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Higher Order Strong Convexity and Global Strict Minimizers in Multiobjective Optimization*

C. Gutiérrez
Departamento de Matemática Aplicada, E.T.S.I. Informática, Universidad de Valladolid, Edificio de Tecnologías de la Información y las Telecomunicaciones, Campus Miguel Delibes, s/n, 47011 Valladolid, Spain
cesargv@mat.uva.es

B. Jiménez
Departamento de Matemática Aplicada, E.T.S.I. Industriales, Universidad Nacional de Educación a Distancia, Calle Juan del Rosal, 12, Ciudad Universitaria, 28040 Madrid, Spain
bjimenez@ind.uned.es

V. Novo
Departamento de Matemática Aplicada, E.T.S.I. Industriales, Universidad Nacional de Educación a Distancia, Calle Juan del Rosal, 12, Ciudad Universitaria, 28040 Madrid, Spain
vnovo@ind.uned.es

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We extend the scalar concept of strong convex function of order k due to Lin and Fukushima [11] to a vector-valued function, by considering a partial order given by a convex cone. We analyze some properties of higher order strong convex functions, and we give two characterizations of this kind of strong convexity for locally Lipschitz functions, one of them, through a new property of the Clarke generalized Jacobian, called strong monotonicity of order k. Similar results are obtained for Fréchet differentiable functions. In the second part, we study connections between strong convexity of order k and global strict minimizers of order k, and we establish sufficient optimality conditions for this class of minimizers in multiobjective optimization problems involving strong cone-convex functions.

Keywords: Vector optimization, higher order strong convexity, optimality conditions, strict minimizers, generalized Jacobian, higher order strong monotonicity, ∂-quasiconvexity

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1. Introduction and preliminaries

Lin and Fukushima [11] introduced the following concept of strong convexity of order k for a scalar function, where $k \geq 1$ is a real number. We suppose that C is a

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nonempty convex set of the n-dimensional Euclidean space $\mathbb{R}^n$ endowed with the Euclidean norm $\|\cdot\|$.

Definition 1.1. A function $f : C \to \mathbb{R}$ is said to be strongly convex of order k, if there exists a constant $c > 0$ such that for all $x, y \in C$ and $t \in [0,1]$,

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - ct(1-t)\|x-y\|^k. \tag{1}$$

Lin and Fukushima [11] applied this concept to obtain some exact penalty results for nonlinear programs and mathematical programs with equilibrium constraints. For $k = 2$, the function f is usually called strongly convex (see Rockafellar and Wets [14, Definition 12.58] or Vial [15, 16]). It is clear that strong convexity of any order implies convexity, but the reciprocal implication is not true in general (the function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x$ is a counterexample).

Given a pointed, solid and convex cone $D \subset \mathbb{R}^p$, we consider that $\mathbb{R}^p$ is partially ordered by the relation: $z \leq_D z' \Leftrightarrow z' - z \in D$.

Recall that a convex cone D is pointed if $D \cap -D = \{0\}$ and it is solid if its interior is nonempty. A function $f : C \to \mathbb{R}^p$ is called D-convex on C if for all $x, y \in C$ and $t \in [0,1]$,

$$f(tx + (1-t)y) \leq_D tf(x) + (1-t)f(y).$$

When $p = 1$, we will usually omit the cone $\mathbb{R}_+$ in the expression "$\mathbb{R}_+$-convex".

As usual we denote by $B(x_0, \delta)$ the open ball centered at x_0 and radius $\delta > 0$, by $\operatorname{int} M$ the interior of the set $M \subset \mathbb{R}^n$ and by $\operatorname{co} M$ the convex hull of M. We denote by D^+ the (positive) polar cone to D, that is, $D^+ = \{\lambda \in \mathbb{R}^p : \langle \lambda, v\rangle \geq 0,\ \forall v \in D\}$, where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbb{R}^p$. To shorten the notation, we will write λv instead of $\langle \lambda, v \rangle$ in some parts of this paper.

In this work, we introduce the concept of strong D-convexity of order k, that extends Definition 1.1 to vector-valued functions. With this new concept, if D is the nonnegative orthant $\mathbb{R}^p_+$ (Pareto case) and the components of f are $(f_1, \ldots, f_p)$, then f is strongly $\mathbb{R}^p_+$-convex of order k if and only if for each $i = 1, \ldots, p$, f_i is strongly convex of order k. Although this case has been studied by Gupta et al. [6] and Bhatia [1], these authors only use the scalar version of strong convexity, since they consider the notion by components, not globally.

The paper is organized as follows. In Section 2, we introduce the concept of strongly D-convex vector-valued function of order k, we prove several properties of these functions, and we provide two characterizations of strong D-convexity for Lipschitz functions. One of them is stated through a new higher order strong monotonicity notion for the Clarke generalized Jacobian, introduced in this paper. Similar characterizations are also given for a Fréchet differentiable function. Moreover, an additional sufficient condition is stated for twice Fréchet differentiable functions, which is a characterization for the case $k = 2$. In Sections 3 and 4, we study links between strong convexity of order k and global strict minimizers of order k. To be precise, in Section 3, we establish a characterization of strict minimizer of order k, and we prove that a local strict minimizer of order k for a strongly D-convex function is also a global strict minimizer of order k. Then, in Section 4, we establish sufficient optimality conditions for this class of minimizers in a multiobjective optimization problem involving locally Lipschitz or differentiable strong cone-convex functions.

2. Higher order strong cone-convexity

We extend the definition of strong convexity of order k due to Lin and Fukushima [11], to a vector-valued map as follows.

Definition 2.1. Let C be a convex subset of $\mathbb{R}^n$. It is said that $f : \mathbb{R}^n \to \mathbb{R}^p$ is strongly D-convex of order k on C, denoted $f \in \mathrm{SCo}(k, C)$, if there exists $e \in \operatorname{int} D$ such that for all $x, y \in C$ and $t \in [0, 1]$,

$$f(tx + (1-t)y) \leq_D tf(x) + (1-t)f(y) - t(1-t)\|x-y\|^k e. \tag{2}$$

Sometimes, when (2) is satisfied we will say that f is strongly D-convex of order k on C with constant e.

This definition does not depend on the norm and it is a natural extension of the scalar strong convexity of order k to a vector-valued map, because for a scalar function, taking $D = \mathbb{R}_+$ and $e = c$ we obtain Definition 1.1.

When $\lambda \in D^+$ we will write λf instead of $\lambda \circ f$. Some immediate properties are the following. Let $f \in \mathrm{SCo}(k, C)$, then:

(i) f is D-convex on C,
(ii) $\alpha f, \lambda f \in \mathrm{SCo}(k, C)$ for all $\alpha > 0$, $\lambda \in D^+ \setminus \{0\}$,
(iii) $f + g \in \mathrm{SCo}(k, C)$ for all D-convex function g.

Other property about the composition is the following.

Proposition 2.2. *Let $K \subset \mathbb{R}^q$ be a solid convex cone, C be a convex set of $\mathbb{R}^n$ and let $e \in \operatorname{int} D$. Assume that the function $\psi : \mathbb{R}^p \to \mathbb{R}^q$ satisfies the following conditions* (3)*:*

$$\psi \textit{ is increasing, i.e., } y \leq_D z \ \Rightarrow\ \psi(y) \leq_K \psi(z), \tag{3a}$$

$$\psi \textit{ is } K\textit{-convex on } \mathbb{R}^p, \tag{3b}$$

$$\psi(y + \alpha e) \leq_K \psi(y) + \alpha\psi(e) \quad \forall y \in \mathbb{R}^p, \forall \alpha \in \mathbb{R}, \tag{3c}$$

$$\psi(e) \in \operatorname{int} K. \tag{3d}$$

If f is strongly D-convex of order k on C with constant e, then $\psi \circ f$ is strongly K-convex of order k on C with constant $\psi(e)$.

Proof. By assumption, for all $x, y \in C$ and $t \in [0, 1]$, inequality (2) holds. Applying ψ to this inequality and using properties (3a) and (3c), we obtain

$$\psi(f(tx + (1-t)y)) \leq_K \psi(tf(x) + (1-t)f(y)) - t(1-t)\|x-y\|^k \psi(e).$$

As ψ is K-convex, it follows that

$$\psi(f(tx + (1-t)y)) \leq_K t\psi(f(x)) + (1-t)\psi(f(y)) - t(1-t)\|x-y\|^k \psi(e),$$

and the proof is finished. □

Some examples of functions ψ satisfying conditions (3), which are very used in optimization, are the following:

(a) Case $q = 1$, $K = \mathbb{R}_+$.

1. The functionals $\psi = \lambda \in D^+ \setminus \{0\}$.
2. The Gerstewitz function [5] $\varphi_e : \mathbb{R}^p \to \mathbb{R}$ is given by

$$\varphi_e(y) = \inf\{t \in \mathbb{R} : \ y \in te - D\}.$$

   Property (3c) is proved, for example, in [7, Lemma 4.4].
3. If $\psi_1, \ldots, \psi_r$ are functionals from $\mathbb{R}^p$ to $\mathbb{R}$ satisfying properties (3), then $\sum_{i=1}^r \alpha_i \psi_i$ with $\alpha_i > 0$ also satisfies (3). If, in addition, for some constant $c > 0$, $\psi_i(e) = c$ for all i, then $\max_{1 \le i \le r} \psi_i$ satisfies properties (3) too.

(b) Case $q > 1$.

1. A linear function $\psi : \mathbb{R}^p \to \mathbb{R}^q$ that is positive, i.e., $\psi(D) \subset K$, and $\psi(e) \in \operatorname{int} K$ satisfies (3). In particular, the function $\psi(y) = \lambda(y)e'$, where $\lambda \in D^+ \setminus \{0\}$ and $e' \in \operatorname{int} K$.
2. If $\psi_1, \ldots, \psi_q$ are functionals from $\mathbb{R}^p$ to $\mathbb{R}$ satisfying properties (3) for $K = \mathbb{R}_+$, then $\psi = (\psi_1, \ldots, \psi_q) : \mathbb{R}^p \to \mathbb{R}^q$ satisfies (3) with $K = \mathbb{R}^q_+$.

Recall that given a subset C of $\mathbb{R}^n$, a function $f : \mathbb{R}^n \to \mathbb{R}^p$ is said to be Lipschitz on C if, for some $L > 0$, one has $\|f(x) - f(x')\| \le L\|x - x'\|$ $\forall x, x' \in C$. The function is said to be Lipschitz near x if f satisfies a Lipschitz condition on a neighborhood of x, and f is locally Lipschitz on C if f is Lipschitz near x, for each $x \in C$.

Cone-convex functions have interesting Lipschitz properties. In particular, if f is D-convex on C and the closure of D is pointed, then f is locally Lipschitz on the relative interior of C (see Luc [12, Corollary 5.1]). In the sequel, *we assume that the cone D is closed*.

Rademacher's theorem (see [4]) asserts that f is Fréchet differentiable almost everywhere on a neighborhood of a point x when f is Lipschitz near x. Taking into account this fact, Clarke introduced the concept of generalized Jacobian as follows (see [3]). We denote by $Jf(x)$ the usual Jacobian matrix of partial derivatives when f is Fréchet differentiable at x.

Definition 2.3. Let $f : \mathbb{R}^n \to \mathbb{R}^p$ be locally Lipschitz, the generalized Jacobian of f at x_0 is

$$\partial f(x_0) = \operatorname{co}\left\{\lim_{n\to\infty} Jf(x_n) : \ x_n \to x_0, \ Jf(x_n) \text{ exists}\right\}.$$

Below we will use the following lemma whose proof is easy and left to the reader.

Lemma 2.4. *Let $f : \mathbb{R}^n \to \mathbb{R}^p$ be a strongly D-convex function of order k on C with constant $e \in \operatorname{int} D$. If f is Fréchet differentiable at $x_0 \in C$, then*

$$f(x) \ge_D f(x_0) + Jf(x_0)(x - x_0) + \|x - x_0\|^k e \quad \forall x \in C.$$

The following result is a characterization of strong D-convexity of order k for Lipschitz functions.

Theorem 2.5. *Let $C \subset \mathbb{R}^n$ be an open convex set and let $f : C \to \mathbb{R}^p$ be locally Lipschitz. Then $f \in \mathrm{SCo}(k, C)$ if and only if there exists $e \in \operatorname{int} D$ such that*

$$f(x) \geq_D f(y) + A(x-y) + \|x-y\|^k e \quad \forall x, y \in C, \forall A \in \partial f(y). \tag{4}$$

Proof. Suppose that $f \in \mathrm{SCo}(k, C)$ with constant $e \in \operatorname{int} D$, let Ω be the set of points $x \in C$ such that $Jf(x)$ exists, and let $A \in \partial f(y)$, with $y \in C$. Then, there are two possible situations:

(i) $A = \lim_{n\to\infty} Jf(y_n)$, with $y_n \to y$ and $y_n \in \Omega$, and
(ii) $A = \sum_{i=1}^r \alpha_i A_i$, with $\alpha_i > 0$, $\sum_{i=1}^r \alpha_i = 1$ and for each $i = 1, \dots, r$, $A_i = \lim_{n\to\infty} Jf(y_{i,n})$, with $y_{i,n} \to y$ as $n \to \infty$ and $y_{i,n} \in \Omega$.

Case (i). Since $y_n \in \Omega \subset C$, by Lemma 2.4 we have that

$$Jf(y_n)(x-y_n) \leq_D f(x) - f(y_n) - \|x-y_n\|^k e \quad \forall x \in C,\ \forall n \in \mathbb{N},$$

and taking the limit when $n \to \infty$

$$A(x-y) \leq_D f(x) - f(y) - \|x-y\|^k e \quad \forall x \in C.$$

Case (ii). As a consequence of (i), for each $i = 1, \dots, r$, we have that

$$A_i(x-y) \leq_D f(x) - f(y) - \|x-y\|^k e \quad \forall x \in C.$$

Multiplying by α_i and adding up, one has

$$\big(\textstyle\sum_{i=1}^r \alpha_i A_i)(x-y) \leq_D (\sum_{i=1}^r \alpha_i)(f(x) - f(y) - \|x-y\|^k e),$$

and the conclusion follows because $A = \sum_{i=1}^r \alpha_i A_i$ and $\sum_{i=1}^r \alpha_i = 1$.

For the reciprocal implication, let $x, y \in C$, $t \in (0,1)$ and $z = tx + (1-t)y$. Then by assumption

$$\begin{aligned} f(x) - f(z) &\geq_D A(x-z) + \|x-z\|^k e \quad \forall A \in \partial f(z), \\ f(y) - f(z) &\geq_D A(y-z) + \|y-z\|^k e \quad \forall A \in \partial f(z). \end{aligned}$$

Multiplying the first inequality by t, the second one by $1-t$ and adding up, one has

$$tf(x) + (1-t)f(y) - f(z) \geq_D (t\|x-z\|^k + (1-t)\|y-z\|^k)e.$$

Since $x - z = (1-t)(x-y)$ and $y - z = t(y-x)$ it follows that

$$\begin{aligned} tf(x) + (1-t)f(y) - f(z) &\geq_D (t(1-t)^k + (1-t)t^k)\|x-y\|^k e \\ &= \ t(1-t)((1-t)^{k-1} + t^{k-1})\|x-y\|^k e. \end{aligned}$$

Using that for all $t \in (0,1)$,

$$t^{k-1} + (1-t)^{k-1} \geq \varphi(k) := \begin{cases} 1 & \text{if } 0 < k \leq 2 \\ (1/2)^{k-2} & \text{if } k > 2 \end{cases}$$

(see, for example, the proof of Theorem 4.2 in [11] or Theorem 3.1 in [1]), we obtain

$$tf(x) + (1-t)f(y) - f(z) \geq_D t(1-t)\|x-y\|^k \varphi(k) e,$$

and so

$$f(tx + (1-t)y) \leq_D tf(x) + (1-t)f(y) - t(1-t)\|x-y\|^k ce,$$

where $c = \varphi(k) > 0$ is independent of x, y and t. As $ce \in \operatorname{int} D$, f is strongly D-convex of order k on C. □

Theorem 2.5 reduces to [1, Theorem 3.1] when $p = 1$.

The following result is a characterization of strong D-convexity of order k for differentiable functions.

Theorem 2.6. *Let $f : \mathbb{R}^n \to \mathbb{R}^p$ be Fréchet differentiable at each point of the convex set $C \subset \mathbb{R}^n$. Then $f \in \mathrm{SCo}(k, C)$ if and only if there exists $e \in \operatorname{int} D$ such that*

$$f(y) \geq_D f(x) + Jf(x)(y-x) + \|y-x\|^k e \quad \forall x, y \in C.$$

Proof. ($\Rightarrow$) It follows from Lemma 2.4.

($\Leftarrow$) The proof of this part is similar to that of Theorem 2.5 with $A = Jf(x)$, and so we omit it. □

For $p = 1$, Theorem 2.6 improves Theorem 4.2 in [11], since we do not require f to be a C^1 function.

In the next example, an application of the previous theorem is given.

Example 2.7. Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined by $f(x) = \|x\|^k$, with $k \geq 2$. We are going to prove that this function is strongly convex of order k on $\mathbb{R}^n$. For this aim, let us apply Theorem 2.6. We have to find a constant $c > 0$ such that $f(a) \geq f(b) + Jf(b)(a-b) + c\|a-b\|^k \ \forall a, b \in \mathbb{R}^n$, i.e.,

$$\|a\|^k \geq \|b\|^k + k\|b\|^{k-2}\langle b, a-b\rangle + c\|a-b\|^k. \tag{5}$$

If $b = 0$, this inequality is satisfied for all $c \in (0, 1]$. If $a = b$, (5) is satisfied for all $c > 0$. So we can suppose $a \neq b$ and $b \neq 0$. Now, dividing by $\|b\|^k$ and setting $t = \frac{\|a\|}{\|b\|}$, it results

$$t^k \geq 1 + k\left(\frac{1}{\|b\|^2}\langle a, b\rangle - \frac{1}{\|b\|^2}\langle b, b\rangle\right) + c\left(\frac{\|a-b\|^2}{\|b\|^2}\right)^{k/2}. \tag{6}$$

As $\langle a, b\rangle = \|a\|\|b\|\cos(\widehat{a, b})$, we may write $\langle a, b\rangle / \|b\|^2 = wt$, where $w = \cos(\widehat{a, b}) \in [-1, 1]$, and as $\|a-b\|^2 = \|a\|^2 + \|b\|^2 - 2\langle a, b\rangle$, we have that $\|a-b\|^2/\|b\|^2 = t^2 + 1 - 2wt$. Therefore (6) becomes

$$t^k \geq 1 + k(wt-1) + c(t^2 + 1 - 2wt)^{k/2}.$$

We have to prove that the function

$$g(w, t) = \frac{t^k - kwt + k - 1}{(t^2 + 1 - 2wt)^{k/2}}$$

has a lower bound $c > 0$ for $t \geq 0$ and $w \in [-1, 1]$. If $k = 2$ the function $g(w, t)$ is constant, i.e., $g(w, t) = 1$ for all $(w, t) \in [-1, 1] \times [0, +\infty)$. This means that, for

$k = 2$, (5) is satisfied with equality for all $a, b \in \mathbb{R}^n$ choosing $c = 1$. In consequence, we can suppose that $k > 2$.

If $t = 0$, $g(w, 0) = k - 1 > 0$.

For each fixed $t > 0$, the function $w \to g(w, t)$ is increasing for $w \in [-1, 1]$, because the derivative

$$\frac{dg(w,t)}{dw} = \frac{-k(k-2)t^2}{(t^2+1-2wt)^{\frac{k}{2}+1}} \left(w - \frac{t^k - t^2 + k - 2}{(k-2)t} \right)$$

is positive since its root is $w_0(t) = \frac{t^k - t^2 + k - 2}{(k-2)t} > 1$ for all $t \neq 1$. If $t = 1$,

$$g(w,1) = \frac{-kw + k}{(2-2w)^{k/2}} = \frac{k(1-w)}{2^{k/2}(1-w)^{k/2}} = \frac{k}{2^{k/2}} \frac{1}{(1-w)^{k/2-1}}$$

is increasing for $w < 1$ ($(w, t) = (1, 1)$ corresponds to $a = b$ and we know that the inequality (5) is true for these values).

To prove $w_0(t) > 1$ we write $w_0(t)$ as $w_0(t) = \frac{1}{k-2} \left(t^{k-1} - t + \frac{k-2}{t} \right)$. Its derivative is given by

$$w_0'(t) = \frac{d}{dt} w_0(t) = \frac{1}{k-2} \left[(k-1)t^{k-2} - \left(1 + \frac{k-2}{t^2} \right) \right].$$

This function satisfies

$$w_0'(t) < 0 \quad \text{for } t \in (0,1), \quad w_0'(1) = 0 \quad \text{and} \quad w_0'(t) > 0 \quad \text{for } t \in (1, +\infty).$$

Therefore $w_0(t) > w_0(1) = 1$ for all $t \neq 1$ (let us observe that the equation $(k - 1)t^{k-2} = 1 + \frac{k-2}{t^2}$ has a single root $t = 1$ for $t > 0$).

Now, using that $w \to g(w, t)$ is increasing on $[-1, 1]$ we have

$$g(w,t) \geq g(-1,t) = \frac{t^k + kt + k - 1}{(t^2+1+2t)^{k/2}} = \frac{t^k + kt + k - 1}{(t+1)^k} \quad \forall w \in [-1, 1].$$

The real function $\varphi(t) = \frac{t^k + kt + k - 1}{(t+1)^k}$, $t \geq 0$, is lower bounded by a positive constant c because:

(i) $\lim_{t \to +\infty} \varphi(t) = 1$, and so there exists $R > 0$ such that $\varphi(t) > 1/2$ for all $t > R$.

(ii) $\varphi(t)$ is continuous and positive on $[0, R]$, and by the Weierstrass Theorem there exists $m > 0$ such that $\varphi(t) \geq m \ \forall t \in [0, R]$.

Therefore $\varphi(t) \geq c \ \forall t \geq 0$ where $c = \min\{m, 1/2\} > 0$.

It is well-known that there is a strong relation between convexity and monotonicity (see [14]). Now we extend the notion of monotonicity and give a characterization of strong convexity of order k via this notion.

We denote the set of all $p \times n$ real matrices by $L_{p \times n}$.

Definition 2.8. A set-valued map $T : \mathbb{R}^n \rightrightarrows L_{p\times n}$ is said to be strongly monotone of order k on $C \subset \mathbb{R}^n$ if there exists $e \in \operatorname{int} D$ such that

$$(B - A)(y - x) \geq_D \|y - x\|^k e \quad \forall x, y \in C, A \in T(x), B \in T(y). \tag{7}$$

This definition extends the notion of strong monotonicity for a set-valued map T from $\mathbb{R}^n$ to $\mathbb{R}^n$ (see, for example, [14, Definition 12.53], where only the case $k = 2$ is considered) and it reduces to [11, Definition 4.1] when T is single-valued and $p = 1$.

Theorem 2.9. *Let C be an open convex set of $\mathbb{R}^n$ and $f : C \to \mathbb{R}^p$ be a locally Lipschitz function. Then $f \in \mathrm{SCo}(k, C)$ if and only if the set-valued map $\partial f : \mathbb{R}^n \rightrightarrows L_{p\times n}$ is strongly monotone of order k on C.*

Proof. Suppose that $f \in \mathrm{SCo}(k, C)$. By Theorem 2.5 there exists $e \in \operatorname{int} D$ such that (4) holds. In consequence, for all $x, y \in C$, $A \in \partial f(x)$ and $B \in \partial f(y)$ we have

$$\begin{aligned} f(x) - f(y) &\geq_D B(x - y) + \|y - x\|^k e, \\ f(y) - f(x) &\geq_D A(y - x) + \|y - x\|^k e. \end{aligned}$$

Adding up these inequalities, it results

$$0 \geq_D B(x - y) + A(y - x) + 2\|y - x\|^k e,$$

which proves that ∂f is strongly monotone of order k on C with constant $2e \in \operatorname{int} D$.

Reciprocally, assume that ∂f is strongly monotone of order k on C, i.e., (7) holds with $T = \partial f$. We are going to apply Proposition 2.6.5 in Clarke [3] which establishes that for $a, b \in C$, $f(b) - f(a) \in \operatorname{co} \partial f([a, b])(b - a)$, where $[a, b] = \operatorname{co}\{a, b\}$, i.e., there exist $q \geq 0$, $z_j \in [a, b]$, $\alpha_j > 0$, $A_j \in \partial f(z_j)$, $j = 0, 1, \ldots, q$ such that $\sum_{j=0}^{q} \alpha_j = 1$ and $f(b) - f(a) = \sum_{j=0}^{q} \alpha_j A_j (b - a)$.

Given $x, y \in C$, consider $m \in \mathbb{N}$, $t_i = \frac{i}{m+1}$, $x_i = x + t_i(y - x)$, $i = 0, 1, \ldots, m + 1$. By Proposition 2.6.5 in [3] there exist $n_i \geq 0$, $z_{ij} \in [x_i, x_{i+1}]$, $\alpha_{ij} > 0$, $A_{ij} \in \partial f(z_{ij})$, $j = 0, 1, \ldots, n_i$ such that $\sum_{j=0}^{n_i} \alpha_{ij} = 1$ and

$$f(x_{i+1}) - f(x_i) = A_i(x_{i+1} - x_i) = (t_{i+1} - t_i)A_i(y - x),$$

where $A_i = \sum_{j=0}^{n_i} \alpha_{ij} A_{ij}$, $z_{ij} = x + r_{ij}(y - x)$ with $t_i \leq r_{ij} \leq t_{i+1}$ and $i = 0, 1, \ldots, m$. It follows for all $A \in \partial f(x)$,

$$\begin{aligned} f(y) - f(x) = \sum_{i=0}^{m} (f(x_{i+1}) - f(x_i)) &= \sum_{i=0}^{m} (t_{i+1} - t_i) A_i (y - x) \\ &= A(y - x) + \sum_{i=0}^{m} (t_{i+1} - t_i)(A_i - A)(y - x). \end{aligned} \tag{8}$$

By hypothesis, $(A_{ij} - A)(z_{ij} - x) \geq_D \|z_{ij} - x\|^k e = r_{ij}^k \|y - x\|^k e$. Therefore, for each

$i = 1, 2, \ldots, m$

$$\begin{aligned}(A_i - A)(y-x) = &\sum_{j=0}^{n_i} \alpha_{ij}(A_{ij} - A)(y-x) = \sum_{j=0}^{n_i} \alpha_{ij}(A_{ij}-A) r_{ij}^{-1}(z_{ij} - x) \\ &\geq_D \sum_{j=0}^{n_i} \alpha_{ij} r_{ij}^{-1} r_{ij}^{k} \|y-x\|^k e = \left(\sum_{j=0}^{n_i} \alpha_{ij} r_{ij}^{k-1}\right) \|y-x\|^k e \\ &\geq_D \left(\sum_{j=0}^{n_i} \alpha_{ij} t_i^{k-1}\right) \|y-x\|^k e = t_i^{k-1} \|y-x\|^k e.\end{aligned}$$

We have excluded $i = 0$ because some r_{0j} could be 0.

In view of (8), since $t_0 = 0$,

$$\begin{aligned}&f(y) - f(x) \\ \geq_D\ & A(y-x) + (t_1 - t_0)(A_0 - A)(y-x) + \left(\sum_{i=1}^{m} (t_{i+1} - t_i) t_i^{k-1}\right) \|y-x\|^k e \\ =\ & A(y-x) + \frac{1}{m+1}(A_0 - A)(y-x) + \left(\sum_{i=0}^{m} (t_{i+1} - t_i) t_i^{k-1}\right) \|y-x\|^k e.\end{aligned}$$

Let us observe that A_0 belongs to the set $\operatorname{co} \partial f([x, x_1])$, which is bounded because the set-valued map $z \mapsto \partial f(z)$ is upper semicontinuous at x by [3, Proposition 2.6.2] and $\partial f(x)$ is a nonempty convex compact set. Therefore, taking the limit as $m \to +\infty$, we obtain

$$f(y) - f(x) \geq_D A(y-x) + \left(\int_0^1 t^{k-1} dt\right) \|y-x\|^k e = A(y-x) + \frac{1}{k}\|y-x\|^k e.$$

By Theorem 2.5, taking into account that $(1/k)e \in \operatorname{int} D$, we see that $f \in \mathrm{SCo}(k, C)$, and the proof is finished. □

Theorem 2.9 is also new, to our knowledge, for scalar functions (i.e., $p = 1$), and in this case, it generalizes Theorem 4.3 in Lin and Fukushima [11], where it is assumed that $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable.

Next we state a similar result for differentiable functions.

Theorem 2.10. *Let $f : \mathbb{R}^n \to \mathbb{R}^p$ be continuously differentiable at each point of the convex set $C \subset \mathbb{R}^n$. Then $f \in \mathrm{SCo}(k, C)$ if and only if the map $Jf : \mathbb{R}^n \to L_{p\times n}$ is strongly monotone of order k on C.*

Proof. The proof of "only if" part is similar to that of Theorem 2.9 using Theorem 2.6 instead of Theorem 2.5.

Thus, suppose the map $Jf : \mathbb{R}^n \to L_{p\times n}$ is strongly monotone of order k on C, i.e., there exists $e \in \operatorname{int} D$ such that $(Jf(y) - Jf(x))(y-x) \geq_D \|y-x\|^k e\ \forall x, y \in C$.

Consider $x, y \in C$ and $t \in [0,1]$. By assumption

$$(Jf(x+t(y-x)) - Jf(x))(t(y-x)) - t^k\|y-x\|^k e \in D \quad \forall t \in [0,1].$$

Hence

$$\psi(t) := (Jf(x+t(y-x)) - Jf(x))(y-x) - t^{k-1}\|y-x\|^k e \in D \quad \forall t \in [0,1].$$

Moreover, if $\varphi(t) := f(x+t(y-x)) - Jf(x)(t(y-x)) - \frac{1}{k}t^k\|y-x\|^k e$ one has that $\varphi'(t) = \psi(t)$. As D is a closed convex cone, $\int_0^1 \psi(t)dt = \lim_{m\to+\infty}\sum_{i=0}^m (t_{i+1}-t_i)\psi(t_i) \in D$, with $t_i = i/(m+1)$, $i = 0, 1, \ldots, m+1$. Therefore

$$\int_0^1 \psi(t)dt = \varphi(1) - \varphi(0) = f(y) - Jf(x)(y-x) - \frac{1}{k}\|y-x\|^k e - f(x) \in D,$$

and by Theorem 2.6 the conclusion is obtained. □

This theorem generalizes Theorem 4.3 in [11], where the scalar case is considered.

We finish this section with a sufficient condition for a twice differentiable function to be strongly cone-convex of order k. Such a condition is a characterization for the case $k = 2$ and it is based on the notion of strongly positive definite map of order k introduced in Definition 2.11. We denote by $J^2f(x)$ the second order Fréchet derivative of f at x, which is considered as a bilinear function from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}^p$. The set of all bilinear functions from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}^p$ is denoted by $B(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^p)$.

Definition 2.11. A map $H : \mathbb{R}^n \to B(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^p)$ is strongly positive definite of order k on C if there exists $e \in \operatorname{int} D$ such that

$$H(z)(y-x, y-x) \geq_D \|y-x\|^k e \quad \forall x, y \in C, \ \forall z \in [x, y].$$

Theorem 2.12. *Let $f : \mathbb{R}^n \to \mathbb{R}^p$ be twice continuously Fréchet differentiable at each point of the convex set $C \subset \mathbb{R}^n$. Consider the following statements:*

(a) $f \in \mathrm{SCo}(k, C)$.

(b) *The map $J^2f : \mathbb{R}^n \to B(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^p)$ is strongly positive definite of order k on C.*

Then

(i) (b) ⇒ (a).

(ii) *If $k = 2$, then* (a) ⇔ (b).

Proof. (i) Suppose that (b) holds. Given $x, y \in C$, define the function $\psi : [0,1] \to \mathbb{R}^p$ by

$$\psi(t) = Jf(x+t(y-x))(y-x) - Jf(x)(y-x) - t\|y-x\|^k e.$$

Its derivative is

$$\psi'(t) = J^2f(x+t(y-x))(y-x, y-x) - \|y-x\|^k e.$$

By hypothesis, $\psi'(t) \geq_D 0 \ \forall t \in [0,1]$, and therefore $\psi(1) - \psi(0) = \int_0^1 \psi'(t)dt \geq_D 0$, i.e., $Jf(y)(y-x) - Jf(x)(y-x) - \|y-x\|^k e \geq_D 0$. By Theorem 2.10 the conclusion follows.

(ii) We only have to prove $(a) \Rightarrow (b)$. Suppose that (a) holds with $k = 2$. By Theorem 2.6 there exists $e \in \operatorname{int} D$ such that

$$f(y) \geq_D f(x) + Jf(x)(y-x) + \|y-x\|^2 e \quad \forall x, y \in C.$$

Given $x, y \in C$ and $z \in [x, y)$, for all $t > 0$ small enough we have $z + tv \in C$ where $v = y - x$. By applying the above inequality, one has $f(z+tv) - f(z) - tJf(z)v \geq_D t^2\|v\|^2 e$. Dividing by $\frac{1}{2}t^2$ and taking the limit as $t \to 0^+$ we obtain $J^2f(z)(v,v) \geq_D 2\|v\|^2 e$, which is the desired conclusion. If $z = y$, we choose $v = x - y$ and the conclusion is the same $J^2f(z)(v,v) \geq_D 2\|v\|^2 e$, i.e., $J^2f(z)(y-x, y-x) \geq_D 2\|y-x\|^2 e$ because $J^2f(z)(-v,-v) = J^2f(z)(v,v)$. □

Example 2.13. As an application of this theorem, if $f(x) = \langle x, Ax\rangle + \langle b, x\rangle + c$ is a quadratic function, where A is a symmetric $n \times n$ matrix, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$, then $f \in \mathrm{SCo}(2, \mathbb{R}^n)$ if and only if A is positive definite, because $J^2f(z) = A$ for all $z \in \mathbb{R}^n$, and the function $v \to \langle v, Av\rangle$ is continuous and positive on the compact set $S^1 := \{v \in \mathbb{R}^n : \|v\| = 1\}$, so there exists $\alpha > 0$ such that $\langle v, Av\rangle \geq \alpha$ for all $v \in S^1$, and therefore $\langle v, Av\rangle \geq \alpha\|v\|^2 \ \forall v \in \mathbb{R}^n$.

Example 2.14. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^4$ is strongly convex of order 4 on $\mathbb{R}$ (according to Example 2.7), but it is not strongly convex of order 2 on $\mathbb{R}$ by Theorem 2.12(ii) since $f''(0) = 0$. This simple example shows the necessity of studying strong convexity of order k, with $k \neq 2$.

3. Strong convexity and strict minimizers

Given a function $f : \mathbb{R}^n \to \mathbb{R}^p$ and a nonempty set $C \subset \mathbb{R}^n$ we are interested in strict minimizers to the following multiobjective optimization problem:

$$D - \operatorname{Min}\{f(x) : \ x \in C\}. \tag{9}$$

We consider the notion of strict minimizer of order k according to Definition 3.1 due to Jiménez [8] (see, also, [9] and [10]).

Definition 3.1. (a) We say that a point $x_0 \in C$ is a local strict minimizer of order k for f on C, denoted $x_0 \in \mathrm{LStr}(k, f, C)$, if there exist a constant $\alpha > 0$ and a neighborhood U of x_0 such that

$$(f(x) + D) \cap B(f(x_0), \alpha\|x - x_0\|^k) = \emptyset \quad \forall x \in C \cap U \setminus \{x_0\}. \tag{10}$$

(b) We say that a point $x_0 \in C$ is a local strict (resp. weak) minimizer for f on C, denoted $x_0 \in \mathrm{LStr}(f, C)$ (resp. $x_0 \in \mathrm{LWMin}(f, C)$), if there exists a neighborhood U of x_0 such that

$$f(x) - f(x_0) \notin -D \ (\text{resp. } f(x) - f(x_0) \notin -\operatorname{int} D) \quad \forall x \in C \cap U \setminus \{x_0\}. \tag{11}$$

It is clear that we have global notions of strict minimizer of order k, strict minimizer and weak minimizer on C, if U is replaced by the whole space $\mathbb{R}^n$. In these cases we write $x_0 \in \mathrm{GStr}(k, f, C)$, $x_0 \in \mathrm{GStr}(f, C)$ and $x_0 \in \mathrm{GWMin}(f, C)$, respectively. Also it is clear that any local (resp. global) strict minimizer of order k is a local (resp. global) strict minimizer.

Next, we state a characterization of strict minimizer of order k.

Proposition 3.2. *(i) A point $x_0 \in C$ is a local strict minimizer of order k for f on C if and only if there exist a neighborhood U of x_0 and $e \in \operatorname{int} D$, such that*

$$f(x) - f(x_0) - \|x - x_0\|^k e \notin -\operatorname{int} D \quad \forall x \in C \cap U \setminus \{x_0\}, \tag{12}$$

which is equivalent to

$$f(x) \not< f(x_0) + \|x - x_0\|^k e \quad \forall x \in C \cap U \setminus \{x_0\},$$

where the relation $a \not< b$ means that $a - b \notin -\operatorname{int} D$.

(ii) A point $x_0 \in C$ is a global strict minimizer of order k for f on C if and only if there exists $e \in \operatorname{int} D$ such that

$$f(x) - f(x_0) - \|x - x_0\|^k e \notin -\operatorname{int} D \quad \forall x \in C \setminus \{x_0\}.$$

Proof. We only prove part (i) because the proof of part (ii) is similar. Suppose that $x_0 \in \mathrm{LStr}(k, f, C)$, then condition (10) is true, and this is equivalent to

$$\frac{f(x) - f(x_0)}{\|x - x_0\|^k} \in (B(0, \alpha) - D)^c \quad \forall x \in C \cap U \setminus \{x_0\}, \tag{13}$$

where M^c denotes the complement of the set M. It is clear that, given $\alpha > 0$ and $e \in \operatorname{int} D$, there exists $\beta > 0$ such that $\beta e \in B(0, \alpha)$, and, consequently $\beta e - \operatorname{int} D \subset B(0, \alpha) - D$. So, from (13) it follows that

$$\frac{f(x) - f(x_0)}{\|x - x_0\|^k} \in (\beta e - \operatorname{int} D)^c \quad \forall x \in C \cap U \setminus \{x_0\},$$

that is

$$f(x) - f(x_0) - \|x - x_0\|^k \beta e \notin -\operatorname{int} D \quad \forall x \in C \cap U \setminus \{x_0\},$$

and we have the conclusion, because $\beta e \in \operatorname{int} D$.

Now, for the reciprocal implication, suppose that condition (12) holds with $e \in \operatorname{int} D$. Then there exists $\alpha > 0$ such that $B(0, \alpha) \subset e - \operatorname{int} D$. From (12) it follows that

$$\frac{f(x) - f(x_0)}{\|x - x_0\|^k} - e \notin -\operatorname{int} D \quad \forall x \in C \cap U \setminus \{x_0\}. \tag{14}$$

Suppose that (10) is not satisfied, then for some $x \in C \cap U \setminus \{x_0\}$,

$$\frac{f(x) - f(x_0)}{\|x - x_0\|^k} \in B(0, \alpha) - D \subset e - \operatorname{int} D - D \subset e - \operatorname{int} D,$$

which contradicts (14). $\square$

This proposition extends Proposition 2.11 by Gupta et al. [6] who consider the Pareto case and local strict minimizers.

It is well-known that a local minimizer of a convex (scalar) function is also a global minimizer. Next, we extend this classical result to a vector optimization problem. First we show that, for a D-convex function, a local strict (resp. weak) minimizer is also a global strict (resp. weak) minimizer and, second, we prove that for a strongly D-convex function of order k, a local strict minimizer of order k is also a global strict minimizer of order k. For a Pareto problem (i.e., $D = \mathbb{R}^p_+$), our results reduce to several ones obtained by Bhatia [1].

Proposition 3.3. *Consider problem* (9) *and assume that C is convex and f is D-convex on C.*

(*i*) *If* $x_0 \in \mathrm{LStr}(f, C)$*, then* $x_0 \in \mathrm{GStr}(f, C)$*.*

(*ii*) *If* $x_0 \in \mathrm{LWMin}(f, C)$*, then* $x_0 \in \mathrm{GWMin}(f, C)$*.*

Proof. We only prove part (*i*) because the proof of part (*ii*) follows in a similar way (see also [12, Proposition 5.20] and [13]).

By hypothesis condition (11) is satisfied. Suppose that the conclusion is false. Then there exists $\bar{x} \in C \setminus \{x_0\}$ such that

$$f(\bar{x}) - f(x_0) \in -D. \tag{15}$$

As U is a neighborhood of x_0 and C is convex, there exists $t_0 > 0$ such that $x_t = t\bar{x} + (1-t)x_0 \in C \cap U \setminus \{x_0\}$ $\forall t \in (0, t_0]$. As f is D-convex, we have

$$f(x_t) \leq_D tf(\bar{x}) + (1-t)f(x_0) = f(x_0) + t(f(\bar{x}) - f(x_0)).$$

From here and using (15) we derive

$$f(x_t) - f(x_0) \leq_D t(f(\bar{x}) - f(x_0)) \leq_D 0 \quad \forall t \in (0, t_0],$$

which contradicts (11). □

Theorem 3.4. *Consider problem* (9) *and assume that C is a closed and convex set and $f \in \mathrm{SCo}(k, C)$ is continuous on C. If $x_0 \in \mathrm{LStr}(k, f, C)$, then $x_0 \in \mathrm{GStr}(k, f, C)$.*

Proof. By hypothesis, we can suppose that (10) holds for $U = B(x_0, r)$ with $r > 0$. Suppose that the conclusion is false. Then $\forall n \in \mathbb{N}$ $\exists x_n \in C \setminus \{x_0\}$ such that

$$(f(x_n) + D) \cap B(f(x_0), \tfrac{1}{n}\|x_n - x_0\|^k) \neq \emptyset.$$

Hence, there exists $d_n \in D$ such that

$$b_n := \frac{f(x_n) + d_n - f(x_0)}{\|x_n - x_0\|^k} \in B(0, 1/n). \tag{16}$$

We have two cases.

Case (i). The sequence (x_n) is bounded. Then we can suppose that $x_n \to x \in C$ since C is closed. From (16) it follows that

$$y_n := f(x_n) + d_n - f(x_0) = \|x_n - x_0\|^k b_n \to 0.$$

From here, as f is continuous, we have that $f(x_n) - f(x_0) = y_n - d_n \to f(x) - f(x_0) \in -D$, which is a contradiction if $x \neq x_0$ by Proposition 3.3 (since a local strict minimizer of order k is a local strict minimizer).

If $x = x_0$, as $x_n \to x_0$, $\exists n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$, $1/n < \alpha$ and $x_n \in B(x_0, r) \setminus \{x_0\}$. Using (16) we have

$$\begin{aligned} f(x_n) - f(x_0) = \|x_n - x_0\|^k b_n - d_n &\in B(0, \tfrac{1}{n}\|x_n - x_0\|^k) - D \\ &\subset B(0, \alpha\|x_n - x_0\|^k) - D, \end{aligned}$$

which contradicts (10).

Case (ii). The sequence (x_n) is unbounded. Then we can suppose that $\|x_n - x_0\| \to +\infty$ and $v_n := \frac{r}{2} \cdot \frac{x_n - x_0}{\|x_n - x_0\|} \to v$ for some $v \in \mathbb{R}^n$. Let us observe that $\|v\| = r/2$ and so $v \in B(0, r)$.

As $f \in \mathrm{SCo}(k, C)$, there exists $e \in \operatorname{int} D$ such that (2) holds, and so we deduce that

$$f(x_0 + t(x_n - x_0)) \leq_D f(x_0) + t(f(x_n) - f(x_0)) - t(1-t)\|x_n - x_0\|^k e.$$

From here

$$\begin{aligned} \frac{f(x_0 + t(x_n - x_0)) - f(x_0)}{t} &\leq_D (f(x_n) - f(x_0)) - (1-t)\|x_n - x_0\|^k e \\ &= \quad - d_n + \|x_n - x_0\|^k b_n - (1-t)\|x_n - x_0\|^k e. \end{aligned}$$

Hence

$$\begin{aligned} \frac{f(x_0 + t(x_n - x_0)) - f(x_0)}{t\|x_n - x_0\|} &\leq_D \frac{-d_n}{\|x_n - x_0\|} + \|x_n - x_0\|^{k-1}(b_n - (1-t)e) \\ &\leq_D \|x_n - x_0\|^{k-1}(b_n - (1-t)e). \end{aligned}$$

As the above inequality is true for all $t \in (0, 1)$ we can apply it to the sequence $t_n = \frac{r}{2} \cdot \frac{1}{\|x_n - x_0\|} \to 0$. Then, we have $t_n(x_n - x_0) = \frac{r}{2} \cdot \frac{x_n - x_0}{\|x_n - x_0\|} = v_n \to v$. Therefore

$$\frac{f(x_0 + t_n(x_n - x_0)) - f(x_0)}{r/2} \leq_D \|x_n - x_0\|^{k-1}(b_n - (1-t_n)e).$$

Now, $\lim_{n\to\infty}(b_n - (1-t_n)e) = -e \in -\operatorname{int} D$, and so $b_n - (1-t_n)e \in -\operatorname{int} D$ for all n large enough. Let $x'_n := x_0 + t_n(x_n - x_0) = x_0 + v_n \in C \cap B(x_0, r)$ ($x'_n \in C$ because $t_n \to 0$ and x'_n is a convex linear combination of points of C). For all n large enough, one has

$$\frac{f(x'_n) - f(x_0)}{r/2} \in \|x_n - x_0\|^{k-1}(b_n - (1-t_n)e) - D \subset -\operatorname{int} D - D \subset -\operatorname{int} D.$$

Hence, $f(x'_n) - f(x_0) \in -\operatorname{int} D$ which is a clear contradiction, because if x_0 is a strict minimizer of order k of f on $C \cap B(x_0, r)$ then x_0 is a weak minimizer of f on $C \cap B(x_0, r)$. $\square$

Let us observe with respect to the hypotheses of Theorem 3.4 that f is continuous on the relative interior of C because f is D-convex, and so f is continuous on C if and only if f is continuous on the relative boundary of C.

Theorem 3.4 extends Theorem 4.1 in Bhatia [1], who uses the Pareto order.

Theorem 3.4 is also true for a constrained vector optimization problem if the constraint function is cone-convex on C. Let $g : \mathbb{R}^n \to \mathbb{R}^m$ and let $K \subset \mathbb{R}^m$ be a convex cone. Consider the constrained optimization problem

$$D - \mathrm{Min}\{f(x) : \ x \in S\}, \tag{17}$$

where $S = \{x \in C : g(x) \in -K\}$.

Corollary 3.5. *Consider problem* (17) *and assume that* C *is a closed and convex set,* $f \in \mathrm{SCo}(k, C)$ *is continuous on* C*,* K *is closed and* g *is continuous on* C*. If* g *is* K*-convex on* C *and* $x_0 \in \mathrm{LStr}(k, f, S)$*, then* $x_0 \in \mathrm{GStr}(k, f, S)$*.*

Proof. If we see that S is a convex set, the corollary is an immediate consequence of Theorem 3.4 since S is closed. In fact, if $x, \bar{x} \in S$, then $g(x), g(\bar{x}) \in -K$ and since g is K-convex on C we have that for $x_t = tx + (1 - t)\bar{x}$,

$$g(x_t) \leq_K tg(\bar{x}) + (1 - t)g(x) \in (-K) + (-K) \subset -K$$

because K is a convex cone. Consequently $x_t \in S$, for each $t \in [0, 1]$. □

4. Optimality conditions

In this section we obtain optimality conditions for a point to be a strict minimizer of order k. For this aim we will use higher order strong cone-convexity assumptions. From our study it is clear that this kind of convexity is appropriate to obtain optimality conditions for higher order strict minimizers.

Lemma 4.1. *If for some* $\lambda \in D^+ \setminus \{0\}$*,* $x_0 \in \mathrm{GStr}(k, \lambda f, C)$*, then* $x_0 \in \mathrm{GStr}(k, f, C)$*.*

Proof. By assumption there exists $c > 0$ such that $\lambda f(x) \geq \lambda f(x_0) + c\|x - x_0\|^k$ $\forall x \in C$. As $\mathrm{int}\, D$ is a nonempty cone, there exists $e_0 \in \mathrm{int}\, D$ satisfying $\lambda(e_0) < c$. Suppose that $x_0 \notin \mathrm{GStr}(k, f, C)$. Then by Proposition 3.2, for e_0 there exists $\bar{x} \in C$ such that $f(\bar{x}) - f(x_0) - \|\bar{x} - x_0\|^k e_0 \in -\mathrm{int}\, D$. As $\lambda \in D^+ \setminus \{0\}$ it follows that

$$\lambda f(\bar{x}) < \lambda f(x_0) + \|\bar{x} - x_0\|^k \lambda(e_0) < \lambda f(x_0) + c\|\bar{x} - x_0\|^k,$$

which is a contradiction. □

In Proposition 4.2 and Theorems 4.5 and 4.6, we assume the function $f : \mathbb{R}^n \to \mathbb{R}^p$ is strongly D-convex of order k on an open convex set C. According to [12, Corollary 5.1], f is locally Lipschitz on C, and so the generalized Jacobian $\partial f(x)$ exists for all $x \in C$.

Proposition 4.2. *Let* $C \subset \mathbb{R}^n$ *be an open convex set, let* $f \in \mathrm{SCo}(k, C)$ *and let* $x_0 \in C$*. If* $0 \in \lambda\partial f(x_0)$ *for some* $\lambda \in D^+ \setminus \{0\}$ *then* $x_0 \in \mathrm{GStr}(k, f, C)$*.*

Proof. By Theorem 2.5, there exists $e \in \operatorname{int} D$ such that

$$f(x) \geq_D f(x_0) + A(x - x_0) + \|x - x_0\|^k e \quad \forall x \in C,\ \forall A \in \partial f(x_0). \tag{18}$$

As $0 \in \lambda \partial f(x_0)$, for some $\bar{A} \in \partial f(x_0)$ one has $0 = \lambda \bar{A}$. Taking this into account, using the inequality (18) with $A = \bar{A}$, and by applying λ to both sides it results

$$\lambda f(x) \geq \lambda f(x_0) + \lambda \bar{A}(x - x_0) + \|x - x_0\|^k \lambda(e) = \lambda f(x_0) + \|x - x_0\|^k \lambda(e)$$

$\forall x \in C$. By Lemma 4.1, the conclusion follows since $\lambda(e) > 0$. □

Now we study problem (17) with explicit constraint.

Definition 4.3. It is said that a locally Lipschitz function $g : \mathbb{R}^n \to \mathbb{R}^m$ is ∂-quasiconvex by scalarization on C if $\forall \mu \in K^+$, $\forall x, y \in C$ one has

$$(\mu g)(y) \leq (\mu g)(x) \ \Rightarrow\ \langle \eta, y - x \rangle \leq 0 \quad \forall \eta \in \partial(\mu g)(x).$$

If $m = 1$ we say simply g is ∂-quasiconvex on C instead of g is ∂-quasiconvex by scalarization on C. In this case, this notion has been considered, for example, in [2, Definition 2.2].

The notion of ∂-quasiconvexity by scalarization is more general than the notion of cone-convexity as the following result shows.

Proposition 4.4. *If g is K-convex on an open convex set C, then g is ∂-quasiconvex by scalarization on C.*

Proof. Suppose that $x, y \in C$, $\mu \in K^+$ and $(\mu g)(y) \leq (\mu g)(x)$. As μg is convex, it is regular in the Clarke sense (see [3]) and so

$$(\mu g)^\circ(x, y - x) = (\mu g)'(x, y - x) = \lim_{t \to 0^+} \frac{(\mu g)(x + t(y - x)) - (\mu g)(x)}{t}, \tag{19}$$

where $(\mu g)^\circ$ is the Clarke generalized derivative and $(\mu g)'$ is the usual one-sided directional derivative. Since μg is convex

$$(\mu g)(x + t(y - x)) \leq t(\mu g)(y) + (1 - t)(\mu g)(x) = (\mu g)(x) + t[(\mu g)(y) - (\mu g)(x)],$$

and so $t^{-1}[(\mu g)(x + t(y - x)) - (\mu g)(x)] \leq (\mu g)(y) - (\mu g)(x)$. In view of this inequality, from (19) it follows that

$$(\mu g)^\circ(x, y - x) \leq (\mu g)(y) - (\mu g)(x) \leq 0.$$

As $(\mu g)^\circ(x, y - x) = \max_{\eta \in \partial(\mu g)(x)} \langle \eta, y - x \rangle$, we derive that $\langle \eta, y - x \rangle \leq 0$ $\forall \eta \in \partial(\mu g)(x)$, which completes the proof. □

Theorem 4.5. *Consider problem (17). Suppose that $x_0 \in S$, C is an open convex set, $f : \mathbb{R}^n \to \mathbb{R}^p$ is strongly D-convex of order k on C, and $g : \mathbb{R}^n \to \mathbb{R}^m$ is ∂-quasiconvex by scalarization on C. If there exist $\lambda \in D^+ \setminus \{0\}$ and $\mu \in K^+$ such that*

$$0 \in \lambda \partial f(x_0) + \mu \partial g(x_0) \quad \text{and} \quad (\mu g)(x_0) = 0,$$

then $x_0 \in \operatorname{GStr}(k, f, S)$.

Proof. As $f \in \mathrm{SCo}(k, C)$, by Theorem 2.5 there exists $e \in \operatorname{int} D$ such that (18) holds. By applying λ to both sides of (18) it results

$$\lambda f(x) \geq \lambda f(x_0) + \lambda A(x - x_0) + \|x - x_0\|^k \lambda(e) \quad \forall x \in C,\ \forall A \in \partial f(x_0). \tag{20}$$

If $x \in S$, then $g(x) \leq_K 0$, and so $(\mu g)(x) \leq 0 = (\mu g)(x_0)$. As g is ∂-quasiconvex by scalarization on C, it follows that

$$0 \geq \langle \eta, x - x_0 \rangle \quad \forall \eta \in \partial(\mu g)(x_0). \tag{21}$$

By hypothesis, there exist $\bar{A} \in \partial f(x_0)$ and $\bar{B} \in \partial g(x_0)$ satisfying

$$\lambda \bar{A} + \mu \bar{B} = 0. \tag{22}$$

As $\mu \partial g(x_0) = \partial(\mu g)(x_0)$ by [3, Theorem 2.6.6], from (21) one has

$$0 \geq \mu \bar{B}(x - x_0) \quad \forall x \in S. \tag{23}$$

Adding up (20) with $A = \bar{A}$ and (23), we obtain

$$\lambda f(x) \geq \lambda f(x_0) + (\lambda \bar{A} + \mu \bar{B})(x - x_0) + \|x - x_0\|^k \lambda(e) \quad \forall x \in S.$$

Using (22) it results

$$\lambda f(x) \geq \lambda f(x_0) + c\|x - x_0\|^k \quad \forall x \in S, \tag{24}$$

where $c = \lambda(e) > 0$. Equation (24) says that $x_0 \in \mathrm{GStr}(k, \lambda f, S)$. From Lemma 4.1 the conclusion follows. □

In the case $K = \mathbb{R}^m_+$ we can weaken the hypothesis obtaining the following result, whose proof is similar to that of Theorem 4.5 and so it is omitted. We denote $I(x_0) = \{j \in \{1, \ldots, m\} : \ g_j(x_0) = 0\}$.

Theorem 4.6. *Consider problem (17) with $K = \mathbb{R}^m_+$. Suppose that $x_0 \in S$, C is an open convex set, $f : \mathbb{R}^n \to \mathbb{R}^p$ is strongly D-convex of order k on C, and $g_j : \mathbb{R}^n \to \mathbb{R}$, $j \in I(x_0)$, are ∂-quasiconvex on C. If there exist $\lambda \in D^+ \setminus \{0\}$ and $\mu_j \geq 0$, $j \in \{1, \ldots, m\}$, such that*

$$0 \in \lambda \partial f(x_0) + \sum_{j=1}^{m} \mu_j \partial g_j(x_0) \quad \text{and} \quad \mu_j g_j(x_0) = 0,\ j = 1, \ldots, m,$$

then $x_0 \in \mathrm{GStr}(k, f, S)$.

Bathia [1, Definition 3.3] uses the following notion (for scalar functions). A locally Lipschitz function $g : \mathbb{R}^n \to \mathbb{R}$ is said to be strongly quasiconvex of order k on C if there exists $c > 0$ such that for all $x, y \in C$,

$$g(y) \leq g(x) \ \Rightarrow \ \langle \eta, y - x \rangle + c\|y - x\|^k \leq 0 \quad \forall \eta \in \partial g(x).$$

It is clear that if g is strongly quasiconvex of order k then g is ∂-quasiconvex. Taking this into account Theorem 4.6 generalizes Theorem 4.4 in [1].

In the above results (Proposition 4.2 and Theorems 4.5 and 4.6), the involved functions are locally Lipschitz. If we suppose the functions are differentiable, we obtain the following results whose proofs are omitted because are similar (we apply Theorem 2.6 instead of Theorem 2.5).

In the sequel, we assume the functions $f : \mathbb{R}^n \to \mathbb{R}^p$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ are Fréchet differentiable and $C \subset \mathbb{R}^n$ is a convex set.

Proposition 4.7. *Let $f \in \mathrm{SCo}(k, C)$ and $x_0 \in C$. If $\lambda Jf(x_0) = 0$ for some $\lambda \in D^+ \setminus \{0\}$ then $x_0 \in \mathrm{GStr}(k, f, C)$.*

Definition 4.8. It is said that $g : \mathbb{R}^n \to \mathbb{R}^m$ is quasiconvex by scalarization on C if $\forall \mu \in K^+$, $\forall x, y \in C$ one has

$$(\mu g)(y) \leq (\mu g)(x) \;\Rightarrow\; \langle \nabla(\mu g)(x), y - x\rangle \leq 0,$$

i.e., if μg is quasiconvex on C in the ordinary sense (here $\nabla(\mu g)(x)$ is the gradient of μg at x). If $m = 1$ we say simply g is quasiconvex on C.

Theorem 4.9. *Consider problem (17). Suppose that $x_0 \in S$, $f \in \mathrm{SCo}(k, C)$ and g is quasiconvex by scalarization on C. If there exist $\lambda \in D^+ \setminus \{0\}$ and $\mu \in K^+$ such that*

$$\lambda Jf(x_0) + \mu Jg(x_0) = 0 \quad \textit{and} \quad (\mu g)(x_0) = 0,$$

then $x_0 \in \mathrm{GStr}(k, f, S)$.

Theorem 4.10. *Consider problem (17) with $K = \mathbb{R}^m_+$. Suppose that $x_0 \in S$, $f \in \mathrm{SCo}(k, C)$ and $g_j : \mathbb{R}^n \to \mathbb{R}$, $j \in I(x_0)$, are quasiconvex on C. If there exist $\lambda \in D^+ \setminus \{0\}$ and $\mu_j \geq 0$, $j \in \{1, \ldots, m\}$, such that*

$$\lambda Jf(x_0) + \sum_{j=1}^{m} \mu_j \nabla g_j(x_0) = 0 \quad \textit{and} \quad \mu_j g_j(x_0) = 0,\ j = 1, \ldots, m,$$

then $x_0 \in \mathrm{GStr}(k, f, S)$.

Theorem 4.10 generalizes and improves Theorem 2.13 in Gupta et al. [6], who consider the Pareto case ($D = \mathbb{R}^p_+$), f and g are continuously differentiable and strongly convex of order k and $\lambda \in \operatorname{int} \mathbb{R}^p_+$.

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