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Tangency vis-à-vis Differentiability by Peano, Severi and Guareschi

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Commemorating the 150th Birthday of Giuseppe Peano (1858–1932).

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Peano defined *differentiability* of functions and *lower tangent cones* in 1887, and *upper tangent cones* in 1903, but uses the latter concept already in 1887 without giving a formal definition. Both cones were defined for arbitrary sets, as certain limits of appropriate homothetic relations. Around 1930 Severi and Guareschi, in a series of mutually fecundating individual papers, characterized differentiability in terms of *lower tangent cones* and strict differentiability in terms of *lower paratangent cones*, a notion introduced, independently, by Severi and Bouligand in 1928. Severi and Guareschi graduated about 1900 from the University of Turin, where Peano taught till his demise in 1932.

1. Preamble

In 2008 mathematical community celebrated the 150th anniversary of the birth of Giuseppe Peano, as well as the 100th anniversary of the last (fifth) edition of *Formulario Mathematico*. Taking part in the commemoration, we have been reviewing Peano's foundational contributions to various branches of mathematics: optimization [19], Grassmann geometric calculus [38], derivation of measures [37], definition of surface area [36], general topology [20], infinitesimal calculus [35], as well as to tangency and differentiability (in the present paper). Peano contributed in an essential way to several other fields of mathematics: set theory[1], ordinary differential equations, arithmetic, convexity and, maybe most significantly, he introduced a completely rigorous formal language of mathematics. Also these contributions should and hopefully will be discussed in future papers.

[1]In 1914 Hausdorff wrote in *Grundzüge der Mengenlehre* [46, (1914), p. 369] of Peano's filling curve: *das ist eine der merkwürdigsten Tatsachen der Mengenlehere, deren Entdeckung wir G. Peano verdanken* ⟦this is one of the most remarkable facts of set theory, the discovery of which we owe to G. Peano⟧. It is less known that Peano formulated the axiom of choice in [68, (1890)] (c.f. Appendix 8), fourteen years before Zermelo [111, (1904)].

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Peano acquired an international reputation soon after his graduation[2]. Recognized as one of the leading mathematical authorities of the epoch, he was invited to publish in prestigious mathematical journals[3]. He was at the summit of fame at the break of the 20th century when he took part in the International Congress of Philosophy and the International Congress of Mathematicians in Paris in 1900. Bertrand Russell, who also participated in the philosophy congress, noted in [85, (1967), pp. 217–218]

> The Congress was a turning point in my intellectual life, because I there met Peano. [...] In discussions at the Congress I observed that he was always more precise than anyone else, and that he invariably got the better of any argument upon which he embarked.

In *The Principles of Mathematics* [84, (1903) p. 241] Russell said that Peano had a rare immunity from error.

Peano was associated with the University of Turin during his whole mathematical career, from October 1876, when he became a student, till 19th of April 1932, when he taught his classes as usual, a day before his death. From 1903 on, following the example of Méray, with whom he corresponded, Peano dedicates himself more and more to auxiliary international languages (postulated as *lingua rationalis* by Leibniz [15, (1901), Ch. III]) in company with a philosopher and logician Louis Couturat, linguists Otto Jespersen and Jan Baudouin de Courtenay, and a chemist Wilhem Ostwald[4]. This interest becomes his principal passion after the completion of the last edition of *Formulario Mathematico* in 1908, written in a (totally rigorous) mathematical formal language[5] and commented in an auxiliary language, *latino sine flexione*,

[2]Already in [70, (1882)] he observed that the definition of surface measure of the famous *Cours de calcul différentiel et intégral* of Serret [90] was inadequate.

[3]For example, he was invited by Klein to contribute to *Mathematische Annalen* (see Segre [89, (1997)] and the letters from Mayer to Klein [59, n. 125 p. 161, n. 126 p. 163, n. 148 p. 181]). As a result, Peano published three papers: on the resolvent (in particular, the exponential of a matrix) of a system of linear differential equations [66, (1888)], on the existence of solutions of a system of differential equations with the sole hypothesis of continuity [68, (1890)], and on a filling curve [71, (1890)].

[4]Wilhem Ostwald (1853–1932), Nobel Prize in Chemistry in 1909, in his *Selbstbiographie* [60, (1927)] describes Peano as follows:

> Eine Personalität besonderer Art war der italienische Mathematiker Peano. Lang, äußerst mager, nach Haltung und Kleidung ein Stubengelehrter, der für Nebendinge keine Zeit hat, mit gelbbleichem, hohlem Gesicht und tiefschwarzem, spürlichem Haar und Bart, erschien er ebenso abstrakt, wie seine Wissenschaft. Er hatte eigene Vorschläge zu vertreten, nämlich sein latino sine flexione, ein tunlichst vereinfachtes latein, für welches er mit unerschütterlicher Hingabe eintrat, da er als Italiener das Gefühl hatte, im Latein ein uraltes Erbe zu verteidigen.

⟦An Italian mathematician Peano was a personality of peculiar kind. Tall, extremely slim, by attitude and clothes, a scientist, who has no time for secondary things, with his pale yellowish hollow face and sparse deeply black hair and beard, looked so abstract as his science. He had his proper proposal to present, namely his latino sine flexione, a simplified, as much as possible, Latin, which he presented with imperturbable devotion, since, as an Italian, he had the feeling to defend in Latin a primordial heritage.⟧

[5]Hilbert and Ackermann write in the introduction to [48, (1928)]: *G. Peano and his co-workers began in 1894 the publication of the Formulaire de Mathématiques, in which all the mathematical disciplines were to be presented in terms of the logical calculus.*

both conceived by Peano.

It should be emphasized that the formal language conceived and used by Peano was not a kind of shorthand adapted for a mathematical discourse, but a collection of ideographic symbols and syntactic rules with univocal semantic interpretations, which produced precise mathematical propositions, as well as inferential rules that ensure the correctness of arguments.

Peano's fundamental contributions to mathematics are numerous. Yet, nowadays, only few mathematical achievements are commonly associated with his name. It is dutiful to reconstitute from (partial) oblivion his exceptional role in the development of science (see Appendix 8). In the present paper we intend to delineate the evolution, in the work of Peano, of the concept of tangency and of its relation to differentiability[6].

By respect for historical sources and for the reader's convenience, the quotations in the sequel will appear in the original tongue with a translation in square brackets (usually placed in a footnote). All the biographical facts concerning Peano are taken from H. C. Kennedy, *Life and Works of Giuseppe Peano* [50, 51, (1980, 2006)]. On the other hand, we have checked all the reported bibliographic details concerning mathematical aspects.

2. Introduction

In *Applicazioni Geometriche* of 1887 [64], Peano defined *differentiability* of functions, *lower tangent cone*, and (implicitly in [64] and explicitly in *Formulario Mathematico* of 1903 [76]) *upper tangent cone*, both for arbitrary sets, as certain limits of appropriate homothetic relations. Around 1930 Francesco Severi (1879–1961) and Giacinto Guareschi (1882–1976), in a series of mutually fecundating individual papers, characterized differentiability in terms of tangency without referring to Peano.

Following Peano [77, (1908) p. 330], a function $f : A \to \mathbb{R}^n$ is *differentiable at* an accumulation point $\hat{x}$ of $A \subset \mathbb{R}^m$ if $\hat{x} \in A$ and there exists[7] a linear function $Df(\hat{x}) : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$\lim_{A \ni x \to \hat{x}} \frac{f(x) - f(\hat{x}) - Df(\hat{x})(x - \hat{x})}{\|x - \hat{x}\|} = 0. \tag{1}$$

It is *strictly differentiable at* $\hat{x}$ (Peano [73, (1892)] for $n = 1$, Severi in [93, (1934) p. 185][8]) if (1) is strengthened to

[6] In his reference book [58, (1973)] K. O. May discusses a role of direct and indirect sources in historiography of mathematics. He stresses the importance of primary sources, but acknowledges also the usefulness of secondary (and n-ary sources) under the provision of critical evaluation. As mathematicians, we are principally interested in development of mathematical ideas, so that we use almost exclusively primary sources, that is, original mathematical papers. On the other hand, one should not neglect the biography of the mathematicians whose work one studies, because it provides information about effective and possible interactions between them.

[7] In his definition Peano assumes uniqueness, which we drop because of the prevalent contemporary use that we adopt in the sequel of the paper.

[8] As we will see later, Severi uses the term *hyperdifferentiable*.

$$\lim_{A \ni x, y \to \hat{x}, x \neq y} \frac{f(y) - f(x) - Df(\hat{x})(y - x)}{\|y - x\|} = 0. \tag{2}$$

These are exactly the definitions that we use nowadays. The first notion is often called *Fréchet differentiability* (referring to Fréchet [24, 25, (1911)]) and the second is frequently referred to Leach [52, (1961)], where it is called *strong differentiability* and to Bourbaki [10, (1967), p. 12].

Currently an assortment of tangent cones have been defined by a variety of limits applied to homothetic relations. Peano gave an accomplished definition of tangency in *Formulario Mathematico* [77, (1908)], as was noticed in Dolecki, Greco [19, (2007)]; he defined what we call respectively, the *lower* and the *upper tangent cones* of F at x (traditionally denominated *adjacent* and *contingent* cones)[9]

$$\mathrm{Tan}^-(F, x) := \operatorname*{Li}_{t \to 0^+} \tfrac{1}{t}\left(F - x\right), \tag{3}$$

$$\mathrm{Tan}^+(F, x) := \operatorname*{Ls}_{t \to 0^+} \tfrac{1}{t}\left(F - x\right), \tag{4}$$

where $\mathrm{Li}_{t \to 0^+}$ and $\mathrm{Ls}_{t \to 0^+}$ denote the usual lower and upper limits of set-valued maps. Here, we adopt the modern definition of lower and upper limits in metric spaces, both introduced by Peano, the first in *Applicazioni geometriche* [64, (1887), p. 302] and the second in *Lezioni di analisi infinitesimale* [74, (1893), volume 2, p. 187] (see Dolecki, Greco [19, (2007)] for further details). Let d denote the Euclidean distance on $\mathbb{R}^n$ and let A_t be a subset of $\mathbb{R}^n$ for $t > 0$. According to Peano,

$$\mathrm{Li}_{t \to 0^+} A_t := \{x \in \mathbb{R}^n : \lim_{t \to 0^+} d(x, A_t) = 0\} \tag{5}$$

$$\mathrm{Ls}_{t \to 0^+} A_t := \{x \in \mathbb{R}^n : \liminf_{t \to 0^+} d(x, A_t) = 0\}. \tag{6}$$

Since $d(v, \frac{1}{t}(F - x)) = \frac{1}{t} d(x + tv, F)$, from (3) and (4) it follows that

$$v \in \mathrm{Tan}^-(F, x) \quad \text{if and only if} \quad \lim_{t \to 0^+} \frac{1}{t} d(x + tv, F) = 0 \tag{7}$$

$$v \in \mathrm{Tan}^+(F, x) \quad \text{if and only if} \quad \liminf_{t \to 0^+} \frac{1}{t} d(x + tv, F) = 0. \tag{8}$$

The *upper paratangent cone* (traditionally called *paratingent cone*) of F at x

$$\mathrm{pTan}^+(F, x) := \operatorname*{Ls}_{t \to 0^+,\, F \ni y \to x} \tfrac{1}{t}\left(F - y\right) \tag{9}$$

was introduced later by Severi [91, (1928) p. 149] and Bouligand in [6, (1928) pp. 29–30][10]. The *lower paratangent cone*

$$\mathrm{pTan}^-(F, x) := \operatorname*{Li}_{t \to 0^+,\, F \ni y \to x} \tfrac{1}{t}\left(F - y\right) \tag{10}$$

[9]Actually Peano defined affine variants of these cones.

[10]Successively in [8, (1930) pp. 42–43], Bouligand introduces the terms of *contingent* and *paratingent* to denote upper tangent and paratangent cones. In contrast to definitons (4) and (9), for Severi and Bouligand, an upper tangent (resp. upper paratangent) cone is a family of half-lines (resp. straight lines); consequently, they are empty at isolated points and, on the other hand, they consider closedness in the sense of half-lines (resp. straight lines).

is usually called the *Clarke tangent cone* (see Clarke [14, (1973)]). In [19, pp. 499–500] we listed the properties of the upper tangent cone observed by Peano. Of course,

$$\mathrm{pTan}^-(F,x) \subset \mathrm{Tan}^-(F,x) \subset \mathrm{Tan}^+(F,x) \subset \mathrm{pTan}^+(F,x). \tag{11}$$

In the works of Peano there are no occurrences of sets for which the upper and lower tangent cones are different. Here we furnish an easy one.[11]

Example 2.1. If $S := \left\{\frac{1}{n!} : n \in \mathbb{N}\right\}$, then $\mathrm{Tan}^+(S,0) = \mathbb{R}_+$ and $\mathrm{Tan}^-(S,0) = \{0\}$.

It is surprising, but it seems that so far in the literature there have been no such examples. The pretended instances:

$$A := \left\{\left(t, t\sin\left(\frac{1}{t}\right)\right) : t \in \mathbb{R} \setminus \{0\}\right\}$$

given by Rockafellar and Wets in *Variational Analysis* [82, (1998), p. 199], and

$$B := \{(t,-t) : t < 0\} \cup \left\{\left(\frac{1}{n}, \frac{1}{n}\right) : n \in \mathbb{N}\right\}$$

provided by Aubin and Frankowska in *Set-Valued Analysis* [4, (1990), p. 161] are not pertinent, because in both of them the upper and the lower tangent cones coincide[12].

In the literature there are numerous examples of sets, for which other inclusions in (11) are strict.

The remarkable fact that the coincidence of the upper and lower paratangent cones at every point of a locally closed subset F of Euclidean space is equivalent to the fact that F is a C^1-submanifold, has not been observed till now. It will be an object of [39], in which a mathematical and historical account on the subject will be provided.[13]

Intrinsic notions of tangent straight line to a curve and of tangent plane to a surface were clear to Peano (see Section 4) and even prior to him, before the emergence of the concept of tangent cone to an arbitrary set. On rephrasing these special notions

[11] In [19, (2007), p. 499, footnote 21] we observed that $v \in \mathrm{Tan}^-(S,x)$ if and only if

($*$) there exists a sequence $\{x_n\}_n \subset S$ such that $\lim_n x_n = x$ and $\lim_n n(x_n - x) = v$.

On the other hand it is well known that $v \in \mathrm{Tan}^+(S,x)$ if and only if

($**$) there exist sequences $\{\lambda_n\}_n \subset \mathbb{R}_{++}$ and $\{x_n\}_n \subset S$ such that $\lim_n \lambda_n = 0$, $\lim_n x_n = x$ and $\lim_n (x_n - x)/\lambda_n = v$.

In [105, (1929)] von Neumann shows that a closed matrix group G is a Lie group by proving the following three fundamental facts: (a) "$\mathrm{Tan}^+(G,E)$ at unit E of G is a matrix Lie algebra", (b) "($**$) implies ($*$)", (c) "$\exp A \in G$ for every $A \in \mathrm{Tan}^+(G,E)$". The second claim (b), which amounts to $\mathrm{Tan}^+(G,E) = \mathrm{Tan}^-(G,E)$, is the crucial step in his proof.

[12] In fact, by footnote 11, $\mathrm{Tan}^+(A,(0,0)) = \mathrm{Tan}^-(A,(0,0)) = \left\{(h,k) \in \mathbb{R}^2 : |k| \le |h|\right\}$ and $\mathrm{Tan}^+(B,(0,0)) = \mathrm{Tan}^-(B,(0,0)) = \{(t,|t|) : t \in \mathbb{R}\}$.

[13] Although Severi and Guareschi characterized C^1 manifolds in Euclidean space in terms of tangency, their definitions and reasonings are not entirely transparent; see Greco [39] for further details.

in terms of a vector space H, tangent to a set F at an accumulation point $\hat{x}$ of F, we recover the following condition:

$$\lim_{F\ni x\to\hat{x},x\neq\hat{x}} \frac{d(x,H+\hat{x})}{d\,(x,\hat{x})}=0. \tag{12}$$

Geometrically, (12) means that the vector space H and the half-line passing through $\hat{x}$ and x in F form an angle that tends to zero as x tends $\hat{x}$.

From 1880 Peano taught at the University of Turin. Among the students of that university at the very end of 19th century were Beppo Levi, Severi and Guareschi (see the biography in Appendix 10). They were certainly acquainted with the famous *Applicazioni Geometriche* [64, (1887)] of Peano, so that their writings on tangency and differentiability could not abstract from the achievements of Peano. Yet neither Severi nor Guareschi cite Peano[14]. By the bye, in [53, (1932)] Beppo Levi acknowledges explicitly the influence of *Calcolo Geometrico* [65, (1888)] of Peano on his understanding of the work of Grassmann; Beppo Levi recalls his enthusiastic interest in *Calcolo Geometrico* and difficulty in reading *Ausdehnungslehre* [32]:

> [interesse] quasi entusiastico che, giovane principiante, mi prese alla lettura del *Calcolo geometrico secondo l'Ausdehnungslehre di Grassmann*; e ricordo all'opposto, l'impressione di malsicura astrattezza che il medesimo principiante ricevette volendo affrontare la fonte, l'*Ausdehnungslehre* del 1844.[15]

In [26, p. 241] of 1937, Fréchet comments[16]:

[14]Severi however mentions in [92, (1930)] a paper [11, (1930)] of Cassina (who, by the way, became later the editor of the collected works of Peano [78]). It was on browsing through Severi's citation of Cassina that the second author (G. H. Greco) of this paper discovered the immensity of Peano's contributions to scientific culture. Parenthetically, Severi reproaches to Cassina for having failed to quote him:

> [Cassina] ha ultimamente considerato allo stesso mio modo la figura tangente ad un insieme, ignorando certo i precedenti sull'argomento.

⟦[Cassina] recently considered, in the same way of mine, the tangent figure of a set, apparently ignoring the precedents in this topic.⟧
This surprising oblivion of Peano's work by Severi can be perhaps explained by a merely sporadic interest in mathematical analysis by this algebraic geometer.
Another algebraic geometer, Beniamino Segre (a coauthor with Severi of a paper on tangency [96, (1929)], and, on the other hand, an author of a historical paper on Peano [88, (1955)]), presented to *Accademia dei Lincei* a paper on tangency [103, (1973)] that ignored the contributions of Peano, Severi and Segre himself, without reacting to this unawareness.
It is also surprising that Boggio, one of the best known pupils of Peano, did not recall in [22, (1936)] the famous contribution to tangency of his mentor, when he recommended for publication in *Memorie dell'Accademia delle Scienze di Torino* a paper of Guareschi [41, (1936)] that begins: "*Il concetto di semitangente [...] introdotto nell'analisi da F. Severi.*" ⟦The concept of semitangent [...] introduced in analysis by F. Severi.⟧

[15]⟦Almost entusiatic [interest] that took me, a young beginner, at the lecture of *Calcolo geometrico secondo l'Ausdehnungslehre di Grassmann*; and I remember, in contrast, an impression of insecure abstractness that the same beginner received attempting to confront the source, *Ausdehnungslehre* of 1844.⟧

[16]We believe that Fréchet, who never investigated tangency, took this information either from his

> On doit à M. Bouligand et à ses élèves d'avoir entrepris l'étude systématique [de] cette théorie des "contingents et paratingents" dont l'utilité a été signalée d'abord par M. Beppo Levi, puis par M. Severi.[17]

Following the guidelines of Fréchet, we initiated to study the writings of Severi (see, for example, Dolecki [18, (1982)]) and, thanks to a reference in Severi [93, (1934)], also those of Guareschi.

An exhaustive historical study of the work of Bouligand and his pupils is also dutiful, and we hope that it will be done before long[18].

3. Tangency

The notion of tangency originated from geometric considerations in antiquity. On the emergence of the coordinates of Descartes, analytic aspect prevailed over the geometric view in tangency, also because of the growth of infinitesimal calculus.

In *Formulario Mathematico* [77, (1908), p. 313], a compendium of mathematics known at the epoch, edited and mostly written by Peano[19], the tangents of Euclid and Descartes are described in these terms:

> Euclide [...], dice que recta es tangente ≪$\varepsilon\varphi\alpha\pi\tau\varepsilon\sigma\vartheta\alpha\iota$≫ ad circulo [...] si habe uno solo puncto commune cum circulo.
>
> Nos pote applica idem Df [definition] ad ellipsi, etc.; sed non ad omni curva.
>
> Descartes, *La Géométrie* a. 1637 Œuvres, t. 6, p. 418 dice que tangente es recta que seca curva in duo puncto ≪ioins en un≫; id es, si æquatione que determina ce punctos de intersectione habe duo ≪racines entièrement ésgales≫.
>
> Df [definition] considerato se transforma in P·0 [usual definition], si nos considera per duo puncto ≪juncto in uno≫, ut limite de recto per duo puncto distincto.[20]

A drawback of the predominance of analytic approach in geometry was that tangency concepts were defined through an auxiliary system and not intrinsically (that

friend Bouligand or, directly, from a paper of Severi [92, (1931)] where B. Levi, Bouligand and his pupils Rabaté and Durand are quoted. To our knowledge Bouligand neither refers to nor quotes Severi.

[17]⟦We owe to Bouligand and his pupils a systematic study [of] this theory of contingents and paratingents, the usefulness of which was pointed out first by Beppo Levi, then by Severi.⟧

[18]Among those who refer to Bouligand in their study of tangency we recall Durand, Rabaté (1931), Mirguet (1932), Marchaud (1933), Blanc (1933), Charpentier (1933), Vergnères (1933), Zaremba (1936), Pauc (1936–41), Ward (1937), Saks (1937), Roger (1938), Choquet (1943–48).

[19]In contrast to former versions that were written in French, the last (fifth) version of ≪Formulario Mathematico≫ (1908) was written in "latino sine flexione".

[20]⟦Euclid [...] says that a straight line is tangent to a circle [...] if it has only one common point with the circle. One can apply the same definition to an ellipse, and so on, but not to every curve. Descartes, *La Géométrie* a. 1637 Œuvres, t. 6, p. 418, says that a tangent is a straight line that cuts a curve in two points ≪joined in one≫; that is, the equation that determines these points of intersection has two ≪entirely equal roots≫[The definition] considered [by Descartes] becomes [the usual definition] if we mean by the points ≪joined in one≫ the limit of straight lines passing through two distinct points [when these tend to one point].⟧

is, independently of a particular coordinate system). Analytic approach to tangency requires that a figure, like a line or a surface be defined via equations or parametrically, hence with the aid of functions of some regularity. This constitutes another drawback, excluding, for instance, figures defined by inequalities. On the other hand, geometrically defined figures necessitate analytic translation before they could be investigated for tangency.

The comeback to the geometric origin of tangency, and actually to a synergy of both (geometric and analytic) aspects, is operated by the definitions of tangency of arbitrary sets that use limits of homothetic figures. This breakthrough was done by Peano in *Applicazioni Geometriche* [64, (1887)].

Synthetic geometry started with Euclid, was axiomatized by Pasch, later by Peano and finally by Hilbert. *Analytic geometry* (in the original sense) was initiated by Descartes and enabled mathematicians to reduce geometric problems to algebraic equalities, and thus to use algebraic calculus to solve them. *Vector geometry* of Grassmann potentiates the virtues of both, synthetic and analytic, aspects of geometry.

In comparison with analytic methods, the classical geometric approach had certainly an inconvenience of the lack of a system of standard operations obeying simple algebraic rules, that is, of a calculus. In a letter of 1679 to Huygens, Leibniz postulated the need of a *geometric calculus*, similarly to the already existing *algebraic calculus*. This postulate was realized by Grassmann in *Geometrische Analyse* [33, (1847)] and in *Ausdehnungslehre* [32, (1844, 1862)]. In *Applicazioni Geometriche* [64, (1887)] Peano presented the geometric calculus of Grassmann in order to treat geometric objects directly (without coordinates), and in *Calcolo Geometrico* [65, (1888)] refounded the affine exterior algebra of Grassmann in three-dimensional spaces (see Greco, Pagani [38, (2009)] for further details). In this way Peano eliminated the inconvenience of the geometric approach mentioned above. This achievement enabled him to develop a simple and sharp tangency theory abounding with applications. Although Peano's framework was that of 3-dimensional Euclidean space, his method can be extended in an obvious way to arbitrary dimensions (for example, the notion of angle between two subspaces can be expressed in terms of the inner product multi-vectors).

Peano's works permitted an easy access to the geometric calculus of Grassmann by the mathematical community at the end of 19th century[21], in particular to the mathematicians of the Turin University.

4. Evolution of concepts of tangency in the work of Peano

The interest of Peano in tangency goes back to 1882, two years after he graduated from the university, when he discovered that the definition of area of surface, given by Serret in his *Cours de calcul différentiel et intégral* [90, p. 293 (5th edition 1900)] was defective. Indeed, Serret defined the area of a given surface as the limit of the areas of polyhedral surfaces inscribed in that surface. Peano found a sequence of

[21] See section 35: *Begründung der Punktrechnung durch G. Peano* in [56, (1923)] of the celebrated *Encyklopädie der mathematischen Wissenschaften.*

polyhedral surfaces inscribed in a bounded cylinder so that the corresponding areas tend to infinity [76, (1902–1903), pp. 300–301][22]. As Peano comments in that note

> On ne peut pas définir l'aire d'une surface courbe comme la limite de l'aire d'une surface polyédrique inscrite, car les faces du polyèdre n'ont pas nécessairement pour limite les plans tangents à la surface.[23]

Lower (3) and upper (4) tangent cones constitute a final achievement of Peano's investigations started in *Applicazioni Geometriche* [64, (1887)], where the lower tangent cone was already defined explicitly as in (3), while the upper tangent cone was implicitly used in [64, (1887)] in the proof of necessary optimality conditions, and explicitly defined in *Formulaire Mathématique* [76, p. 296] of 1902–3 and in *Formulario Mathematico* [77, (1908) p. 331] as in (4). Apart from [64, (1887)] and [77, (1908)] Peano studies and uses tangency concepts in several other works: *Teoremi su massimi e minimi geometrici e su normali a curve e superficie*} [67, (1888)], *Sopra alcune curve singolari* [69, (1890)], *Elementi di calcolo geometrico* [72, (1891)], *Lezioni di analisi infinitesimale* [74, (1893)] and *Saggio di calcolo geometrico* [75, (1895–96)].

Following this list we will trace the development of his ideas on tangency, describing not only definitions and properties, but also his methods, calculus rules and applications.

Peano managed to maintain exceptional coherence and precision during a quarter of century of investigations on various and changing aspects of tangency. Only a particular care, with which we perused his work, enabled us to discern a couple of slight variations in the definitions, which, however, did not induce Peano to any erroneous statement. For instance, Peano gives an *intrinsic definition* of tangent straight line to a curve, and also another definition that is the tangent vector to the function representing that curve. He underlines that the two notions are slightly different [77, (1908), p. 332 (see properties P69.4, P70.1)]

In *Applicazioni Geometriche* [64, (1887)], after having presented elements of the geometric calculus of Grassmann (point, vector, bi-vector, tri-vector[24], scalar product and linear operations on them), Peano defines limits of points and vector-type objects (vectors, bi-vectors, tri-vectors) and proves the continuity and differentiability of the operations of addition, scalar multiplication, scalar product and products of vectors (see pages 39–56 of *Applicazioni Geometriche*).

Moreover he defines limits of straight lines and of planes. Straight lines and planes are seen by Peano as sets of points, so that their limits are instances of a general concept of convergence of variable sets: the *lower limit* (3). Accordingly, a variable straight line (a variable plane) A_t converges to a straight line (plane) A as a parameter t tends

[22]On reporting this discovery to his teacher Genocchi, Peano (24 years old) learned with disappointment that Genocchi was already informed by Schwarz about the defect of Serret's definition in 1882 (see [51, p. 9]).

[23]⟦One cannot define the area of a curved surface as the limit of the area of an inscribed polyhedral surface, because the faces of the polyhedron do not necessarily tend to the tangent planes of that surface.⟧

[24]A bi-vector is the exterior product of 2 vectors, a tri-vector is the exterior product of 3 vectors. Vectors, bi-vectors and tri-vectors are used by Peano in 1888 in replacement of the corresponding terms of segment, area and volume adopted in *Applicazioni Geometriche* [64, (1887)].

to some finite or infinite quantity, if

$$A \subset \mathrm{Li}_t A_t, \tag{13}$$

that is, if the distance $d(x, A_t)$ converges to 0 for each $x \in A$. Then he checks meticulously (without using coordinates) the continuity of various relations involving points, straight lines and planes. For instance,

(*i*) A variable straight line L_t converges to a straight line L if and only if for two distinct points $x, y \in L$ the distances of L_t to x and y tend to 0.
(*ii*) A variable plane P_t converges to a plane P if and only if for non-colinear points $x, y, z \in P$ the distances of P_t to x, y and z tend to 0.
(*iii*) If two variable straight lines L_t and M_t converge to the non-parallel straight lines L and M, respectively, then the straight line N_t which meets perpendicularly both L_t and M_t, converge to the straight line N which meets perpendicularly both L and M.

In *Applicazioni Geometriche* [64, (1887), p. 58] Peano defines

Definition 4.1. A *tangent straight line* of a *curve* C at a point $x \in C$ is the limit of the straight line passing through x and another point $y \in C$ as y tends to x.

For Peano, a *curve* C is a subset of Euclidean space such that C is homeomorphic to an interval I of the real line, so that $C = \{C(t) : t \in I\}$ can be seen as depending on a parameter $t \in I$. He gives a description of the tangent straight line in the case where the derivatives $C^{(k)}(\hat{t})$ are null for $k < p$ and $C^{(p)}(\hat{t}) \neq 0$. Moreover,

Proposition 4.2 ([64, (1887), Teorema II, p. 59]). *If C is continuously differentiable and $C'(\hat{t}) \neq 0$, then the tangent straight line L is the limit of the lines passing through $x, y \in C$ as x, y tend to $C(\hat{t})$ and $x \neq y$.*

Notice that Proposition 4.2 makes transparent the relation between paratangency and the continuity of derivative (see (21) for a sequential description of paratangent vector). Paratangency to curves and surfaces was used by Peano also in other instances in *Applicazioni geometriche* [64, (1887), p. 163, 181–184] to evaluate the infinitesimal quotient of the length of an arc and its segment or its projection.

After a study of mutual positions of a curve and its tangent straight lines, Peano gives rules for calculating the tangent straight line to the graph of a function of one variable and to a *curve* given by an equation $f(x, y) = 0$, or by two equations

$$f(x, y, z) = 0 \quad \text{and} \quad g(x, y, z) = 0,$$

for which he needs the implicit function theorem. Incidentally, he presented, for the first time in 1884 in a book form [28], the implicit function theorem proved by Dini in 1877–78 in his lectures [17, pp. 153–207] and provided a new proof, much shorter than the original demonstration of Dini.

Peano gave numerous examples of application of these calculus rules, among others, to parabolas of arbitrary order, logarithmic curve, Archimedean spiral, logarithmic spiral, concoids (e.g., limacon of Pascal, cardioid), cissoids (e.g., lemniscate).

Successively Peano defines

Definition 4.3. A *tangent plane* to a surface S at a given point $x \in S$ is the plane α such that the acute angle between α and each straight line passing through x and another point $y \in S$ tends to 0 as y tends to x.

A *surface* is assumed to be a subset (of Euclidean space) homeomorphic to a rectangle. Several properties of tangent planes are then proved intrinsically, by geometric calculus, without the use of coordinates or parametric representations.

He also calculates intrinsically the tangent planes of many classical surfaces, like cones, cylinders and revolution figures, and more generally, surfaces obtained by a rigid movement of a curve. As he did before with curves, Peano calculates tangent planes to the graphs of functions of two variables as well as to surfaces given by equations and parametrizations. As for curves, he gives analytic criteria on the position of a surface with respect to its tangent planes.

The novelty does not consist of a description of particular cases of tangency, but of the precision and the refinement of the analysis of conditions that are necessary for tangency, which characterize the methods of geometric calculus.

5. Remarks on relationship between tangency and differentiability

Most sophisticated examples of calculation of tangent planes come from geometric operations, like geometric loci (described in terms of distance functions from points, straight lines and planes). They are based on the notion of differentiability introduced by Peano (called nowadays *Fréchet differentiability*). An essential tool is the following theorem on differentiability of distance functions[25].

Theorem 5.1 ([64, (1887), pp. 139–140]). *Let F be a subset of the Euclidean space X such that there exists a continuous function $\gamma : X \to F$ so that $d(x, \gamma(x)) = d(x, F)$. Then the distance function $x \mapsto d(x, F)$ is differentiable at each point $\hat{x} \notin F$ and the derivative is equal to $\frac{\hat{x}-\gamma(\hat{x})}{\|\hat{x}-\gamma(\hat{x})\|}$.*

Finally, in the last chapter of *Applicazioni Geometriche*, Peano introduces lower affine tangent cone of an arbitrary subset of the Euclidean space X [64, (1887), p. 305]. The *lower affine tangent cone* $\mathrm{tang}(F, x)$ of F at x (for arbitrary $x \in X$) is given by the blowup

$$\mathrm{tang}(F, x) = \mathrm{Li}_{h\to+\infty} \left(x + h(F - x)\right), ^{26} \tag{14}$$

hence, by (5)

$$y \in \mathrm{tang}(F, x) \Longleftrightarrow \lim_{t\to 0^+} \tfrac{1}{t} d(x + t(y - x), F) = 0. \tag{15}$$

Peano claims that $\mathrm{tang}(F, x)$ "generalizes" the tangent straight line of a curve and the tangent plane of a surface. Actually, there is a discrepancy between (15) and

[25] A detailed study of regularity of distance function was carried out for the first time by Federer in [23, (1959)].

[26] Observe that the lower affine tangent cone is an affine version of the lower tangent cone, since $\mathrm{tang}(F, x) = x + \mathrm{Tan}^-(F, x)$.

Definitions 4.1 and 4.3, because the tangent defined above is a cone that need not be a straight line (resp. a plane).[27]

Tangency was principally used by Peano for the search of maxima and minima with the aid of necessary conditions of optimality. Many of optimization problems considered in *Applicazioni Geometriche* are inspired by geometry, for example: "Find a point that minimizes the sum of the distances from given three points" [64, (1887), p. 148].

Necessary optimality conditions (see Theorem 5.2 below) given in *Applicazioni Geometriche*, reappear in *Formulario Mathematico* formulated with the aid of the upper affine tangent cone. The *upper affine tangent cone* is defined by the blowup

$$\operatorname{Tang}(F,x) = \operatorname{Ls}_{h\to+\infty}\left(x + h(F-x)\right).^{28} \tag{16}$$

Hence, by (6),

$$y \in \operatorname{Tang}(F,x) \Longleftrightarrow \liminf_{t\to 0^+} \tfrac{1}{t} d(x+t(y-x),F) = 0. \tag{17}$$

Theorem 5.2 (Peano's Regula). *If $f\colon \mathbb{R}^n \to \mathbb{R}$ is differentiable at $x \in A \subset \mathbb{R}^n$ and $f(x) = \max\{f(y) : y \in A\}$, then*

$$\langle Df(x), y-x\rangle \le 0 \quad \textit{for each } y \in \operatorname{Tang}(A,x), \tag{18}$$

where $Df(x)$ denotes the gradient of f at x.

This theorem was formulated in *Formulario Mathematico* [77, (1908), p. 335] exactly as above, but was proved informally already in *Applicazioni Geometriche* [64, (1908), p. 143–144] (without an explicit definition of the upper affine tangent cone). Condition (18) is best possible in the following sense:[29]

$$\{Df(x) : f \text{ is differentiable at } x \text{ and } \max_A f = f(x)\} = \operatorname{Nor}(A,x), \tag{19}$$

where the usual *normal cone* (defined by Federer [23] in 1959) is

$$\operatorname{Nor}(A,x) := \{w \in \mathbb{R}^n : \langle w, y-x\rangle \le 0 \text{ for each } y \in \operatorname{Tang}(A,x)\}.$$

The equivalence of differentiability and of the existence of tangent straight line was considered as evident from the very beginning of infinitesimal calculus.

In case of functions of several variables however relationship between differentiability and tangency remained vague, partly because the very notion of tangency was imprecise.

[27]If $F := \{(x,y) \in \mathbb{R}^2 : y = \sqrt{|x|}\}$, then the tangent straight line to F at the origin in the sense of Definition 4.1 is $\{(x,y) \in \mathbb{R}^2 : x = 0\}$, while $\operatorname{tang}(F,(0,0)) = \{(x,y) \in \mathbb{R}^2 : x = 0, y \ge 0\}$.

[28]Observe that the upper affine tangent cone is an affine version of the upper tangent cone, since $\operatorname{Tang}(F,x) = x + \operatorname{Tan}^+(F,x)$.

[29]Indeed, if $w \in \operatorname{Nor}(A,x)$ then we define $f : \mathbb{R}^n \to \mathbb{R}$ as follows $f(y) = \langle w, y\rangle$ for each y with the exception of $y \in A \cap \{y : \langle w, y-x\rangle \ge 0\}$, for which $f(y) = f(x)$.

Ways to a definition of tangency were disseminated with pitfalls as witness several unsuccessful attempts. For instance, Cauchy confused partial differentiability and differentiability, that is, the existence of total differential[30]. Thomae was the first to distinguish the two concepts in [99, (1875), p. 36] by supplying simple counter-examples.

Differentiability of a function of several variables was defined by Peano in [64, (1887)], as it is defined today under the name of *Fréchet differentiability* and reappears in his *Formulario Mathematico* in [77, (1908) p. 330]. With the exception of [72, (1891), p. 39], where he observes that the existence of total differential could be taken as a definition of differentiability, Peano uses, in numerous applications, the continuity of partial derivatives, which amounts to strict differentiability. He notices in [73, (1892)] that strict differentiability is equivalent to the uniform convergence of the difference quotient to the derivative, as he also does in an epistolary exchange (see [62, (1884)] and [63, (1884)]), concerning the hypotheses of the mean value theorem in the book of Jordan [49, (1882)][31]. The idea of strict differentiability is extended by Peano in a spectacular way to the theory of differentiation of measures (see Greco, Mazzucchi and Pagani [37] for details).

Peano criticizes various existent definitions of tangency [77, (1908) p. 333]:

> Plure Auctore sume ce proprietate ut definitione. ≪Plano tangente ad superficie in suo puncto p≫ es definito ut ≪plano que contine recta tangente in p ad omni curva, descripto in superficie, et que i trans p≫.[32]

As counter-examples to this definition, Peano quotes a logarithmic spiral at its pole[33] and a loxodrome at its poles. He continues

> Aliquo Auctore corrige præcedente, et voca plano tangente ≪plano que contine tangente ad dicto curvas, que habe tangente≫[34]

He constructs a counter-example[35] to this definition that was adopted, among others, by Serret [90, p. 370]. Bertrand, one of the most famous and influential French

[30]Also the relation between separate and joint continuity was elucidated long after erroneous claims of Cauchy in 1821 in [12]. A classical example of function of two variables that is separately continuous but not continuous was provided by Peano in [28, (1884) p. 173]: $(x, y) \mapsto xy/(x^2 + y^2)$.

[31]Peano points out that it is enough to assume differentiability, and not continuous differentiability as did Jordan and Cauchy.

[32]⟦Several authors take this property as a definition: ≪a tangent plane to a surface at its point p≫ is defined as ≪a plane that contains the tangent straight line at p of every curve traced on the surface and passing through p≫.⟧

[33]called also a *miraculous spiral* (*spira mirabile* in latino sine flexione), after the Latin name *spira mirabilis* given to it by J. Bernoulli, that is, a curve described in polar coordinates (r, θ) by $r = ae^{b\theta}$. The pole is the origin of $\mathbb{R}^2$.

[34]⟦Other authors correct the preceding [definition], and call a tangent plane ≪the plane that contains the tangent to those [said] curves that have a tangent [straight line]≫.⟧

[35]By rotating around the x-axis in the space of (x, y, z), the function

$$y = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases}$$

that he had introduced. Recall that at that epoch, a *curve* is assumed to be continuous.

mathematicians of 19th century, writes in [5, (1865), p. 15]

> Le plan tangent d'une surface en un point est le plan qui, en ce point, contient les tangentes à toutes les courbes tracées sur la surface.[36]

The literature abounds with observations, mostly in view of didactic use, on the relation between the notion of tangent plane at the graph of the function

$$z = f(x, y) \tag{20}$$

and the differentiability of f at interior points of the domain of f. For example, in [24, (1911)] Fréchet observes[37]

> Une fonction $f(x, y)$ a une différentielle à mon sens au point (x_0, y_0), si la surface $z = f(x, y)$ admet en ce point un plan tangent unique non parallèle à Oz: $z - z_0 = p(x - x_0) + q(y - y_0)$. Et alors cette différentielle est par définition l'expression
>
> $$p\Delta x + q\Delta y, \tag{*}$$
>
> où Δx, Δy sont des accroissements arbitraires de x, y[...] La forme analytique de cette définition est la suivante: [...] Une fontion $f(x, y)$ admet une différentielle à mon sens au point (x_0, y_0) s'il existe une fonction linéaire et homogène (*) des accroissements, qui ne diffère de l'accroissement Δf [...] que d'un infiniment petit par rapport á l'écart Δ des points (x_0, y_0), $(x_0 + \Delta x, y_0 + \Delta y)$,[38]

Surely, this definition would be certainly more precise if Fréchet had defined his concept of tangency[39].

Wilkosz characterizes in [106, (1921)] differentiability in terms of non-vertical tangent half-lines that form a single plane and are uniform limits of the corresponding secants.

[36]⟦The tangent plane of a surface at a point is the plane that, at this point, includes all the tangent lines to all the curves drawn on the surface.⟧

[37]⟦A function $f(x, y)$ has a differential in my sense at (x_0, y_0), if the surface $z = f(x, y)$ admits at this point a unique tangent plane non-parallel to Oz: $z - z_0 = p(x - x_0) + q(y - y_0)$. Then this differential is, by definition,

$$p\Delta x + q\Delta y, \tag{*}$$

where Δx, Δy are arbitrary increments of x, y. [...] The analytic form of this definition is the following: [...] A function $f(x, y)$ has a differential in my sense at (x_0, y_0) if there exists a linear homogeneous function (*) of increments that differs from Δf [...] by an infinitesimal with respect to the distance Δ of the points (x_0, y_0), $(x_0 + \Delta x, y_0 + \Delta y)$.⟧

[38]Fréchet forgets that in order that a tangent plane imply differentiability, it is necessary to assume the continuity of f at (x_0, y_0).

[39]In [27, (1964), p. 189] Fréchet gives the following definition of the tangent plane that slightly differs from that of Bertrand:

> Précisons d'abord que nous entendons par plan tangent à [une surface] S au point (a, b, c) un plan qui soit lieu des tangentes aux courbes situées sur S et passant par ce point (s'entendant de celles de ces courbes qui ont effectivement une tangente en ce point).

⟦Let us first make precise that by tangent plane to [a surface] S at a point (a, b, c), we mean a plane that is the locus of tangent lines to the curves lying on S and passing through this point (that is, to those curves that have effectively a tangent line at that point).⟧

It is notable that he acknowledges Stolz and Peano as creators of the notion of total differential.

Saks defines in [86, (1933)] differentiability as the existence of a tangent plane at (20) in the sense of Definition 4.3. Consequently, a tangent plane of Saks can contain vertical lines.

Tonelli defines in [100, (1940)] differentiability as the existence of a tangent plane in the sense of Definition 4.3, provided that the orthogonal projection of (20) on the tangent plane is open at the point of tangency. His notion of differentiability coincides with the modern concept of differentiability.

6. Characterizations of differentiability

Guareschi and Severi characterized differentiability in terms of tangency of their graphs (for functions defined on subsets of Euclidean space). At the same period also Bouligand studied tangency, but his perception of the relationship between differentiability and tangent cones remained vague [9, (1932), pp. 68–71].

Guareschi and Severi stress that the originality of their approach consists in defining a *total differential* of a function f defined on an *arbitrary* subset A of Euclidean space at an accumulation point of A. Consequently, their definition cannot hinge on traditional *partial derivatives*. In [40, (1934)], Guareschi, using a notion of *tangent figure* of Severi [96, 92, (1929, 1931)], introduces a *linear tangent space* in order to characterize existence and uniqueness of total differentials. Both refer to the notion of differentiability of Stolz [97, (1893)].

The *tangent figure* of Severi is defined (only at accumulation points) as the union of all tangent half-lines (that he called *semi-tangents*), in the same way as Saks describes in [87, (1933), p. 262] the Bouligand *contingent cone* [9]. As observed in [20, p. 501], Severi's tangent figure is precisely the *upper tangent cone* (4) of Peano; as we have already noted, although Severi cites Bouligand and Saks, he never quotes Peano (see footnote 14). Nevertheless in [94, p. 23 (footnote)] Severi writes in 1949

> [...] nostro grande logico matematico Giuseppe Peano, che fu mio maestro ed amico e della cui intuizione conobbi tutta la forza.[40]

As we mentioned, neither Guareschi cited Peano. He however did not forget to send the following telegram on the 70th birthday of Peano.

> Esprimo illustre scienziato ammirazione augurio lunga feconda attività.[41]

Guareschi [40, (1934), p. 177] reformulates the Severi's definition of *upper affine tangent cone* with the aid of conical neighborhoods. If $\hat{x}$ is a point and h is a nonzero vector of Euclidean space, then a *conical neighborhood* $C(\hat{x}, h, r, \alpha)$ of a half-line, starting at $\hat{x}$ in the direction h, is the intersection of a sphere (of a radius $r > 0$) centered at $\hat{x}$ with a revolution cone of solid angle α around the axis h. A half-line

[40] ⟦[...] our great logician and mathematician Giuseppe Peano, who was my mentor and friend, of whose intuition I knew all the strength.⟧

[41] ⟦I express, illustrious scientist, admiration [and] wishes of long [and] fertile activity.⟧

at $\hat{x}$ in the direction h is *tangent* to A at $\hat{x}$ if and only if $C(\hat{x}, h, r, \alpha) \cap A \setminus \{\hat{x}\} \neq \varnothing$ for every $r > 0$ and $\alpha > 0$.

In fact, this definition had been already given by Cassina in [11, (1930)]. Cassina presented it as an alternative description of the *lower tangent cone* (3) from *Applicazioni Geometriche*; Cassina's definition is however equivalent to the *upper tangent cone* (4), for which Cassina proves the following new fact[42] that includes a later result of Severi [92, (1931)].

Theorem 6.1 (Cassina [11, (1930)]). *There exists a tangent half-line of A at $\hat{x}$ if and only if $\hat{x}$ is an accumulation point of A.*

Guareschi's characterization of differentiability is as follows. By $\mathrm{graph}(f)$ we denote the *graph* of a function $f : A \to \mathbb{R}$, where $A \subset \mathbb{R}^n$. Of course, a hyperplane H in $\mathbb{R}^n \times \mathbb{R}$ is a graph of an affine function from $\mathbb{R}^n$ to $\mathbb{R}$, whenever H does not include vertical lines.

Theorem 6.2 (Guareschi [40, (1934), p. 181]). *Let $A \subset \mathbb{R}^n$ and let $\hat{x} \in A$ be an accumulation point of A. A function $f : A \to \mathbb{R}$, continuous at $\hat{x}$, is differentiable at $\hat{x}$ if and only if $\mathrm{Tan}^+(\mathrm{graph}(f), (\hat{x}, f(\hat{x})))$ is included in a hyperplane without vertical lines.*

The *linear tangent space* of Guareschi at an accumulation point $\hat{x}$ of A is exactly the affine space spanned by the upper affine tangent cone of A at $\hat{x}$; its dimension is called by Guareschi, *accumulation dimension* of A at point $\hat{x}$ [40, (1934), p. 184].

The *total differential* of a function $f : A \to \mathbb{R}$ at an accumulation point $\hat{x}$ of A with $\hat{x} \in A$ is defined as a linear map $L : \mathbb{R}^n \to \mathbb{R}$ such that

$$\lim_{A \ni y \to x} \frac{|f(y) - f(\hat{x}) - L(y - \hat{x})|}{\|y - \hat{x}\|} = 0.$$

Using these notions, Guareschi reformulates Theorem 6.2:

Theorem 6.3 (Guareschi [40, (1934), p. 183]). *Let f be a real function on a subset of Euclidean space of dimension n. If the linear hull of $\mathrm{Tan}^+(\mathrm{graph}(f), (\hat{x}, f(\hat{x})))$ does not include vertical lines, then the following properties hold:*

(*1*) *there exists a total differential of f at $\hat{x}$ if and only if the accumulation dimension of $\mathrm{graph}(f)$ at $(\hat{x}, f(\hat{x}))$ is not greater than n;*

(*2*) *a total differential of f at $\hat{x}$ is unique if and only if the accumulation dimension of $\mathrm{graph}(f)$ at $(\hat{x}, f(\hat{x}))$ is n.*

Therefore there is a one to one correspondence between total differentials and hyperplanes without vertical lines that include the tangent figure $\mathrm{Tan}^+(\mathrm{graph}(f), (\hat{x}, f(\hat{x})))$.

Severi presented the paper [40, (1934)] of Guareschi to the *Reale Accademia d'Italia* on the 10th November 1933, having suggested to the author several simplifications

[42] We regret to have forgot to cite in [19] this contribution of Cassina, which is parallel to those of Bouligand and Severi.

and generalizations. Subsequently, Severi reconsidered the topic in [93, (1934)] and extended the results of Guareschi; he presented in a clear way the ideas of Guareschi, which originally were introduced with complex technicalities.

The differentiability results of [93, (1934)] can be restated (and partially reinforced) in the following, more modern way.

Theorem 6.4 (Severi-Guareschi). *Let $f : A \to \mathbb{R}^k$ where $A \subset \mathbb{R}^m$, and let $\hat{x} \in A$ be an accumulation point of A. Let $L : \mathbb{R}^m \to \mathbb{R}^k$ be a linear map. Then the following properties are equivalent:*

(*1*) *f is differentiable at $\hat{x}$ and L is a total differential of f at $\hat{x}$;*
(*2*) *f is continuous at $\hat{x}$ and $\mathrm{Tan}^+(\mathrm{graph}(f), (\hat{x}, f(\hat{x}))) \subset \mathrm{graph}(L)$;*
(*3*) *$\lim_n \frac{f(x_n)-f(\hat{x})}{\|x_n-\hat{x}\|} = L(v)$ for each $v \in \mathbb{R}^m$ and for every sequences $\{x_n\}_n \subset A$ such that $\lim_n x_n = \hat{x}$ and $\lim_n \frac{x_n-\hat{x}}{\|x_n-\hat{x}\|} = v$;*
(*4*) *$L(v) = \lim_{\substack{w\to v\\ t\to 0^+}} \frac{f(\hat{x}+tw)-f(\hat{x})}{t}$ for every $v \in \mathrm{Tan}^+(A, \hat{x})$.*

Condition (2) of the theorem above encompasses Theorem 6.3. Condition (3) corresponds to [93, (1934), pp. 183–184] of Severi. Condition (4) represents the total differential in terms of the directional derivatives along tangent vectors [93, (1934), p. 186], called *perfect derivatives* by Guareschi [40, (1934) p. 201]. These derivatives are usually formulated in terms of (just mentioned) conical neighborhoods, and called *Hadamard derivatives.*[43]

Another condition equivalent to those of Theorem 6.4 turns out to be very instrumental in effective calculus of total differential[44].

Proposition 6.5 (Cyrenian Lemma). *A function f is differentiable at $\hat{x}$ and L is a total differential of f at $\hat{x}$ if and only if $\lim_n \frac{f(x_n)-f(\hat{x})}{\lambda_n} = L(v)$ for each $v \in \mathbb{R}^m$ and for every sequences $\{x_n\}_n \subset A$ and $\{\lambda_n\}_n \subset \mathbb{R}_{++}$ such that $\lim_n \lambda_n = 0, \lim_n x_n = \hat{x}$ and $\lim_n \frac{x_n-\hat{x}}{\lambda_n} = v$.*[45]

[43]In spite of our efforts, we were unable to find these derivatives in Hadamard's papers. The reference [45, (1923)] usually mentioned in this context does not contain any pertinent fact.

[44]Because of his pedagogical experience, in which the condition was frequently of great help, the second author named it the *Cyrenian Lemma*, referring to Simon of Cyrene who helped to carry the Christ's cross.

[45]As an instance of its usefulness, let us calculate the total differential at $(0,0)$ of

$$f(x,y) := x + y + \sqrt[2]{y^3(x-y)^3}, \qquad \mathrm{dom}\, f := \left\{(x,y) \in \mathbb{R}^2 : y^3(x-y)^3 \geq 0\right\},$$

that was calculated (over several pages) by Guareschi in [40, (1934), p. 190–194]. In fact if $\lambda_n \to 0^+, \mathrm{dom}\, f \ni (x_n, y_n) \to (0,0)$ and $\frac{1}{\lambda_n}[(x_n,y_n)-(0,0)] \to (v,w)$, then the function $L : \mathbb{R}^2 \to \mathbb{R}$ is well defined by

$$L(v,w) := \lim_n \frac{1}{\lambda_n}[f(x_n,y_n) - f(0,0)] = \lim_n \left(\frac{x_n}{\lambda_n} + \frac{y_n}{\lambda_n} + \sqrt[2]{\frac{y_n^2}{\lambda_n^2} y_n (x_n-y_n)^3}\right) = v + w.$$

ant it is linear. Hence, by *Cyrenian Lemma*, L is a total differential of f at $(0,0)$.

Theorem 6.4 reformulates certain ingredients of the characterizations above in a (hopefully) comprehensive way. For instance, the non-verticality condition is incorporated in each of the conditions (*2*)–(*4*). It is worthwhile to make explicit the particular case of differentiability at interior points of the domain.

Proposition 6.6. *Let $A \subset \mathbb{R}^m$ and let $\hat{x} \in \operatorname{int} A$. A map $f : A \to \mathbb{R}^k$ is differentiable at $\hat{x}$ if and only if*

(*1*) *f is continuous at $\hat{x}$;*
(*2*) *For each $v \in \mathbb{R}^m$ the directional derivative $\frac{\partial f}{\partial v}(\hat{x})$ exists and is linear in v;*
(*3*) *$\mathrm{Tan}^+(f, (\hat{x}, f(\hat{x}))$ is a vector space of dimension m.*

Observe that Condition (*2*) is usually referred to as *Gâteaux differentiability.* In Proposition 6.6 above none of the three conditions can be dropped.

Example 6.7. Let $m := 2,\ k := 1,\ A := \mathbb{R}^2,\ \hat{x} := (0, 0)$.

(1) $f(x, y) := 1$ if $y = x^2 \neq 0$, 0 otherwise, fulfills (*2*) and (*3*) but does not fulfill (*1*).
(2) $f(x, y) := \sqrt[3]{x}$ fulfills (*1*) and (*3*) but not (*2*).
(3) $f(x, y) := x$ if $y = x^2$, 0 otherwise, fulfills (*1*), (*2*) but not (*3*).

7. Characterizations of strict differentiability

Till the installation of the today concept of differentiability, the continuity of partial derivatives had been used to affirm the existence of total differential. As it turned out that this condition is sufficient but not necessary, Severi wanted to find an additional property of the total differential corresponding to the continuity of partial derivatives. He discovered that, for the internal points of the domain, *strict differentiability* (2) (that Severi calls *hyperdifferentiability*) was such a property, the fact recognized by Peano already in 1884 for the functions of one variable in [62, 63], and presented later in [73, (1892)] as an alternative to usual *differentiability.*

Theorem 7.1 (Severi [93, (1934)]). *If A is open, then $f \in C^1(A)$ if and only if f is strictly differentiable at every point of A.*

The next step of Severi was to characterize *strict differentiability* geometrically for functions with arbitrary (closed) domains. This task was carried out with the aid of a new concept of tangency, following the same scheme of geometric characterization of *differentiability*, on replacing the role of *tangent half-lines* by *improper chords.* Bouligand gave these interrelations in [9, (1932), pp. 68–71, 87] (in the special case where the domain is the Euclidean plane) without furnishing any precise and complete mathematical formulation[46].

[46]Bouligand says in in [9, (1932), p. 87]

> De même que l'hypothèse : réduction du contingent à un plan pour la surface $z = f(x, y)$, correspond à la différentielle prise au sens de Stolz, de même l'hypothèse : réduction du paratingent à un plan pour la surface $z = f(x, y)$, correspond à la différentielle au sens classique, la fonction f ayant des dérivées partielles continues.

⟦As the hypothesis of reduction of the contingent to a plane for the surface $z = f(x, y)$ corresponds

A linear map $L : \mathbb{R}^m \to \mathbb{R}^n$ is a *total strict differential* of f at an accumulation point $\hat{x}$ of $\mathrm{dom}(f) \subset \mathbb{R}^m$ provided that $\hat{x} \in \mathrm{dom}(f)$ and

$$\lim_{x \neq y, x, y \to \hat{x}} \frac{f(y) - f(x) - L(y - x)}{\|y - x\|} = 0.$$

Severi provides examples of functions that admit multiple total differentials and a unique total strict differential[47]. In order to give a geometric interpretation of total strict differential, Severi makes use of improper chords, that were also introduced independently by Bouligand [6, 7, (1928, 1930)] and called by him *paratingents.* Both Severi and Bouligand consider the *upper paratangent cone* (9) as a family of straight lines (paratingents, improper chords). The upper paratangent cone $\mathrm{pTan}^+(F, \hat{x})$ can be characterized in terms of sequences, as follows: a vector $v \in \mathrm{pTan}^+(F, \hat{x})$ whenever there exist $\{t_n\}_n \to 0^+, \{y_n\}_n, \{x\}_n \subset F$ that tend to $\hat{x}$ such that

$$\lim_n \frac{x_n - y_n}{t_n} = v. \tag{21}$$

Following Guareschi [42, (1941), p. 154], the *linear paratangent space* of F at x is defined as the linear hull of the upper paratangent cone of F at $\hat{x}$.

Theorem 7.2 (Severi [93, (1934), p. 189]). *Let $A \subset \mathbb{R}^n$ and $\hat{x} \in A$ be an accumulation point of A. A function $f : A \to \mathbb{R}$, continuous at $\hat{x}$, is strictly differentiable at $\hat{x}$ if and only if* $\mathrm{pTan}^+(\mathrm{graph}(f), (\hat{x}, f(\hat{x})))$ *is included in a hyperplane without vertical lines.*

The *chordal dimension* of Guareschi at an accumulation point $\hat{x}$ of a set F is the dimension of $\mathrm{pTan}^+(F, \hat{x})$.

Theorem 7.3 (Guareschi [42, (1941), p. 161]). *If the linear paratangent space of* $\mathrm{graph}(f)$ *at* $(\hat{x}, f(\hat{x})))$ *does not include vertical lines, then there exists a total strict differential if and only if the chordal dimension of* $\mathrm{graph}(f)$ *at* $(\hat{x}, f(\hat{x}))$ *is not greater than n.*

to the [total] differential taken in the sense of Stolz, the hypothesis of reduction of the paratingent to a plane for the surface $z = f(x, y)$ corresponds to the differential in the classical sense, that is, the function f admits continuous partial derivatives.⟧

[47]For instance [95, (1944), p. 283], let $A := \left\{(x_1, x_2) \in \mathbb{R}^2 : |x_2| \leq x_1^2\right\}$ and $f(x_1, x_2) := 0$ for $(x_1, x_2) \in A$. Then a total differential L of f at $(0, 0)$ fulfills

$$\lim_{A \ni (x_1, x_2) \to (0,0)} \frac{L(x_1, x_2)}{\|(x_1, x_2)\|} = 0,$$

hence $|L(x_1, x_2)| \leq \varepsilon\, |x_1|$ for each $\varepsilon > 0$, showing that every linear form such that $L(x_1, 0) = 0$ is a total differential. A total hyperdifferential L of f at $(0, 0)$ satisfies

$$\lim_{A \ni (y_1, y_2), (x_1, x_2) \to (0,0)} \frac{L(y_1 - x_1, y_2 - x_2)}{\|(y_1 - x_1, y_2 - x_2)\|} = 0.$$

As for every $\varepsilon > 0$ and each (h_1, h_2) there exist $(y_1, y_2), (x_1, x_2) \in A$ and $t > 0$ such that $(th_1, th_2) = (y_1 - x_1, y_2 - x_2)$, we infer that $|L(h_1, h_2)| \leq \varepsilon\, \|(h_1, h_2)\|$, so that $L = 0$ is the only total hyperdifferential of f at $(0, 0)$.

Analogously to Theorem 6.4,

Theorem 7.4 (Severi [93, (1934), p. 190]). *Let $f : A \to \mathbb{R}^k$ where $A \subset \mathbb{R}^m$, and let $\hat{x} \in A$ be an accumulation point of A. Let $L : \mathbb{R}^m \to \mathbb{R}^k$ be a linear map. Then the following properties are equivalent:*

(*1*) f *is strictly differentiable at* $\hat{x}$ *and* L *is a total strict differential of* f *at* $\hat{x}$*;*

(*2*) f *is continuous at* $\hat{x}$ *and* $\mathrm{pTan}^+(\mathrm{graph}(f), (\hat{x}, f(\hat{x}))) \subset \mathrm{graph}(L)$*;*[48]

(*3*) $\lim_n \frac{f(x_n)-f(y_n)}{\|x_n-y_n\|} = L(v)$ *for each* $v \in \mathbb{R}^m$ *and for all sequences* $\{x_n\}_n, \{y_n\}_n \subset A$ *such that* $\lim_n x_n = \hat{x} = \lim_n y_n$, $\lim_n \frac{x_n-y_n}{\|x_n-y_n\|} = v$*;*

(*4*) $L(v) = \lim_{\substack{w\to v, x\to\hat{x}\\ t\to 0^+}} \frac{f(x+tw)-f(x)}{t}$ *for every* $v \in \mathrm{pTan}^+(A, \hat{x})$.

Condition (*3*) and (*4*) can be found in [93, (1934), p. 190] where $L(v)$ fulfilling (*3*) is called by Severi the *directional hyperderivative* of f at $\hat{x}$ along v.

8. Appendix: Turin mathematical community toward Peano

Peano's interest in logic and in international auxiliary languages coincided with his progressive marginalization among Turin mathematicians. His colleagues could not recognize a vital role of Peano's formal language[49] in the development of mathematics, and were opposed to his teaching methods. Occurrence of influence groups hostile to Peano's scientific views led to his deprivation of the course of calculus, thus of his habitual contacts with students. Local denigration however did not affect Peano's worldwide reputation. He continued to receive highest national distinctions[50]. Eminent scientists continued to value him very highly (Appendix 9). Nevertheless the persistence of anti-Peano ambience during his last years, and also for half a century or so after his death, inescapably left its traces.

Tricomi (1897–1978) joined the faculty of the University of Turin in 1925. His candidature was strongly supported by Peano's group and opposed by the group of Corrado Segre (see Tricomi [102, (1967), pp. 18–19]). Here we reproduce a postcard (and its English translation[51]) sent by Tricomi to Peano on the 9th of March 1924.

[48]This condition does not appear in Severi, but we evoke it for the sake of comparison with Theorem 6.4.

[49]which, among other things, enabled Peano to discover the axiom of choice.

[50]In 1921 the government promoted Peano to *Commendatore* of the Crown of Italy (see Kennedy [51, (2006), p. 215]).

[51]⟦Most illustrious professor Giuseppe Peano, of the Royal University of Turin, 4, Barbaroux Street. Rome, 9th of March 1924

Illustrious Professor, At the same time that I warmly thank you for the cordial reception that you wanted to reserve to me [during my visit in Turin], I have the honour to communicate to you that during the yesterday meeting of our seminar I spoke to inform the audience about the conversation, which I was fortunate to have with you on the so called Zermelo postulate. By the way, I read the passage of your work from the volume 37 of *Mathematische Annalen* that refers to it, and I had an impression that all the present were struck by the fact that, eighteen years before the memoir of Zermelo, you had already formulated, in the very terms that we use today, the axiom of choice. Moreover Dr. Zariski, who studies here with acuity these things, considered the bibliographical indications that I got from you, and suggested to relaunch the due revendication of the contribution of yours and of your school in this difficult area of mathematics.

Illmo Sig$^{\underline{\text{r}}}$.
Prof. Giuseppe Peano
della R. Università di Torino
Via Barbaroux, 4

Roma, 9 marzo 1924

Illustre Professore,

Nel tempo stesso che vivamente La ringrazio per le cordiali accoglienze che ha voluto farmi costà, mi pregio informarLa che, nella seduta di ieri del nostro Seminario, ho preso la parola per ragguagliare i presenti sulla conversazione che ho avuto la fortuna di avere con Lei, sul così detto postulato di Zermelo.

Fra l'altro ho letto quel passo del Suo lavoro del t. 37 dei Mathem. Annalen che vi si referisce, e mi è parso che tutti i presenti siano rimasti colpiti dal fatto che Ella, diciotto anni prima della Memoria di Zermelo, aveva già formulato, e con le stesse parole che ancora oggidì usiamo, il principio di scelta.

Inoltre il Dr. Zarinschi [*sic*], che con acume si occupa qui di queste cose, ha preso nota delle indicazioni bibliografiche da Lei fornitemi, e si propone di ritornare su questa doverosa rivendicazione del contributo portato da Lei e dalla Sua scuola, in questo difficile campo delle matematiche.

Voglia gradire, Sig$^{\underline{\text{r}}}$ Professore, i più distinti ossequi del

Suo devoto F. Tricomi

In spite of Zariski's awareness of Peano's authorship of the axiom of choice, we have not found any hint of this fact in the writings of Zariski [107, 108, 109, 110, 16, (1924–1926)].

Tricomi exercised considerable influence in Turin mathematical community (and beyond it) till his death. In his writings sarcastic and disdainful opinions on Italian mathematicians [101, 102, (1961, 1967)] are profuse. Tricomi played a decisive role in the discrimination of Peano and used to denigrate Peano and his school also long after Peano's death. As reports in [51, pp. 235–236] Kennedy, the biographer of Peano,

> Even later [after 1966] while President of the Academy of Sciences of Turin, F. G. Tricomi continued to publicly make anti-Peano statements. [...] the continued attacks on his [Peano] reputation thirty five years later [after Peano's death] are inexplicable.

For a long time the ambiance in Turin (and in Italy) was such that many preferred to not to reveal their scientific affiliation with the Peano heritage. Others were simply unaware of the importance of this heritage.

Geymonat (1908–1991), who was graduated in philosophy in 1930 and in mathematics in 1932 with Fubini, and became an assistant of Tricomi, reports in [30, (1986)]:

> Quando nel lontano 1934 mi recai a Vienna per approfondire il neopositivismo di Schlick, portai con me diverse lettere di presentazione (fra le

Please accept the finest homages from your devoted F. Tricomi⟧

> quali anche una di Guido Fubini); esse vennero accolte favorevolmente e valsero a creare subito intorno a me una certa cordialità. Ma, con mia sopresa, ciò che pesò più di tutti a mio vantaggio fu il fatto che nel 1930–1931 io ero stato allievo di Peano. Mi sono permesso di ricordare questo fatto in sé stesso di nessun rilievo a due scopi: 1) per sottolineare l'altissima stima di cui Peano godeva, anche dopo la sua morte, fuori d'Italia; 2) per confessare che purtroppo io pure, come molti altri giovani appena usciti dall'Università di Torino, non mi rendevo conto dell'eccezionale valore dell'uomo di cui tuttavia avevo seguito le lezioni per un intero anno accademico, e col quale avevo avuto tante occasioni per discorrere anche fuori delle aule accademiche.[52]

University of Turin has showed little enthusiasm in commemoration of one of his most illustrious members. Kennedy reports [51, (2006), p. 236]:

> A few months after his death, the faculty of sciences at the university considered the possibility of publishing a selection of his writings and appointed a commission consisting of Carlo Somigliana, Guido Fubini, and F. G. Tricomi, who worked out a project in 1933. The presence of Tricomi on this commission practically guaranteed, however, that nothing would come of the project, and in fact the project was abandoned until after the Second World War when, Tricomi being in the U.S.A., an analogous project was again planned by T. Boggio, G. Ascoli, and A. Terracini. In the meantime the Unione Matematica Italiana [UMI] had decided to publish Peano's work - but delayed so as not to interfere with the plans of the university. The latter, however, abandoned this project in 1956 (Tricomi had in the meantime returned to Turin), so that the UMI then asked Ugo Cassina to propose a project for publishing Peano's works and on 5 October 1956 named a commission consisting of Giovanni Sansone, president of the UMI, A. Terracini, and U. Cassina to make the final selection of works to be published.

The first conference in memory of Peano was organized in 1953 [98] by *Liceo Scientifico* of Cuneo, the capital of the province of birth of Peano.

In 1982 University of Turin organized conference in memory of Peano for the first time (on the 50th anniversary of Peano's death). Kennedy, the biographer of Peano, asked, to no avail, for an invitation [51, (2006), p. IX]. A booklet of the conference proceedings appeared four years later [2, (1986)]In one of the papers [30, (1986), p.

[52]⟦When, in the remote 1934, I went to Vienna to study more thoroughly the neopositivism of Schlick, I carried several recommendation letters (among which that of Guido Fubini); they were favorably received and created certain cheerfulness around me. But, to my surprise, what favored me most by everybody, was the fact that I was Peano's student in 1930–1931. I am quoting this fact, which is insignificant in itself, for two reasons: 1) to stress the highest esteem in which Peano was held abroad, also after his death; 2) to confess that I too, as many other young people graduated from University of Turin, was not aware of the exceptional worthiness of the man, the lessons of whom I attended for a whole academic year, and with whom I had many opportunities to discuss also out of the courses.⟧
Presenting himself as a great expert of Peano's person and works, Geymonat oscillates between clumsy admiration and commiseration of Peano.

12] of [2] Geymonat recalls the following facts[53]:

> Per poter salvare i meriti di Peano nel campo matematico, alcuni avevano cercato di distinguere nettamente due fasi [...]. Nella prima fase Peano sarebbe stato un valente matematico, mentre nella seconda (o fase della decadenza) egli si sarebbe ridotto a occuparsi di logica simbolica, passando poi a problemi linguistici connessi alla ricerca di un linguaggio universale [*sic*] (ricerca già promossa da Leibniz negli anni a cavallo fra il Sei e il Settecento), problemi che egli ritenne di poter risolvere con il suo latino sine flexione [...]. Questa all'incirca fu la tesi sostenuta da Fubini, il suo grande avversario nella Facoltà di Torino, in una conferenza tenuta al Seminario matematico di tale Facoltà, non ricordo più esattamente se poco prima o poco dopo il 1930, comunque mentre Peano era ancora in vita. Ma neanche questa conferenza riuscì a conciliare le due posizioni di Fubini e Peano [...].

Recalling events of that conference in [54, (1982)], Lolli, who graduated with Tricomi in 1965 and became an assistant of Geymonat in 1967, alludes to a *curtain of silence* of the Turin mathematical community around the *embarrassing and bizarre personage who, for about fifty years, disturbed and discomfitted, and in the last thirty years almost dishonored the whole profession*[54]. In his book [55, (1985), p. 8], Lolli qualifies Peano as a *pathetic inventor of symbols* and, in the same book [55, (1985), p. 50], *who made through cowardice the great refusal*[55] in reference to Dante's *Divina Commedia.* [56]

The persistence of anti-Peano ambience in Turin Mathematical Community a half century after Peano's death, was nourished and reinforced by a surprisingly poor knowledge of his works. In [29, 1959] Geymonat, an authoritative member of that community, on the occasion of the edition of Peano's *Selected Works* by Cassina, wrote[57]:

[53]⟦In order to save Peano's merits in the area of mathematics, certain persons tried to distinguish two periods [...]. In the first Peano was a talented mathematician, while in the second (decadence phase) his activity was reduced to symbolic logic, passing to linguistic problems related to a search of a universal language [sic] (the pursuit promoted already by Leibniz between seventeenth and eighteenth centuries), the problems that he pretended to able to solve with his latino sine flexione [...]. This was approximately a thesis defended by Fubini, his great antagonist at the Faculty of Turin, during a talk held at the mathematical Seminar of this faculty about 1930, I do not remember exactly, but in any case when Peano was still alive. But even that talk did not succeed to reconcile the positions of Fubini and Peano [...].⟧

[54]Lolli's words:

> la cortina di silenzio [of the Turin mathematical cummunity around the] [. . .] scomodo e bizzarro personaggio che per circa cinquanta anni aveva disturbato ed imbarazzato, e negli ultimi trenta quasi disonorato la intera professione.

[55]Dante [3, Inferno, Canto III]: "Colui che fece per viltade il gran rifiuto".

[56]Ironically, in 2000 Lolli was recipient of a Peano Prize, sponsored by Department of Mathematics of Turin.

[57]⟦The second volume [of Peano's Selected Works] [. . .] gathers works in mathematical logic [. . .] [and] in interlingua and algebra of grammar. This juxtaposition [. . .] confirms without doubt Cassina's opinion, after which mathematical logic and linguistic research constitute, in Peano, two phases [. . .] of the same grand program designed to realize [. . .] the teaching of Leibniz.
This thesis is of particular importance, because it undermines the legend [*sic*], following which the

> Il II volume [delle Opere Scelte di Peano] [...] raccoglie lavori di logica matematica [...] [e] lavori di interlingua ed algebra della grammatica. L'accostamento [...] conferma in modo incontestabile l'opinione di Cassina, secondo cui logica matematica e ricerche linguistiche costituiscono, in Peano, due fasi [...] di un medesimo grandioso programma volto a realizzare [...] l'insegnamento leibniziano.
>
> La tesi ha una particolare importanza, perché sfata la leggenda [*sic*] secondo cui gli interessi linguistici peaniani sarebbero stati il frutto di una decadenza senile del Nostro.

Multiple contributions of Mangione on the history of logic to the six volumes of Geymonat's *Storia del pensiero filosofico e scientifico* [31, (1971–1973)] indicate persisting poor knowledge of Peano's works. Mangione's contributions, very much appraised by Italian logicians and philosophers, are completely unknown to mathematicians. They were collected in *Storia della logica* [57, (1993)] a few years ago, without any change of attitude with regard to Peano and his School, who are ridiculed therein.

In *La Stampa*, a daily of Turin, in October 1995 R. Spiegler declared that certainly Peano spent some periods in a madhouse. This news without any basis was belied by Lalla Romano, a Peano's great-niece. A mathematician and our colleague asked Spiegler (who is also a mathematician) where he took this absurd information; Spiegler replied that he had learned this from G.-C. Rota who, in turn, was informed by nobody else but Tricomi in person.[58]

More recently University of Turin edited *Opera omnia* [79, (2002)]; Peano is the celebrity whom *Accademia delle Scienze* of Turin put on its home page

`http://www.torinoscienza.it/accademia/home`.

An international congress *Giuseppe Peano e la sua Scuola, fra matematica, logica e interlingua* commemorating the 150th anniversary of Peano's birth and 100th anniversary of Formulario Mathematico took place in Turin in October 2008 at the Academy of Science of Turin and the Archive of State.

Peano's is not the first case of an ostracism against a mathematical precursor. As in other cases, the resulting prejudice is inestimable. And, as a rule, pupils cannot expect a better destiny.

A famous economist Luigi Einaudi (1874–1961), who was a professor of University of Turin before becoming the president[59] of the Italian Republic, witnesses in 1958 [21]:

> Il professor Peano fu vero maestro, sia per l'invenzione di teoremi, che

linguistic interests of Peano would be a fruit of his senile decadence.⟧

[58]Rota wrote in *Indiscrete Thoughts* (Birkäuser, 1997, page 4): "Several outstanding logicians of the twentieth century found shelter in asylums at some time in their lives: Cantor, Zermelo, Gödel, Peano, and Post are some."
Another example of a disdainful attitude toward Peano was the adjectival use of "peanist" rather than of more standard and graceful "peanian". The word "peanist" was introduced by the renowned historian Grattan-Guinness; it evokes the word "opportunist" that was used in a judgement of Grattan-Guinness on Peano's works: "Both in his mathematics and his logic, he [Peano] seems to me to have been an opportunist" [34, (1986)].

[59]from 1948 to 1955.

ritrovati poi da altri, resero famosi gli scopritori, sia per l'universalità del suo genio. Nemmeno a farlo apposta, taluni suoi assistenti ai quali si pronosticava un grande avvenire nel campo matematico, presero tutt'altra via. [...] Vacca [assistente di Peano], divenne [...] professore universitario di lingua e letteratura cinese [...]. [Vailati] nonostante la crescente estimazione in cui era tenuto nel mondo scientifico italiano e straniero, [...] non ottenne la cattedra alla quale doveva aspirare. [...] Così fu che Vailati scomparve dall'orizzonte torinese per girare l'Italia come insegnante nelle scuole medie.[60]

9. Appendix: International mathematical community toward Peano

Despite the depicted ambience at the University of Turin, Peano was held in high esteem by numerous famous scientists also in that period.[61]

Among the letters and telegrams sent to Peano on his 70th birthday are those of Guareschi, Dickstein, Zaremba, Fréchet, Hadamard, Tonelli and Levi-Civita [1, (1928)]].

We include few samples of letters and other signs of recognition around 1930. They are extracted from a [80, (2002)].

A letter from **Benjamin Abram Bernstein** (1881–1964)

University of California, Department of Mathematics, Berkeley, California, Feb. 8, 1928

My dear Professor Peano -

I am anxious to get the Rivista di Matematica v. 1–8, and the Formulaire Mathématique, v. 1–5. I shall appreciate it greatly if you can tell me if these can be still got from the publishers and at what price.

With keen appreciation of your great work in logic, I am,

Sincerely yours, BABernstein.

A letter[62] from **Jan Lukasiewicz** (1878–1956)

[60]⟦Professor Peano was a real master, as for the invention of theorems, which rediscovered later by others, made them famous, as for the universal character of his genius. Not deliberately, several of his assistants, who had great prospects in mathematics, took completely different ways. [...] Vacca, [an assistant of Peano] became [...] a university professor of Chinese language and literature [...]. [Vailati] who despite the growing esteem in which he was held by Italian and foreign scientists [...] did not obtain a professorship, for which he could legitimately pretend. [...] So Vailati disappeared from the Turin horizon to move around Italy as a secondary school teacher.⟧

[61]A writer Lalla Romano (1906–2001), Peano's great-niece describes the atmosphere of Peano's house, where she was a guest (1924–1928) during her unversity studies [83, (1979), p. 8]:

[...] lo zio [Peano] riceveva le visite: studenti, per lo più stranieri - perfino cinesi - ossequiosissimi, dal sorriso esitante, l'inchino a scatto; e scienziati [...] guardavano lo zio con venerazione. Mentre lui, cupo, la barba arruffata, andava avanti e indietro nella stanza, scuotevano la testa.

⟦[...] my uncle [Giuseppe Peano] received visitors: students, mostly foreigners – even Chinese – obsequious, smiling hesitatingly, bowing snappingly; scientists [...] looked at my uncle with veneration. While he, gloomy, with his ruffled beard, walked to and fro, they shaked their heads.⟧

[62]⟦Warsaw, 31.VII.1928

Warszawa, 31.VII.1928

Sehr Geehrter Herr Professor!

Bitte mich vielmals zu entschuldigen, dass ich deutsch schreibe, aber ich verstehe leider nicht soviel italienisch, um mich mit Ihnen in Ihrer Muttersprache zu verständigen.

Ich habe gar nicht gehofft, dass ich an dem Internationalen Kongresse der Mathematiker in Bologna werde teilnehmen können. Nun hat sich mir die Möglichkeit geboten, nach Bologna zu kommen. Ich bitter daher, Herr Professor, wenn es nur irgendwie möglich ist, meine verspätete Anmeldung von Kommunikaten gütigst berücksichtigen zu wollen. Seit Jahren arbeite ich im Gebiete der mathematischen Logik, doch habe ich meine wichtigsten Ergebnisse aus dem Aussagenkalkül und dessen Geschichte bisher nicht veröffentlicht. Es wäre mir sehr lieb, wenn ich meine Resultate gerade in Italien, das so sehr für die mathematische Logik verdient ist, der internationalen Gelehrtenwelt vorlegen könnte.

Sollte es nicht mehr möglich sein, dass ich am Kongresse aktiv teilnehme, so wäre ich für eine Mitteilung darüber sehr dankbar.

Bitte, Herr Professor, den Ausdruck meiner vorzüglichsten Hochachtung entgegenzunehmen

Dr. Jan Lukasiewicz, Professor für Philosophie und gewesener Rektor der Universität Warschau /Polen/.

Adresse: Prof. Dr. J. Lukasiewicz, Warszawa, Brzozowa 12. /Varsovia [*sic*], Polonia/

In a speech at the Congress of Mathematicians in Bologna on the 3rd of September 1928 [47, p. 4] **David Hilbert** (1862–1943) talks about Peano's symbolic language[63]:

> [...] ein wesentliches Hilfsmittel für meine Beweistheorie [ist] die Begriffsschrift; wir verdanken dem Klassiker dieser Begriffsschrift, Peano, die sorgfältigste Pflege und weitgehendste Ausbildung derselben. Die Form in der ich die Begriffsschrift brauche, ist wesentlich diejenige, die Russell zuerst eingeführt hat.[64]

Dear Professor, Please forgive me that I write in German, but unfortunately I do not know that much Italian in order to communicate with you in your mother tong.
I did not expect at all that I would be able to take part in the International Congress of Mathematicians in Bologna. Only now I have this possibility. Therefore, I ask you, if it were still in some way possible to accept a delayed registration of my communications. For years I have been working in the area of mathematical logic, but I have not yet published my most important results on the propositional calculus and its history. I would be delighted if I could present my results in Italy, that has so many merits in mathematical logic, to the international learned audience.
If it were no longer possible that I actively participate in the congress, I would be very grateful for information about it.
Please, accept the expression of my greatest respect.
Dr. Jan Lukasiewicz, Professor Philosophy and a former Rector of Warsaw University /Poland/.
Address: Prof. Dr. J. Lukasiewicz, Brzozowa street, 12, Warsaw, Poland.⟧

[63] Peano did not participate in that Congress because of his brother's death.

[64] ⟦[...] an essential tool for my proof theory is ideography; we owe to the classical author of this ideography, Peano, most thorough care and utmost cultivation of it. The form, in which I use this

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A letter[65] from **Leonida Tonelli** (1885–1946)

Pisa, 12 gennajo [sic] 1931=IX°

Illustre Professore,

Nel corrente anno 1931, gli "Annali della Scuola Normale Superiore" di Pisa assorbiranno gli "Annali delle Università Toscane" e si trasformeranno in un grande periodico internazionale, del tipo degli "Annales de l'École Normale Supérieure" di Parigi. La parte matematica, che accoglierà Memorie e Note di valorosi scienziati italiani e stranieri, si presenterà, ogni anno, con quattro fascicoli, ciascuno di 100 pagine.

La Scuola Normale spera di poterLa annoverare fra i collaboratori degli Annali così rinnovati; ed io, in particolare, sarei molto lieto se potessi inserire un Suo lavoro nei primi fascicoli della nuova serie.

Vuole essere tanto gentile da accontentarmi?

Con anticipati ringraziamenti e molti ossequi.

Suo devotissimo, L. Tonelli

A letter[66] from **Alfred Tarski** (1902–1983)

Warschau, 2.XI.32[67]

Hoch verehrter Herr Professor!

Ich nehme mir die Freiheit, Sie mit einer privaten Angelegenheit zu behelligen. Ich habe nämlich die Aussicht, für das kommende Jahr 1933/4 das Rockefeller-Stipendium für das Studium in Ausland zu bekommen, und würde mich sehr freuen, wenn ich eine Zeit unter Ihrer Führung in Turin arbeiten dürfte. Würden Sie damit einverstanden sein?

In Erwartung Ihrer freundlichen Antwort verbleibe ich inzwischen in vorzüglicher Hochachtung

ideography, is essentially that Russell has first introduced.⟧

[65]⟦Pisa, 12 January 1931=IX°, Illustrious Professor,

During this year 1931 the "Annals of the Scuola Normale Superiore" of Pisa will absorb the Annals of Tuscan universities and will be transformed in a great international periodical, of the type of "Annals of the École Normale Supérieure" of Paris. The mathematical section, that will receive memoirs and notes of excellent Italian and foreign scientists, will appear, each year, in four volumes of 100 pages each.

The Scuola Normale hopes to count you among the collaborators of the so renewed Annals; and I particularly would be very glad if I could include one of your papers in the first volumes of the new series.

Would you be so kind to gratify me? With anticipated thanks and many homages. Your most devoted, L. Tonelli⟧

[66]⟦Warsaw, 2nd of November 1932. Dear Professor, I take freedom to bother you with my personal affairs. I have namely a prospect, for the coming year 1933/4, to obtain the Rockefeller fellowship to study abroad, and would be very glad if I could work sometime in Turin under your supervision. Would you kindly agree to this?

Looking forward to your kind reply, I remain in deep respect.

Dr. A. Tarski, Private Docent at Warsaw University (Poland, Warsaw XXI, Sułkowskiego street 2 app. 5)⟧

[67]Peano died on the 20th of April 1932.

Dr. A. Tarski, Privat-Dozent a.d. Universität Warschau (Polonia, Warszawa XXI, ul. Sułkowskiego 2 m.5)

10. Appendix: Biography of Giacinto Guareschi

We provide a somewhat detailed biography of Guareschi, because it is not available, except a brief mention in Atti dell'Accademia Ligure [81]. *Biographies of other mathematicians referred to in this paper are easily obtainable.*

Giacinto Guareschi (1882–1976) was born in Turin on the 2nd of October 1882. His father, Icilio (1847–1918) was a famous chemist-pharmacologist [13], a member of Accademia delle Scienze of Turin at the same time as Peano. His mother was Anna Maria Pigorini († 1942). Guareschi had a sister Paolina and a brother Pietro (1888–1965), a distinguished chemical engineer, member of *Accademia Ligure.*

Guareschi studies mathematics at the University of Turin graduating in 1904. In a letter of 1932 [104, p. 87] to Vacca, he recalls the importance of Severi and Vacca (assistants of, respectively, D'Ovidio and Peano) for his mathematical education. He was assistant of projective geometry at the University of Turin (1904–1906), and of analytic geometry at the University of Pavia (1907–1910)[68]. In 1910 he obtained a professorship of high school *(liceo)* to voluntarily retire in 1944 in order not to collaborate with, and to avoid to swear faithfulness to the Fascist regime. During his high school teacher carrier, Guareschi served as a principal and was appointed[69] a *provveditore*[70] in July 1936. From November 1936 Guareschi continued to ask to be exempted[71], and, after several refusals, was finally dismissed in 1938.

On the 21st of November 1914 he was enrolled in the army and participated in the First World War. He left the army on 15th of May 1919 with the grade of captain; in 1921 he was granted a commemorative medal of the First World War. In 1931 he was promoted to the grade of major of artillery, and on 11th of June 1940 was enrolled to the army to be demobilized on the 19th of August of the same year with the grade of lieutenant-colonel.

In 1924 Guareschi started pedagogical activity in projective and analytic geometry at the University of Genoa, where he became a *libero docente*[72] of algebra on 13th of

[68] at a suggestion of Berzolari.

[69] by the minister of National Education, without having asked for it. Guareschi was not happy with this nomination, mainly because it interfered with his research (namely, on differentiability and tangency), but could not refuse due to the legal system at that moment. Soon after he realized that the Mussolini government politicized education. In Gareschi's words:

> [il] pagliaccio di Predappio [aveva reso la carica di Provveditore] squisitamente politica

⟦because the clown of Predappio made this position exquisitly political [The reference to Mussolini who was born in Predappio]⟧.
Contrary to Guareschi, Severi is an enthousiastic follower of Mussolini (see Guerraggio-Nastasi in [44, 43, (1993, 2005)].

[70] a provincial responsible of education.

[71] The reason was primarily political, because Guareschi was opposed to the Fascit regime, however he could not openly evoke it, as this would amount to severe persecution.

[72] The title of *libero docente*, granted on the basis of scientific publication, entitled to teach courses at a university.

March 1929. He kept this position till 1952 (when he became 70, which was the legal retirement age). Due to a derogation, he taught at the University of Genoa till 1959.

In 1927 Guareschi was elected a corresponding member of *Accademia* Ligure di Scienze e Lettere (proposed by Loria and Severini) and in 1957 its effective member. In 1956 he and his brother Pietro donated to *Accademia* manuscripts of their father Icilio.

Guareschi married Gemma Venezian (1897–1975). Their only son, Marco, was born on the 21st of March 1922. In 1944 he joined the underground army, which was, in terms used by Guareschi, *la sola via dell'onore* (the only way of honor). On the 11th of April 1944 Marco was arrested[73] and deported to Germany where he died in a concentration camp in April 1945[74]. The pain of Guareschi and his wife was amplified by uncertainty about their son's fate, as, for a couple of years, they did not have reliable information about his passing. Since then Guareschi dedicated himself to promotion to reconstruction of the history of the *Resistenza* (Italian underground army) and to defence of its values; in doing so, he collaborated with several Italian and international associations[75].

The postwar years were extremely difficult for Guareschi and his wife. Guareschi had neither salary nor pension, because he resigned from the public service during the Fascist period. In January 1946 Guareschi wrote

> Io me ne sono andato [dalla scuola] per non servirla [la repubblica fascista] al tempo dell'obbligo del giuramento, e nemmeno ho giurato agli Ufficiali in congedo; né più ho esercitato l'incarico Universitario, sfidando la fame. [...] Sono agli estremi dal lato finanziario; i mesi arretrati [per il pagamento dello stipendio e della pensione] sono ormai 21.[76]

Guareschi successfully applied to be readmitted as a high school professor, because the political nature of his resignation in 1944 was recognized.

After the war Guareschi had various political commitments. In 1945 he became a mayor of a village Serravalle Scrivia (Alessandria). In 1953 he was an unsuccessful candidate (from the lists of PCI[77]) for senator. In recognition of their intense political activity, Giacinto and Gemma Guareschi received a gold medal in 1956. On his retirement from the secondary education, on the 28th of September 1950, three principal newspapers of Genoa (*Il lavoro nuovo*, *Il secolo XIX* and *l'Unità*) published a paper about Guareschi, writing, among other things,

[73]at the *rastrellamento* (sweep) of *Benedicta*, where more than hundred partisans were executed and other 400 arrested. Guareschi reconstructed the event in [GG38, (1951)], which became a basic source for [61, (1967)] of Pansa.

[74]First to Mauthausen, later in August 1944 to Peggau (near Graz) and finally to the so called Russian Camp where he died between 10 and 12 April 1945.

[75]For example, *Istituto storico della Resistenza* in Liguria, ANED (*Associazione nazionale ex deportati*), ANPI (*Associazione nazionale partigiani d'Italia*), ANCR (*Associazione combattenti e reduci*), ANPPIA (*Associazione Nazionale Perseguitati Politici Italiani Antifascisti*), *Consiglio Federativo della Resistenza*, *Conseil Mondial de la Paix.*

[76]⟦I quit [the teaching] in order not to serve [the fascit republic] at the time of obligation of oath of allegiance, nor I swore as an officer in leave; nor I had a university appointment, defying the hunger. [...] Financially I am destitute. The arrears [of wage and pension] are already for 21 months.⟧

[77]Italian Communist Party.

> Inflessibile nei riguardi delle ingerenze del regime fascista nella vita della scuola, durante la lotta contro i nazifascisti ha offerto alla Patria l'unico figlio barbaramente trucidato a Mauthausen.[78]

Giacinto Guareschi died on the 9th of August 1976 in Serravalle Scrivia near Alessandria, in a poor country house, where he lived his last years. Various scholarships, prizes were founded and monuments were erected in memory of Guareschi.

Mathematical interests of Guareschi are principally geometry and algebra, and starting from 1934, differentiability and tangency (see previous Sections 6 and 7) and, finally, characterization of smooth manifolds (see Greco [39] for details). Guareschi's works are reviewed in *Jahrbuch über die Fortschritte de Mathematik* (JFM), in *Zentralblatt Math* (Zbl) and in *Mathematical Reviews* (MR).[79]

Scientific publications of Guareschi cease with the death of his son. Nevertheles his interest for mathematics persists during all his life. In his nineties he collaborates with G. Rizzitelli on the edition of a collection of applications of mathematics, and announces to the secretary of *Accademia Ligure* his intention to publish a paper on algebra. Guareschi wrote 3 books for didactic use [81] and 35 mathematical papers. The following bibliography contains only mathematical papers and 5 writings on the *Resistenza.*

Acknowledgements. For the biographical reconstruction related to Guareschi we are grateful to dott. Paolo Carrega, responsible of the Archive ISRAL, where we could consult the Fondo Guareschi, to dott. Alessandra Baretta of the Historical Archive of the University of Pavia, to Ms. Anna Rapallo and Ms. Maddalena De Mola of the Historical Archive of the University of Genoa, to Ms. Anna Robbiano of the CSBMI of the Faculty of Sciences of the University of Genoa, and to ing. Giovanni Paolo Peloso, the secretary of the Accademia Ligure delle Scienze e Lettere.

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[78]⟦[Guareschi] inflexible with respect to the intrusions in the school life of the Fascist regime, during the fight against Nazifascists, [he] offered to the Fatherland his only son, barbarically slain in Mauthausen.⟧

[79]The reviewers of papers on differentiability and tangency are H. Kneser, O. Haupt, A. B. Brown, H. Busemann, G. Scorza Dragoni, O. Zariski, T. Viola and A. González Domínguez.

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11. Appendix: A chronological list of mathematicians

For reader's convenience, we provide a chronological list of some mathematicians mentioned in the paper, together with biographical sources.

The `html` file with biographies of mathematicians listed below with an asterisk can be attained at University of St Andrews's web-page

`http://www-history.mcs.st-and.ac.uk/history/{Name}.html`

Descartes, René (1596–1650) (*)

Huygens, Christiaan (1629–1695) (*)

Leibniz, Gottfried Wilhelm (1646–1716) (*)

Grassmann, Hermann (1809–1877) (*)

Genocchi, Angelo (1817–1889) (*)

Serret, Joseph (1819–1885) (*)

Bertrand, Joseph L. F. (1822–1900) (*)

Jordan, Camille (1838–1922) (*)

Thomae, Carl J. (1840–1921) (*)

Stolz, Otto (1842–1905) (*)

D'Ovidio, Enrico (1842–1933), see Kennedy [51]

Schwarz, Hermann A. (1843–1921) (*)

Dini, Ulisse (1845–1918) (*)

Klein, Felix (1849–1925) (*)

Dickstein, Samuel (1851–1939) (*)

Peano, Giuseppe (1858–1932) (*), see Kennedy [51]

Hilbert, David (1862–1943) (*)

Loria, Gino B. (1862–1954) (*)

Segre, Corrado (1863–1924) (*)

Hadamard, Jacques S. (1865–1963) (*)

Saks, Stanislaw (1897–1942) (*)

Couturat, Louis (1868–1914) (*)

Hausdorf, Felix (1868–1942) (*)

Zermelo, Ernst (1871–1953) (*)

Severini, Carlo (1872–1951), see Boll. Unione Mat. Ital. 7 (1952) 98–101

Russell, Bertrand (1872–1970) (*)

Levi-Civita, Tullio (1873–1941) (*)

Levi, Beppo (1875–1961), see Kennedy [51]

Vacca, Giovanni (1875–1953) (*)

Boggio, Tommaso (1877–1963) (*)

Fréchet, Maurice (1878–1973) (*)

Lukasiewicz, Jan (1878–1956) (*)

Fubini, Guido (1879–1943) (*)

Severi, Francesco (1879–1961) (*)

Bernstein, Benjamin A. (1881–1964), see Univ. California: In Memoriam (1965)

Guareschi, Giacinto (1882–1976)

Tonelli, Leonida (1885–1946), see Tonelli, Opere Scelte, Cremonese (1963)

Bouligand, Georges (1889–1979), see `http://catalogue.bnf.fr`

Wilkosz, Wiltold (1891–1941), see `http://www.wiw.pl/matematyka/Biogramy`

Ackermann, Wilhelm (1896–1962) (*)

Cassina, Ugo (1897–1964), see Kennedy [51]

Tricomi, Francesco G. (1897–1978) (*)

Saks, Stanislaw (1897–1942) (*)

Zariski, Oscar (1899–1986) (*)

Tarski, Alfred (1902–1983) (*)

Segre, Beniamino (1903–1977) (*)

Geymonat, Ludovico (1908–1991), see `www.torinoscienza.it/accademia/`

Choquet, Gustave (1915–2006), see Gazette des Math. v111 (2007) 74–76

Mangione, Corrado (1930–2009), see `http://dipartimento.filosofia.unimi.it/`

Rota, Gian-Carlo (1932–1999) (*)

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