 \, dx \right)^{1/2} + \left(\int_{B_\varrho} |\nabla v|^2 \, dx \right)^{1/2} \\ &\le \frac{\alpha - \beta}{\alpha + \beta} \left(\int_{B_r} |\nabla u|^2 \, dx \right)^{1/2} + \left(\frac{\varrho}{r} \right)^{n/2} \left(\int_{B_r} |\nabla v|^2 \, dx \right)^{1/2} \\ &\le \left[\frac{\alpha - \beta}{\alpha + \beta} + \left(\frac{\varrho}{r} \right)^{n/2} \right] \left(\int_{B_r} |\nabla u|^2 \, dx \right)^{1/2} \end{aligned}$$

and thus, for all $0 < \varrho < r$

$$\int_{B_\varrho} |\nabla u|^2 \, dx \le \left[\frac{\alpha - \beta}{\alpha + \beta} + \left(\frac{\varrho}{r}\right)^{n/2} \right]^2 \int_{B_r} |\nabla u|^2 \, dx \,. \tag{18}$$

Let us now find the largest $\chi_n < 1$ such that there exists $\vartheta_n < 1$ for which

$$\left(\chi_n + \vartheta_n^{n/2}\right)^2 = \vartheta_n^{n-1} \,. \tag{19}$$

This equality is equivalent to

$$\chi_n = \vartheta_n^{(n-1)/2} - \vartheta_n^{n/2} \,.$$

Since

$$\max_{\vartheta \in [0,1]} \left(\vartheta^{(n-1)/2} - \vartheta^{n/2} \right) = \left(\frac{n-1}{n} \right)^{n-1} - \left(\frac{n-1}{n} \right)^{n} \,,$$

we have that

$$\chi_n = \left(\frac{n-1}{n} \right)^{n-1} - \left(\frac{n-1}{n} \right)^{n} \,. \tag{20}$$

Assume now that

$$\frac{\alpha - \beta}{\alpha + \beta} < \chi_n \,. \tag{21}$$

From (18), (19) and (21) it follows that there exists $\varepsilon > 0$, depending on α, β, n such that

$$\int_{B_{\vartheta_n r}} |\nabla u|^2 \, dx \le \vartheta_n^{n-1+2\varepsilon} \int_{B_r} |\nabla u|^2 \, dx \,.$$

From this estimate, a standard iteration argument yields that for all $0 < \varrho < r$

$$\int_{B_\varrho} |\nabla u|^2 \, dx \le c(n) \left(\frac{\varrho}{r}\right)^{n-1+2\varepsilon} \int_{B_r} |\nabla u|^2 \, dx$$

and this inequality implies that $u \in C^{0,1/2+\varepsilon}(\Omega)$ whenever (see (21))

$$\frac{\alpha}{\beta} < \gamma_n := \frac{1 + \chi_n}{1 - \chi_n} \,.$$

Step 2. Let us now fix r_0 and a point $x \in \Omega$ such that $\operatorname{dist}(x, \partial\Omega) > 2r_0$. Let us take $0 < r < r_0$ and denote by F any set of finite perimeter such that $E \Delta F \subset\subset B_r(x)$. From Theorem 1.1 we have that

$$\mathcal{F}_{\lambda_0}(u, E) \le \mathcal{F}_{\lambda_0}(u, F) \,.$$

From this inequality and from what we have just proved in *Step 1*, we obtain that

$$P(E, B_r(x)) - P(F, B_r(x)) \le \int_{B_r} \sigma_E(x) |\nabla u|^2 \, dx + \lambda_0 \big| |F| - |E| \big| \le c r^{n-1+2\varepsilon} + c r^n \,.$$

The assertion then follows immediately from Theorem 3.1. □

Remark. Notice that (20) and (21) give an explicit value for γ_n. One gets, for instance, $\gamma_2 = 5/3$ and $\gamma_3 = 31/23$. Moreover,

$$n(\gamma_n - 1) \to \frac{2}{e}$$

as $n \to \infty$. We do not know if Theorem 1.2 still holds with a larger bound on the ratio α/β or even with no bound at all.

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References

[1] L. Ambrosio, G. Buttazzo: An optimal design problem with perimeter penalization, Calc. Var. Partial Differ. Equ. 1 (1993) 55–69.

[2] L. Ambrosio, N. Fusco, D. Pallara: Functions of Bounded Variation and Free Discontinuity Problems, Oxford University Press, Oxford (2000).

[3] E. Bombieri: Regularity theory for almost minimal currents, Arch. Ration. Mech. Anal. 78 (1982) 99–130.

[4] I. Fonseca, N. Fusco, G. Leoni, M. Morini: Equilibrium configurations of epitaxially strained crystalline films: existence and regularity results, Arch. Ratio. Mech. Anal. 186 (2007) 477–537.

[5] M. Gurtin: On phase transitions with bulk, interfacial, and boundary energy, Arch. Ration. Mech. Anal. 96 (1986) 243–264.

[6] C. J. Larsen: Regularity of components in optimal design problems with perimeter penalization, Calc. Var. Partial Differ. Equ. 16 (2003) 17–29.

[7] F. H. Lin: Variational problems with free interfaces, Calc. Var. Partial Differ. Equ. 1 (1993) 149–168.

[8] I. Tamanini: Boundaries of Caccioppoli sets with Hölder-continuous normal vector, J. Reine Angew. Math. 334 (1982) 27–39.