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Symmetry in Multi-Phase Overdetermined Problems

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In this paper we prove symmetry for a multi-phase overdetermined problem, with nonlinear governing equations. The most simple form of our problem (in the two-phase case) is as follows: For a bounded C^1 domain $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ let u^+ be the Green's function (for the p-Laplace operator) with pole at some interior point (origin, say), and u^- the Green's function in the exterior with pole at infinity. If for some strictly increasing function $F(t)$ (with some growth assumption) the condition $\partial_\nu u^+ = F(\partial_\nu u^-)$ holds on the boundary $\partial\Omega$, then Ω is necessarily a ball. We prove the more general multi-phase analog of this problem.

Keywords: Symmetry, overdetermined problems, multi-phases, viscosity solutions, Green's function

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1. Introduction

In this paper, we study a multi-phase version of a symmetry problem, that has been treated by several mathematicians, in various forms. In overdetermined symmetry problem it is usual to consider the problem in a given domain, involving only one function, and an extra boundary condition; e.g. if the Green's function G of a domain, has the property that $|\nabla G|$ is constant on the boundary, then one expects the domain to be a ball. The proof of this theorem, for C^1 domains follows a simple argument, given in [6]. Recently many new mathematical problems, where there are more than one phase entering into the game, or the physical model, have been introduced. One such example is the so-called multi-phase flows, where several liquids are present. In such problems, there is a different governing equation on the "free boundary", that in general, is a nonlinear equation. More exactly, suppose for a bounded C^1 domain $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ the Green's function u^+ with pole at some interior point (origin, say), and the Green's function u^- of the exterior domain with pole at infinity we have

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the boundary gradient condition

$$\partial_\nu u^+ = F(\partial_\nu u^-) \text{ on } \partial\Omega,$$

where F is a given function, with certain properties, and ν is the inward normal direction. Can we conclude that Ω is a ball? In this paper, we answer this question in the affirmative, for certain functions F. Indeed, we prove a multi-phase version of this problem, see Theorem 4.2, with general governing conditions, and with the p-Laplacian as the bulk equation. We also discuss the viscosity definitions of this problem, and state a similar symmetry result, for which the proof is similar. The advantage of the viscosity definition is that we need not to assume any regularity of the boundary.

The paper is written in the following way. In the next section, we give a general overview of existing results, and indicate difficulties and the nature of the problem, for the one-phase case. Then, we give some exact definitions of the Green's functions for the ball, and write down exact form of these. They will be used in Section 4, where our main theorem is formulated. In the last section, we state the viscosity counterpart of our main theorem.

2. One phase case (Exploratory)

Suppose we are given the overdetermined problem

$$\begin{aligned} \Delta u &= -c_0\delta_0 \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \\ \partial_\nu u &= F(|x|) \text{ on } \partial\Omega, \end{aligned} \tag{1}$$

for some bounded C^1 domain $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$.

A question that has challenged several mathematicians is whether Ω is a ball. A natural follow-up question would then be the uniqueness of the solution. If we replace the boundary normal derivative with $|\nabla u|(x) = F(|x|)$, then the smoothness assumption on the boundary can be relaxed considerably. For example one can allow this boundary gradient condition to hold a.e. on the boundary, and still ask the same question (see [3]). However, in the discussion to follow below, we will only consider C^1 domains.

The departing point of any analysis of this problem would be to set the right conditions on F, so that an appropriate ball can be a solution. Indeed, a simple integration gives

$$\int_{\partial\Omega} F = \int_{\partial\Omega} \partial_\nu u = \int_\Omega -\Delta u = c_0.$$

In particular if Ω is the ball $B_R(0)$ then

$$F(R)\ \omega_n\ R^{n-1} = c_0,$$

where ω_n is the surface area of the unit sphere. This suggests that if for any R we have the above condition fulfilled then the ball $B_R(0)$ and its Green's function would be a solution to our problem. In particular there would be no solution if the condition above is not satisfied for any ball $B_R(0)$. The uniqueness also fails when the above condition holds for more than one R. For example if

$$F(t) := c_0 t^{1-n}/\omega_n$$

then all balls with centers at the origin are solutions to our problem.

In proving that a solution must be a ball, one can use standard argument of scaling and comparison between the solution and the scaled version of it. So let us start with a solution u and the corresponding Ω to our problem (1). Let us first set $c_0 = \omega_n$ for simplicity. Then the requirement that the appropriate ball is a solution is $F(R) = R^{1-n}$ for at least one R. Suppose that this condition fails. Then one can easily see that there cannot be any solution (u, Ω) to our problem. Indeed, suppose $F(R) > R^{1-n}$, for all R. Let us take the smallest ball $B_r(0) \supset \Omega$ and its Green's function G_r. Then by comparison principle $u \le G_r$ in Ω and hence

$$r^{1-n} = \partial_\nu G_r(z) \ge \partial_\nu u(z) = F(r)$$

where z is a touching point between the boundary of Ω and the sphere $|x| = r$. This contradicts that $F(R) > R^{1-n}$ for all R. A similar argument (taking largest ball from inside) also shows the failure of existence when the reverse condition $F(R) < R^{1-n}$ holds.

Let us now look for further conditions that forces solutions to be spherical. Assume we have a solution (u, Ω) and also that (G_r, B_r) is the corresponding ball solution. If $B_r \setminus \Omega \ne \emptyset$ then we may scale so that $B_{tr} \subset \Omega$ and it touches the boundary of Ω at z, and that $t > 1$. Then one can easily show that $v(x) := t^{n-2} G_r(tx)$ satisfies $\Delta v = -\omega_n$ and hence by comparison principle $v \le u$ in B_{tr}. In particular

$$\partial_\nu (v - u)(z) \le 0$$

where ν is the inward normal direction, resulting in

$$t^{n-1} F(t|z|) \le F(|z|).$$

If we assume that $r^{n-1}F(r)$ is strictly increasing (or just increasing for $C^{1,\text{dini}}$ domains, by the use of Hopf's lemma) then the above inequality results in

$$t^{n-1} F(t|z|) \le F(|z|) = \frac{|z|^{n-1}}{|z|^{n-1}} F(|z|) < t^{n-1} F(t|z|)$$

which is a contradiction.

It is noteworthy that the above condition on F can be relaxed considerably. Indeed, it would be enough to assume that $T(r) := r^{n-1}F(r) - 1$ vanishes at only one point and that $T(r) < 0$ for small values of r and $T(R) > 0$ for large values of R. To see this we need a different argument than scaling. So let us again consider the largest ball B_r inside our domain and the smallest one B_R containing it. A similar comparison argument as above gives that the inward normal derivative of $u - G_r$ and $G_R - u$ are non-negative ($r < R$). In particular we will have

$$\partial_\nu(u - G_r) \geq 0, \qquad \partial_\nu(G_R - u) \leq 0,$$
$$F(r) \geq r^{1-n}, \qquad F(R) \leq R^{1-n}$$

or that $T(r) := r^{n-1}F(r) - 1 \geq 0$ and $T(R) := R^{n-1}F(R) - 1 \leq 0$. But this along with the conditions on T implies that T vanishes at two points, at least. A contradiction.

If we assume stronger boundary regularity so that the Hopf's lemma holds (we need $C^{1,\text{dini}}$) then one can relax the condition on T to that T vanishes at most at two points.

The authors have not been able to find these new observations in the literature. Observe that if $r^{n-1}F(r)$ is strictly increasing, and it takes one value at least once (this is required to have a ball solution, according to our discussions above) then obviously the corresponding T will satisfy our conditions. In particular our new conditions drastically simplifies and relaxes the earlier conditions as well as it generalizes existing results.

Let us now explore the one phase problem even further. So we consider the cases of non-uniqueness, and ask the question of whether one can expect other solutions than a ball. Let us, for simplicity, take $F(r) = r^{-n}$ and play around with some ideas. The first thing one can see is that this function does not satisfy our conditions. However, we will now by a simple inversion argument show that for smooth domains (C^2) we can conclude that solutions with $F(r) = r^{-n}$ must be a ball. Indeed, the inversion of the solution in the unit sphere gives us a new solution $v(x) = u(x/|x|^2)/|x|^{n-2}$ in the domain D which is the inversion of Ω. The function v is then harmonic in D, it is zero on the boundary of D and it tends to the constant $c_0 > 0$ at infinity. The boundary gradient condition is also invariant, and we obtain $|\nabla v| = A$, for some constant A on the boundary of D. Now by results of Reichel [4] one obtains that D is a ball, and hence Ω is a ball. We formulate this as a theorem.

Theorem 2.1. *A solution to* (1) *with C^2 domain and with $F(t) = t^{-n}$ is necessarily spherical.*

Now reversing this argument and starting with any ball $B_R(x^0)$ with $B_1(0) \subset B_R(x^0)$, and x^0 in $B_1(0)$ we can see that $v(x) := 1 - (R/|x - x^0|)^{n-2}$ is harmonic in $\mathbb{R}^n \setminus B_R(x^0)$, it is nonnegative and $|\nabla v| = n - 2$ on the boundary of the ball $B_R(x^0)$. Inversion in the unit sphere now produces a solution to our problem, that is still a ball with center y^0, but the level sets of the inverted function is not spherical around the origin. It is interesting that the center of the mass (the origin) does not coincide with that of the solution ball $B_r(y^0)$, $r = 1/R$. We refer the reader to Theorem 3.29 in [2], which shows that in $\mathbb{R}^2$ for $F(r) = r^{-k}$ for $k = 3, 4, \cdots$ we have solutions that are not balls.

If we elaborate on the idea of inversion we see that for any solution to our problem with F as boundary gradient condition, one obtains (after inversion) a new problem in the exterior domain. Namely one gets

$$\Delta v = 0 \quad \text{in } \mathbb{R}^n \setminus D$$

where D is the inversion of our domain, under Kelvin transformation. The boundary gradient condition also transfers to $|\nabla v| = A|x|^{-n}F(|x|)$ for some constant A. One can use ideas of Reichel (originally due to Serrin [5] in such problems, and to A. D. Alexandrov in another geometric problem) and apply the moving plane technique to obtain the symmetry and consequently that D is a ball, provided $r^n F(r)$ is increasing. The argument of moving plane technique is a well-known method and outside the scope of this paper. The reader can verify our result by applying the moving plane technique to our problem, but with somewhat stronger assumptions.

3. The Radial Green's and Capacitor function

In this section we compute explicitly the Green's functions and the capacitor potential of a sequence of spherical rings, that we will use as barriers from below and from above in the sequel.

Define the p-Laplacian

$$\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u) \quad (1 < p < \infty).$$

For a given positive integer m, and positive real numbers

$$0 < r_1 < r_2 < \cdots < r_m < \infty, \qquad 0 = \alpha_1 < \alpha_2 < \cdots < \alpha_m < \alpha_{m+1} < \infty$$

consider $B_{r_i}(0)$ and the corresponding Green's/capacitor functions G_i:

$$\Delta_p G_i = 0, \quad \text{in } B_{r_i} \setminus B_{r_{i-1}}, \quad i = 2, \cdots, m$$

$$G_i = -\alpha_i, \quad \text{on } \partial B_{r_i}, \quad G_i = -\alpha_{i-1}, \quad \text{on } \partial B_{r_{i-1}}, \quad i = 2, \cdots, m$$

and for $i = 1$ and $i = m+1$

$$\Delta_p G_1 = -c_0 \delta_0, \quad \text{in } B_{r_1}, \qquad G_1 = 0, \quad \text{on } \partial B_{r_1}$$

and

$$\Delta_p G_{m+1} = 0, \quad \text{in } \mathbb{R}^n \setminus B_{r_m}, \qquad G_{m+1} = -\alpha_m, \quad \text{on } \partial B_{r_m},$$

$$\lim_{x\to\infty} G_{m+1} = -\alpha_{m+1}, \quad p < n, \qquad \lim_{x\to\infty} |x|^{\frac{n-p}{p-1}} G_{m+1} = -\alpha_{m+1}, \quad p > n.$$

We can compute explicitly all these functions, and we obtain:

Case: $n \neq p$.

$$G_1 = \frac{c_0}{\omega_n}\left| |x|^{\frac{p-n}{p-1}} - r_1^{\frac{p-n}{p-1}} \right|, \tag{2}$$

$$G_i = (-\alpha_i + \alpha_{i-1}) \frac{|x|^{\frac{p-n}{p-1}} - r_i^{\frac{p-n}{p-1}}}{r_i^{\frac{p-n}{p-1}} - r_{i-1}^{\frac{p-n}{p-1}}} - \alpha_i, \quad i = 2, \cdots, m, \tag{3}$$

$$G_{m+1} = (\alpha_{m+1} - \alpha_m)\, r_m^{\frac{n-p}{p-1}} |x|^{\frac{p-n}{p-1}} - \alpha_{m+1}, \quad p < n, \tag{4}$$

$$G_{m+1} = -\alpha_{m+1}\, r_m^{\frac{n-p}{p-1}} |x|^{\frac{p-n}{p-1}}, \quad p > n. \tag{5}$$

Case: $n = p$.

$$G_1 = \frac{c_0}{\omega_n} \left(\ln |x|^{-1} - \ln r_1^{-1}\right), \tag{6}$$

$$G_i = (-\alpha_i + \alpha_{i-1}) \frac{\ln |x|^{-1} - \ln r_i^{-1}}{\ln r_i^{-1} - \ln r_{i-1}^{-1}} - \alpha_i, \quad i = 2, \cdots, m, \tag{7}$$

$$G_{m+1} = -\alpha_{m+1} \left(\ln |x|^{-1} - \ln r_m^{-1}\right) - \alpha_m. \tag{8}$$

4. Two- and Multi-phase case

In this section we consider the multi-phase version of the overdetermined problem discussed in Section 2. To start we need some definitions.

Definition 4.1. For the multi phase case we will define $F_i(A,B)$, $i = 1, \cdots, m$ with the properties

$$F_i(A,B) < F_i(A_1,B_1), \quad \text{for } (A,B) < (A_1,B_1), \; i = 1, \cdots, m, \tag{9}$$

where $(A,B) < (A_1,B_1)$ means either $A < A_1$ or $B < B_1$ or both,

$$F_1(A,B) \geq CB^{\frac{1+\alpha}{p-1}} - A \quad (0 < \alpha < n-2) \text{ for } A, B \text{ large enough}, \tag{10}$$

$$F_i(A,B) \geq CB - A \quad \text{for } A, B \text{ large enough}, \; i = 2, \cdots, m, \tag{11}$$

$$F_i(A,B) \leq CB^{\frac{n-1}{p-1}} - A \quad \text{for } A, B \text{ small enough}, \; i = 1, \cdots, m-1, \tag{12}$$

$$F_m(A,B) \leq CB^{\alpha} - A \quad (\alpha < 1) \quad \text{for } A, B \text{ small enough}, \tag{13}$$

where C is a constant.

Theorem 4.2. *Let Ω_i $(i = 1, \cdots, m)$ be bounded smooth domains in $\mathbb{R}^n$ with $\Omega_{i-1} \subset \Omega_i$, $0 = \alpha_1 < \alpha_2 < \cdots < \alpha_{m+1}$, and suppose there exist u_i $(i = 1, \cdots, m)$ solving the following problem*

$$\begin{gathered}
\Delta_p u_1 = -c_0 \delta_0 \quad \text{in } \Omega_1, \\
\Delta_p u_i = 0, \quad \text{in } \Omega_i \backslash \Omega_{i-1}, \; (i = 2, \cdots, m) \\
\Delta_p u_{m+1} = 0, \quad \text{in } \mathbb{R}^n \backslash \Omega_m, \\
u_i = u_{i-1} = -\alpha_{i-1}, \quad \text{on } \partial\Omega_{i-1}, \; (i = 2, \cdots, m+1) \\
u_{m+1} \to -\alpha_{m+1} \quad \text{as } |x| \to \infty, \; 1 < p < n, \\
|x|^{\frac{n-p}{p-1}} u_{m+1} \to -\alpha_{m+1} \quad \text{as } |x| \to \infty, \; n < p, \\
u_{m+1} / \log |x|^{-1} \to -\alpha_{m+1} \quad \text{as } |x| \to \infty, \; n = p,
\end{gathered} \tag{14}$$

along with the boundary gradient condition

$$F_i(\partial_\nu u_i, \partial_\nu u_{i+1}) = 0, \quad i = 1, \cdots, m. \tag{15}$$

Then, Ω_i $(i = 1, \cdots, m)$ are balls centered at the origin.

Remark 4.3. In the above theorem the Dirac source δ_0 can be replaced by a Dirichlet data on $B_s(0) \subset \Omega$ (for some s) and/or Ω_m with ball $B_{r_m}(0)$. Then one may need to modify the assumptions on the functions F_i slightly.

Proof. We split the proof into two cases.

Case A: $n \neq p$. Step 1: (Largest ball from inside). Let us first consider the largest ball $B_{r_i} \subset \Omega_i$ $(i = 1, 2, \cdots, m)$ and denote G_i, the capacitor potential for each ring-shaped region $B_{r_i} \setminus B_{r_{i-1}}$ $(i = 2, 3, \cdots, m)$. For B_{r_1} we let G_1 denote the Green's function with source $-c_0\delta_0$ and for $\mathbb{R}^n \setminus B_{r_m}$ we let G_m be the harmonic function in $\mathbb{R}^n \setminus B_{r_m}$ with $G_{m+1} = -\alpha_m$ on ∂B_{r_m} and $G_{m+1} = u_{m+1}$ at infinity. For G_1 and G_i, we have

$$G_1 \leq u_1 \quad \text{in } B_{r_1}(0),$$
$$G_i \leq u_i \quad \text{in } B_{r_i}(0) \backslash \Omega_{i-1}.$$

Let $x^i \in \partial B_{r_i} \cap \partial\Omega_i$. Then

$$\partial_\nu G_1 \leq \partial_\nu u_1 \quad \text{at } x^1, \tag{16}$$
$$\partial_\nu G_i \geq \partial_\nu u_i \quad \text{at } x^{i-1}, \tag{17}$$
$$\partial_\nu G_i \leq \partial_\nu u_i \quad \text{at } x^i,\ i = 2, 3, \cdots, m. \tag{18}$$
$$\partial_\nu G_{m+1} \geq \partial_\nu u_{m+1} \quad \text{at } x^m. \tag{19}$$

By using these inequalities and considering the monotonicity of F_i, see (9), we get

$$0 = F_i(\partial_\nu u_i, \partial_\nu u_{i+1}) \geq F_i(\partial_\nu G_i, \partial_\nu G_{i+1}), \quad i = 1, \cdots, m. \tag{20}$$

Let us define, for $i = 2, \cdots, m-1$, and $y = (y_1, \cdots, y_m)$

$$T_i(y) := F_i\left(\frac{A_i\ y_i^{-1}}{(y_i/y_{i-1})^{\frac{n-p}{p-1}} - 1}, \frac{B_i\ y_i^{-1}}{1 - (y_i/y_{i+1})^{\frac{n-p}{p-1}}}\right) \tag{21}$$

and

$$T_1(y) := F_1\left(A_1\ y_1^{\frac{1-n}{p-1}},\ \frac{B_1\ y_1^{-1}}{1 - (y_1/y_2)^{\frac{n-p}{p-1}}}\right), \tag{22}$$

$$T_m(y) := F_m\left(\frac{A_m\ y_m^{-1}}{(y_m/y_{m-1})^{\frac{n-p}{p-1}} - 1},\ B_m\ y_m^{-1}\right). \tag{23}$$

where

$$A_1 = \frac{c_0}{\omega_n}\,\frac{|n-p|}{p-1}, \quad B_1 = \alpha_2\,\frac{|n-p|}{p-1},$$
$$A_i = (\alpha_i - \alpha_{i-1})\,\frac{n-p}{p-1}, \quad B_i = (\alpha_{i+1} - \alpha_i)\,\frac{n-p}{p-1}, \quad i = 2, \cdots, m-1,$$
$$A_m = (\alpha_m - \alpha_{m-1})\,\frac{n-p}{p-1}, \quad B_m = (\alpha_{m+1} - \alpha_m)\,\frac{n-p}{p-1}, \quad n > p,$$
$$A_m = (\alpha_m - \alpha_{m-1})\,\frac{p-n}{p-1}, \quad B_m = \alpha_{m+1}\,\frac{p-n}{p-1}, \quad n < p.$$

Finally define, $T(y) = (T_1(y), \cdots, T_m(y))$. We need to remark that

$$T_{i-1}(y) \geq T_{i-1}(y'), \quad T_{i+1}(y) \geq T_{i+1}(y') \ \text{ if } y_i < y'_i \text{ and } y_j = y'_j, \ j \neq i.$$

Let us also use the notation $T(y) \leq 0$ if the inequality holds for all components T_i. From (20) we have that $T(\overline{r}) \leq 0$, where $\overline{r} = (r_1, \cdots, r_m)$ are the radii of the balls. Next consider the domain

$$D := \{y : T(y) \leq 0\}.$$

The idea is to prove that for $y = (y_1, \cdots, y_m) \in D$, we have $y_i > s_0 > 0$ for some s_0, and for all $i = 1, \cdots, m$.

From now on we will also let C_i be constants, that might change value, depending only on the ingredients such as $n, m, c_0, \cdots$.

It should be noted that while working with the largest balls from inside, we will take y_i small, so that y_i^{-1} is large and will use the assumptions on $F_i(A, B)$, $i = 1, \cdots, m$ where A and B are large enough.

Let $y \in D$, then we extract from (11) for F_m and from (20),

$$\frac{A_m \ y_m^{-1}}{(y_m/y_{m-1})^{\frac{n-p}{p-1}} - 1} \leq C \ B_m \ y_m^{-1}. \tag{24}$$

and

$$\frac{1}{(y_m/y_{m-1})^{\frac{n-p}{p-1}} - 1} \leq C_1,$$

so that

$$\frac{1}{1 - (y_{m-1}/y_m)^{\frac{n-p}{p-1}}} \leq 1 + C_1.$$

Now we will use this for the next step, F_{m-1}, and see that

$$\frac{1}{(y_{m-1}/y_{m-2})^{\frac{n-p}{p-1}} - 1} \leq C_2,$$

and as before

$$\frac{1}{1 - (y_{m-2}/y_{m-1})^{\frac{n-p}{p-1}}} \leq 1 + C_2.$$

Iterating this all the way down to $i = 1$ we obtain

$$\frac{1}{1 - (y_1/y_2)^{\frac{n-p}{p-1}}} \leq 1 + C_{m-1}. \tag{25}$$

On the other hand, the equations (2), (3) and (10) gives

$$A_1 \ y_1^{\frac{1-n}{p-1}} \leq C \left(\frac{B_1 \ y_1^{-1}}{1 - (y_1/y_2)^{\frac{n-p}{p-1}}} \right)^{\frac{1+\alpha}{p-1}}, \tag{26}$$

(25) and (26) gives us

$$y_1^{\frac{n-2-\alpha}{p-1}} \geq C_m,$$

and we conclude

$$y_1 \geq C_{m+1}, \tag{27}$$

uniformly for all $r_i,\ i = 1, \cdots, m$. This proves that all y_i are confined within the convex cone

$$D \subset \{y_1 > s_0,\ y_i > s_0 y_{i-1},\ i = 2, \cdots, m\},$$

for some constant $s_0 > 0$.

Let us now take the smallest element ρ in D, i.e. if for any y with $T(y) \leq 0$ we have $\rho \leq y$. In particular this means that there is an element $\rho \in D$ with $T(\rho) = 0$. Indeed, if this fails, then for some i we have $T_i(\rho) \leq 0$. If we decrease ρ_i to $\rho_i - \epsilon$, for small enough $\epsilon > 0$, and set $\rho^\epsilon = (\rho_1, \cdots, \rho_i - \epsilon, \cdots, \rho_m)$, then by continuity $T_i(\rho^\epsilon) < 0$. It is also apparent that changing ρ_i will only give rise to changes of the value T_i, T_{i-1}, T_{i+1}, for $i = 2, \cdots, m-1$. For $i = 1$, the changes occur only for two elements T_1, T_2, and for $i = m$ the changes occur only for two elements T_{m-1}, T_m.

Using monotonicity of T_i, it is seen that we should have $T_{i-1}(\rho^\epsilon) < 0$ and $T_{i+1}(\rho^\epsilon) < 0$. Hence the minimality of ρ is violated. Thus, for an element $\rho \in D$ we must have $T(\rho) = 0$.

Step 2: (Smallest ball from outside). Let us now take a reverse situation. Let B_{R_i} be the smallest ball containing Ω_i, with the corresponding Green's functions G_i. Then a similar argument as in the previous case shows that $G_i \geq u_i$ and considering the monotonicity of F_i, see (9), we get

$$0 = F_i(\partial_\nu u_i, \partial_\nu u_{i+1}) \leq F_i(\partial_\nu G_i, \partial_\nu G_{i+1}), \quad i = 1, \cdots, m. \tag{28}$$

Now we use a similar iteration as we did in the earlier case. As in the previous case, we define $T(y) = (T_1(y), \cdots, T_m(y))$ and $T(\overline{R}) \geq 0$, where $\overline{R} = (R_1, \cdots, R_m)$ are the radii of the balls. We need to show the estimate (24) and the further ones. Next consider the domain

$$D' := \{y : T(y) \geq 0\}.$$

We will take y_i large, so that y_i^{-1} is small. We start with F_1, using (12) and (28), we obtain

$$C \left(\frac{B_1\ y_1^{-1}}{1 - (y_1/y_2)^{\frac{n-p}{p-1}}} \right)^{\frac{n-1}{p-1}} \leq A_1\ y_1^{\frac{1-n}{p-1}}.$$

Hence we have

$$\frac{1}{1 - (y_1/y_2)^{\frac{n-p}{p-1}}} \leq C_1\ ,$$

and consequently

$$\frac{1}{(y_2/y_1)^{\frac{n-p}{p-1}} - 1} \leq C_1 - 1.$$

In analogy with *Step 1*, we can use this estimate, along with (28) to derive a similar estimate

$$\frac{1}{(y_3/y_2)^{\frac{n-p}{p-1}}-1} \leq C_2 - 1.$$

Iterating this up to $i = m$, we obtain

$$\frac{1}{(y_m/y_{m-1})^{\frac{n-p}{p-1}}-1} \leq C_{m-1} - 1.$$

For y_m we use the estimate for F_m, and have

$$C\left(B_m\, y_m^{-1}\right)^{\alpha} \leq \frac{A_m\, y_m^{-1}}{(y_m/y_{m-1})^{\frac{n-p}{p-1}}-1},$$
$$y_m^{1-\alpha} \leq C_m$$

that simplifies to

$$y_m \leq C_{m+1}. \tag{29}$$

Then according to our analysis above we have that

$$D' \subset \{y_m < s_0,\ y_i > s_0 y_{i-1},\ i = m, \cdots, 2\},$$

where the latter cone is bounded. Now, a similar argument used in the previous case, gives us that the largest element $\rho' \in D'$ must be so that $T(\rho') = 0$.

Step 3: (Putting things together). From the above two cases we see that we will have two values ρ, ρ' for which T becomes zero. Since F_i are strictly increasing this gives us a contradiction that we were looking for.

Case B: $n = p$. The same argument can be used for $n = p$. We need to show (27) and (29), the rest of the prove will follow as in the previous steps. For $n = p$, while working with the largest balls from inside, we start with F_m, use (11) and (20), we obtain

$$\frac{(\alpha_m - \alpha_{m-1})y_m^{-1}}{\ln(y_m/y_{m-1})} \leq C\ \bar{c}\ y_m^{-1},$$

which gives

$$\frac{1}{\ln(y_m/y_{m-1})} \leq C_1.$$

Using this for the next step and iterating up to i=1 gives

$$\frac{1}{\ln(y_2/y_1)} \leq C_{m-1},$$

and with the assumption on F_1 we get

$$\frac{c_0}{\omega_n}\, y_1^{-1} \leq C\left(\frac{\alpha_2 y_1^{-1}}{\ln(y_2/y_1)}\right)^{\frac{1+\alpha}{p-1}},$$
$$y_1^{1-\frac{1+\alpha}{p-1}} \geq C_m,$$

which gives $y_1 \geq C_{m+1}$. While working with the smallest balls from outside, we start with F_1, we get

$$\frac{C\ \alpha_2\ y_1^{-1}}{\ln(y_2/y_1)} \leq \frac{c_0}{\omega_n}\ y_1^{-1},$$

which gives

$$\frac{1}{\ln(y_2/y_1)} \leq C_1,$$

by iteration

$$\frac{1}{\ln(y_m/y_{m-1})} \leq C_{m-1}.$$

Finally by using the assumption on F_m, we obtain

$$C\left(\bar{c}\ y_m^{-1}\right)^{\alpha} \leq \frac{(\alpha_m - \alpha_{m-1})y_m^{-1}}{\ln(y_m/y_{m-1})},$$
$$y_m^{1-\alpha} \leq C_m,$$

which gives $y_m \leq C_{m+1}$. Hence the rest of the proof becomes straightforward. □

5. Viscosity Solutions

In this section we will consider a different approach to our problem, using the definition of viscosity solutions, as in the paper of L. Caffarelli (see [1]).

Definition 5.1 (Classical solution). A classical sub-solution to our multi-phase problem (14) has the property $F_i(\partial_\nu u_i, \partial_\nu u_{i+1}) \geq 0,\ i = 1, \cdots, m$. A classical super-solution is the one with $F_i(\partial_\nu u_i, \partial_\nu u_{i+1}) \leq 0,\ i = 1, \cdots, m$. A classical solution is both a classical super- and sub-solution.

Definition 5.2 (Viscosity sub-solution). A viscosity sub-solution to our problem (14) has the property that no classical super-solution can touch u_i (for some i) from above at a free boundary point. A viscosity super-solution is a solution that can not be touched by a classical sub-solution from below, at a free boundary point. A viscosity solution is both super- and sub-solution.

It should be remarked that somehow the ordering of the gradient is encoded in the definition. In other words, if u is viscosity super-solution, touching a viscosity sub-solution from above, then the "normal derivatives" at free boundary point are correctly ordered.

Here, we will have the same definition for the function F_i given by equations (9)–(13).

Theorem 5.3. *If u_i is a viscosity solution to our multi-phase problem given by* (14) *then Ω_i $(i = 1, \cdots, m)$ must be balls with center at the origin.*

Observe that in the theorem, we do NOT assume any regularity of the free boundary. The viscosity definition, somehow pushes the free boundary to be regular.

Proof. The proof of this theorem is much the same as before. Let us take the largest ball from inside as in the previous section and take the corresponding Green's function G_i. Now by the maximum principle $G_i \le u_i$, and by viscosity definition for each G_i around the touching/common point $x^i \in \partial B_i \cap \partial\Omega_i$, we have the same equations as that in (14) for the corresponding balls, and their Green's functions/capacitor potentials. Since equation (15) is satisfied in terms of viscosity for our viscosity solutions, we see that the Green's functions G_i being smaller classical solutions than u_i, and touching the free boundary at some point x^i, means that G_i should satisfy equation

$$0 \ge F_i(\partial_\nu G_i, \partial_\nu G_{i+1})$$

at x^i. This is equation (20). The rest of the proof for the case with largest ball from inside works similarly.

For the case with smallest ball from outside a similar argument, tells us that the Green's function become sub-solutions and we will have

$$0 \le F_i(\partial_\nu G_i, \partial_\nu G_{i+1})$$

at the touching points. Hence we iterate the argument again and obtain the result in this case as well. This ends the proof. □

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