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On the Strong Law of Large Numbers in Spaces of Compact Sets

Francesco S. de Blasi
Dipartimento di Matematica, Università di Roma "Tor Vergata", Via della Ricerca Scientifica, 1, 00133 Roma, Italy
deblasi@mat.uniroma2.it

Luca Tomassini
Dipartimento di Matematica, Università di Roma "Tor Vergata", Via della Ricerca Scientifica, 1, 00133 Roma, Italy
tomassin@mat.uniroma2.it

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Let $\mathfrak{Y}$ be the space of all nonempty compact subsets of $\mathbb{R}^d$ and let $\mathcal{K}(\mathfrak{Y})$ be the space of all nonempty compact subsets of $\mathfrak{Y}$. For a random set with values in $\mathcal{K}(\mathfrak{Y})$, after defining the expectation, we establish a version of the strong law of large numbers. Some related results concerning the case of nonempty compact convex subsets of a Banach space $\mathbb{E}$ are included.

1. Introduction

It is known that random sets have a theoretical and practical interest, since they generalize random variables and random vectors and, on the other hand, they occur in certain models of growth.

The study of the strong law of large numbers for random sets was initiated by Artstein and Vitale with their seminal 1975 paper [2]. Ever since, important extensions have been obtained by several authors, including Cressie [5], Puri and Radulescu [16, 17], Hess [8, 9], Artstein and Hansen [1], Hiai [10], Terán and Molchanov [20]. For a systematic presentation of the status of the theory of random sets, see the recent monograph of Molchanov [14]. Additional results concerning the general theory of set-valued maps and their applications can be found in Castaing and Valadier [4], Hu and Papageorgiou [13] and Rockafellar and Wets [19].

Denote by $\mathfrak{Y}$ the space of all nonempty compact subsets of $\mathbb{R}^d$. This space, equipped with the usual operations of addition and multiplication by non-negative scalars, has an algebraic structure in which the distributive property $(\lambda+\mu)A = \lambda A+\mu A$, for any $A \in \mathfrak{Y}$ and $\lambda, \mu \geq 0$, does not hold. Consequently one cannot have for $\mathfrak{Y}$ a Rådström type embedding into a cone of some Banach space, as in the case of $\mathfrak{X}$, the space of all nonempty compact convex subsets of $\mathbb{R}^d$ [18].

The aim of this paper is to investigate the law of large numbers for compact-valued random sets whose values are nonempty compact subsets of $\mathfrak{Y}$. For these random sets we define the expectation and then we establish a version of the strong law of

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large numbers (Theorem 6.2). Our approach is based on a convexification result in the limit (Theorem 5.2) and on the strong law of large numbers for compact-valued random sets whose values are the nonempty compact subsets of $\mathfrak{X}$ (Theorem 6.1).

It is worth noting that Terán and Molchanov [20] have obtained similar results by using a different approach. We wish to thank Prof. P. Terán for having kindly drawn our attention, while our paer was in print, to the results of [20].

2. Notation and preliminaries

Let (M, d) be a metric space. If $X \subset M$ we denote by $\overline{X}$ the closure of X. For X a nonempty subset of M, $u \in M$ and $r > 0$ set

$$\mathrm{dist}_d(u, X) = \inf_{x \in X} d(u, x), \qquad N[X, r] = \{z \in M : \mathrm{dist}_d(z, X) \leq r\}.$$

The subscript d will be useful in the sequel to emphasize the specific distance of the underlying metric space. Furthermore let $\mathcal{P}(M)$ denote the set of all nonempty subsets of M. For any map $f : M \to N$ and a nonempty subset A of M we put

$$f[A] = \{f(x) : x \in A\}.$$

Let $\mathbb{E}$ be a real Banach space and $B_{\mathbb{E}}$ its closed unit ball centered at zero. Moreover, set

$$\mathfrak{Y}_{\mathbb{E}} = \{A \subset \mathbb{E} : A \text{ compact nonempty}\},$$

$$\mathfrak{X}_{\mathbb{E}} = \{A \subset \mathbb{E} : A \text{ compact convex nonempty}\}.$$

Clearly $\mathfrak{X}_{\mathbb{E}} \subset \mathfrak{Y}_{\mathbb{E}}$. These spaces are equipped with the Pompeiu-Hausdorff distance

$$h(A, B) = \max\left\{\sup_{x \in A} \mathrm{dist}_d(x, B),\ \sup_{y \in B} \mathrm{dist}_d(y, A)\right\}$$

under which each one of them is complete. The spaces $\mathfrak{X}_{\mathbb{E}}$ and $\mathfrak{Y}_{\mathbb{E}}$, endowed with the usual operations of addition $A + B$ and multiplication λA by a scalar $\lambda \geq 0$, have the following properties:

(a) For $A, B, C \in \mathfrak{Y}_{\mathbb{E}}$ and $\lambda, \mu \geq 0$ we have: (i) $A + \{0\} = A$; (ii) $A + B = B + A$; (iii) $A + (B + C) = (A + B) + C$; (iv) $1A = A$; (v) $\lambda(\mu A) = (\lambda\mu)A$; (vi) $\lambda(A + B) = \lambda A + \lambda B$; (vii) $(\lambda + \mu)A \subset \lambda A + \mu A$. If $A, B, C \in \mathfrak{X}_{\mathbb{E}}$ then (i)–(vi) are valid and instead of (vii) we have: (vii') $(\lambda + \mu)A = \lambda A + \mu A$.

(b) For $A, A', B, B' \in \mathfrak{Y}_{\mathbb{E}}$ and $\lambda \geq 0$ we have:

$$h(A + A', B + B') \leq h(A, B) + h(A', B'), \qquad h(\lambda A, \lambda B) = \lambda h(A, B).$$

A set $\mathcal{A} \subset \mathfrak{X}_{\mathbb{E}}$ is said convex if $A, B \in \mathcal{A}$ and $0 \leq \lambda \leq 1$ imply $(1 - \lambda)A + \lambda B \in \mathcal{A}$. Set

$$\mathcal{K}(\mathfrak{Y}_{\mathbb{E}}) = \{\mathcal{A} \subset \mathfrak{Y}_{\mathbb{E}} : \mathcal{A} \text{ compact nonempty}\},$$

$$\mathcal{K}(\mathfrak{X}_{\mathbb{E}}) = \{\mathcal{A} \subset \mathfrak{X}_{\mathbb{E}} : \mathcal{A} \text{ compact nonempty}\},$$

$$\mathcal{C}(\mathfrak{X}_{\mathbb{E}}) = \{\mathcal{A} \subset \mathfrak{X}_{\mathbb{E}} : \mathcal{A} \text{ compact convex nonempty}\}.$$

Evidently, $\mathcal{C}(\mathfrak{X}_{\mathbb{E}}) \subset \mathcal{K}(\mathfrak{X}_{\mathbb{E}}) \subset \mathcal{K}(\mathfrak{Y}_{\mathbb{E}})$. These spaces are equipped with the Pompeiu-Hausdorff metric

$$H(\mathcal{A},\mathcal{B}) = \max\left\{\sup_{A\in\mathcal{A}} \operatorname{dist}_h(A,\mathcal{B}),\ \sup_{B\in\mathcal{B}} \operatorname{dist}_h(B,\mathcal{A})\right\}$$

under which each one of them is complete.

It is worth noting that for $A, B \in \mathfrak{Y}_{\mathbb{E}}$ and $\mathcal{A},\mathcal{B} \in \mathcal{K}(\mathfrak{Y}_{\mathbb{E}})$ we have

$$\begin{aligned} h(A,B) &= \inf\{r > 0 \ :\ A \subset B + rB_{\mathbb{E}},\ B \subset A + rB_{\mathbb{E}}\}, \\ H(\mathcal{A},\mathcal{B}) &= \inf\{r > 0 \ :\ \mathcal{A} \subset N[\mathcal{B}, r],\ \mathcal{B} \subset N[\mathcal{A}, r]\}. \end{aligned}$$

Furthermore, each one of the spaces $\mathfrak{X}_{\mathbb{E}}$, $\mathfrak{Y}_{\mathbb{E}}$, $\mathcal{C}(\mathfrak{X}_{\mathbb{E}})$, $\mathcal{K}(\mathfrak{X}_{\mathbb{E}})$, $\mathcal{K}(\mathfrak{Y}_{\mathbb{E}})$ is separable, if the underlying Banach space $\mathbb{E}$ is so.

For $\mathcal{A},\mathcal{B} \in \mathcal{K}(\mathfrak{Y}_{\mathbb{E}})$ the operations of addition $\mathcal{A}+\mathcal{B}$ and multiplication $\lambda\mathcal{A}$ by a scalar $\lambda \geq 0$ are defined as follows:

$$\mathcal{A} + \mathcal{B} = \{A + B : A \in \mathcal{A}, B \in \mathcal{B}\}, \qquad \lambda\mathcal{A} = \{\lambda A : A \in \mathcal{A}\}.$$

Under these operations the space $\mathcal{K}(\mathfrak{Y}_{\mathbb{E}})$ (and similarly $\mathcal{K}(\mathfrak{X}_{\mathbb{E}})$ and $\mathcal{C}(\mathfrak{X}_{\mathbb{E}})$) is stable, i.e. $\mathcal{A}+\mathcal{B} \in \mathcal{K}(\mathfrak{Y}_{\mathbb{E}})$ and $\lambda\mathcal{A} \in \mathcal{K}(\mathfrak{Y}_{\mathbb{E}})$, if $\mathcal{A},\mathcal{B} \in \mathcal{K}(\mathfrak{Y}_{\mathbb{E}})$ and $\lambda \geq 0$.
Whenever $\mathbb{E} = \mathbb{R}^d$ we omit the subscript $\mathbb{R}^d$ and thus we write $\mathfrak{X}$, $\mathfrak{Y}$, $\mathcal{K}(\mathfrak{X})$, $\mathcal{K}(\mathfrak{Y})$, $\mathcal{C}(\mathfrak{Y})$ instead of $\mathfrak{X}_{\mathbb{R}^d}$, $\mathfrak{Y}_{\mathbb{R}^d}$, $\mathcal{K}(\mathfrak{X}_{\mathbb{R}^d})$, $\mathcal{K}(\mathfrak{Y}_{\mathbb{R}^d})$, $\mathcal{C}(\mathfrak{Y}_{\mathbb{R}^d})$.

The proof of the following proposition is easy and thus it is omitted.

Proposition 2.1. *For $\mathcal{A},\mathcal{B},\mathcal{D} \in \mathcal{K}(\mathfrak{Y}_{\mathbb{E}})$ and $\lambda, \mu \geq 0$ we have:* (*i*) *$\mathcal{A} + \{\{0\}\} = \mathcal{A}$;* (*ii*) *$\mathcal{A}+\mathcal{B} = \mathcal{B}+\mathcal{A}$;* (*iii*) *$\mathcal{A}+(\mathcal{B}+\mathcal{D}) = (\mathcal{A}+\mathcal{B})+\mathcal{D}$;* (*iv*) *$1\mathcal{A} = \mathcal{A}$;* (*v*) *$\lambda(\mu\mathcal{A}) = (\lambda\mu)\mathcal{A}$;* (*vi*) *$\lambda(\mathcal{A}+\mathcal{B}) = \lambda\mathcal{A}+\lambda\mathcal{B}$. If $\mathcal{A},\mathcal{B},\mathcal{D} \in \mathcal{K}(\mathfrak{X}_{\mathbb{E}})$ (resp. $\mathcal{C}(\mathfrak{X}_{\mathbb{E}})$) and $\lambda, \mu \geq 0$, then* (*i*)*–*(*vi*) *are valid and moreover we have:* (*vii*) *$(\lambda+\mu)\mathcal{A} \subset \lambda\mathcal{A}+\mu\mathcal{A}$ (resp.* (*vii′*) *$(\lambda+\mu)\mathcal{A} = \lambda\mathcal{A}+\mu\mathcal{A}$).*

Proposition 2.2. *For $\mathcal{A},\mathcal{A}',\mathcal{B},\mathcal{B}' \in \mathcal{K}(\mathfrak{Y}_{\mathbb{E}})$ and $\lambda \geq 0$ we have:*

$$H(\mathcal{A}+\mathcal{A}',\mathcal{B}+\mathcal{B}') \leq H(\mathcal{A},\mathcal{B}) + H(\mathcal{A}',\mathcal{B}'), \tag{1}$$

$$H(\lambda\mathcal{A},\lambda\mathcal{B}) = \lambda H(\mathcal{A},\mathcal{B}). \tag{2}$$

Proof. Let us prove (1). Let $A \in \mathcal{A}$ and $A' \in \mathcal{A}'$ be arbitrary. Let $B \in \mathcal{B}$ and $B' \in \mathcal{B}'$ be such that $h(A,B) = \operatorname{dist}_h(A,\mathcal{B})$ and $h(A',B') = \operatorname{dist}_h(A',\mathcal{B}')$. We have

$$\begin{aligned} &\operatorname{dist}_h(A+A',\mathcal{B}+\mathcal{B}') \leq h(A+A',B+B') \\ &\leq h(A,B)+h(A',B') = \operatorname{dist}_h(A,\mathcal{B}) + \operatorname{dist}_h(A',\mathcal{B}') \\ &\leq \sup_{X\in\mathcal{A}} \operatorname{dist}_h(X,\mathcal{B}) + \sup_{X\in\mathcal{A}'} \operatorname{dist}_h(X,\mathcal{B}') \leq H(\mathcal{A},\mathcal{B}) + H(\mathcal{A}',\mathcal{B}') \end{aligned}$$

and thus

$$\sup_{A\in\mathcal{A},A'\in\mathcal{A}'} \operatorname{dist}_h(A+A',\mathcal{B}+\mathcal{B}') \leq H(\mathcal{A},\mathcal{B}) + H(\mathcal{A}',\mathcal{B}').$$

From this and the analogous inequality obtained by interchanging the roles of $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{B}$, $\mathcal{B}'$ we obtain (1). The proof of (2) is immediate. $\square$

The *convex hull* $\mathrm{co}_{\mathfrak{X}_{\mathbb{E}}}\mathcal{A}$ of a nonempty set $\mathcal{A} \subset \mathfrak{X}_{\mathbb{E}}$ is defined as

$$\mathrm{co}_{\mathfrak{X}_{\mathbb{E}}}\mathcal{A} = \left\{ X \in \mathfrak{X}_{\mathbb{E}} : X = \sum_{i=1}^{n} \lambda_i A_i \text{ for some } A_i \in \mathcal{A}, \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1 \right\}.$$

The closure of $\mathrm{co}_{\mathfrak{X}_{\mathbb{E}}}\mathcal{A}$ is denoted by $\overline{\mathrm{co}}_{\mathfrak{X}_{\mathbb{E}}}\mathcal{A}$. The sets $\mathrm{co}_{\mathfrak{X}_{\mathbb{E}}}\mathcal{A}$ and $\overline{\mathrm{co}}_{\mathfrak{X}_{\mathbb{E}}}\mathcal{A}$ are convex. Moreover, in view of [6, Prop. 3], it follows that $\overline{\mathrm{co}}_{\mathfrak{X}_{\mathbb{E}}}\mathcal{A} \in \mathcal{C}(\mathfrak{X}_{\mathbb{E}})$ whenever $\mathcal{A} \in \mathcal{K}(\mathfrak{X}_{\mathbb{E}})$.

Let $\gamma : \mathfrak{Y}_{\mathbb{E}} \to \mathfrak{X}_{\mathbb{E}}$ be the map given by $\gamma(A) = \overline{\mathrm{co}}A$, $A \in \mathfrak{Y}_{\mathbb{E}}$. Evidently, $\gamma(A) \in \mathfrak{X}_{\mathbb{E}}$ by Mazur's theorem [7, Vol. I, p. 416]. Define now $\Gamma : \mathcal{P}(\mathfrak{Y}_{\mathbb{E}}) \to \mathcal{P}(\mathfrak{X}_{\mathbb{E}})$ by

$$\Gamma(\mathcal{A}) = \gamma[\mathcal{A}], \quad \mathcal{A} \in \mathcal{P}(\mathfrak{Y}_{\mathbb{E}}).$$

Since $h(\overline{\mathrm{co}}A, \overline{\mathrm{co}}B) \leq h(A, B)$, $A, B \in \mathfrak{X}_{\mathbb{E}}$, one can conclude that $\mathcal{A} \in \mathcal{K}(\mathfrak{Y}_{\mathbb{E}})$ implies $\gamma[\mathcal{A}] \in \mathcal{K}(\mathfrak{X}_{\mathbb{E}})$. Thus the restriction of Γ to $\mathcal{K}(\mathfrak{Y}_{\mathbb{E}})$ takes values in $\mathcal{K}(\mathfrak{X}_{\mathbb{E}})$, i.e.,

$$\Gamma : \mathcal{K}(\mathfrak{Y}_{\mathbb{E}}) \to \mathcal{K}(\mathfrak{X}_{\mathbb{E}}).$$

Proposition 2.3. *For $\mathcal{A}, \mathcal{A}' \in \mathcal{K}(\mathfrak{Y}_{\mathbb{E}})$ we have $H(\Gamma(\mathcal{A}), \Gamma(\mathcal{A}')) \leq H(\mathcal{A}, \mathcal{A}')$.*

Proof. Let $X \in \Gamma(\mathcal{A})$, i.e., $X = \gamma(A)$ for some $A \in \mathcal{A}$. Let $A' \in \mathcal{A}'$ be such that $h(A, A') = \mathrm{dist}_h(A, \mathcal{A}')$. Setting $X' = \gamma(A')$, we have

$$h(X, X') \leq h(A, A') = \mathrm{dist}_h(A, \mathcal{A}') \leq H(\mathcal{A}, \mathcal{A}').$$

Hence $\mathrm{dist}_h(X, \Gamma(\mathcal{A}')) \leq H(\mathcal{A}, \mathcal{A}')$ and thus

$$\sup_{X \in \Gamma(\mathcal{A})} \mathrm{dist}_h(A, \Gamma(\mathcal{A}')) \leq H(\mathcal{A}, \mathcal{A}').$$

The statement follows from the latter inequality and the analogous one obtained by interchanging the role of $\mathcal{A}$ and $\mathcal{A}'$. □

3. Rådström type embedding

In this section we prove a Rådström type embedding for the space $\mathcal{C}(\mathfrak{X}_{\mathbb{E}})$. We start with the following

Proposition 3.1 (Algebraic cancellation law). *Let $\mathcal{A}, \mathcal{B}, \mathcal{U}$ be nonempty subsets of $\mathfrak{X}_{\mathbb{E}}$ and suppose that $\mathcal{B}$ is convex and closed and $\mathcal{U}$ is bounded. Then $\mathcal{A}+\mathcal{U} \subset \mathcal{B}+\mathcal{U}$ implies $\mathcal{A} \subset \mathcal{B}$. Furthermore, if $\mathcal{A}, \mathcal{B}, \mathcal{U} \in \mathcal{C}(\mathfrak{X}_{\mathbb{E}})$ we have:*

$$\mathcal{A} + \mathcal{U} = \mathcal{B} + \mathcal{U} \quad \textit{iff } \mathcal{A} = \mathcal{B}.$$

Proof. As in [18]. □

Proposition 3.2 (Metric cancellation law). *Let $\mathcal{A}, \mathcal{B}, \mathcal{U}$ be nonempty subsets of $\mathfrak{X}_{\mathbb{E}}$ and suppose that $\mathcal{B}$ is convex and $\mathcal{U}$ is bounded. Then, for $r > 0$,*

$$\mathcal{A} + \mathcal{U} \subset N[\mathcal{B} + \mathcal{U}, r] \quad \textit{implies } \mathcal{A} \subset N[\mathcal{B}, r]. \tag{3}$$

Moreover, if $\mathcal{A}, \mathcal{B}, \mathcal{U} \in \mathcal{C}(\mathfrak{X}_{\mathbb{E}})$ we have

$$H(\mathcal{A} + \mathcal{U}, \mathcal{B} + \mathcal{U}) = H(\mathcal{A}, \mathcal{B}). \tag{4}$$

Proof. Let us prove (3). Let $A \in \mathcal{A}$ be arbitrary. Let $\epsilon > 0$ and set $\epsilon_k = 1/2^k$. Fix any $U_0 \in \mathcal{U}$. Evidently

$$A + U_0 \in \mathcal{A} + \mathcal{U} \subset N[\mathcal{B} + \mathcal{U}, r] = \{X \in \mathfrak{X}_{\mathbb{E}} : \mathrm{dist}_h(X, \mathcal{B} + \mathcal{U}) \le r\}$$

and thus there exist $B_1 \in \mathcal{B}$ and $U_1 \in \mathcal{U}$ such that $h(A + U_0, B_1 + U_1) < r + \epsilon_1$. Then by induction one can construct $B_k \in \mathcal{B}$ and $U_k \in \mathcal{U}$ such that

$$h(A + U_{k-1}, B_k + U_k) < r + \epsilon_k, \quad k = 1, 2, \cdots$$

We have

$$h\left(\sum_{k=1}^{n}(A + U_{k-1}), \sum_{k=1}^{n}(B_k + U_k)\right) \le \sum_{k=1}^{n} h(A + U_{k-1}, B_k + U_k) < nr + \sum_{k=1}^{n} \epsilon_k. \quad (5)$$

On the other hand

$$\begin{aligned} & h\left(\sum_{k=1}^{n}(A + U_{k-1}), \sum_{k=1}^{n}(B_k + U_k)\right) \\ &= h\left(nA + U_0 + \sum_{k=2}^{n} U_{k-1}, \sum_{k=1}^{n} B_k + \sum_{k=1}^{n-1} U_k + U_n\right) \\ &= h\left(nA + U_0, \sum_{k=1}^{n} B_k + U_n\right) = nh\left(A + \frac{U_0}{n}, \frac{1}{n}\sum_{k=1}^{n} B_k + \frac{U_n}{n}\right). \end{aligned}$$

The latter and (5) imply

$$h\left(A + \frac{U_0}{n}, \frac{1}{n}\sum_{k=1}^{n} B_k + \frac{U_n}{n}\right) < r + \frac{1}{n}\sum_{k=1}^{n} \epsilon_k < r + \frac{1}{n}.$$

Thus for n sufficiently large, say $n > n_0$, we have

$$h\left(A, \frac{1}{n}\sum_{k=1}^{n} B_k\right) < r + \epsilon.$$

Since $(1/n)\sum_{k=1}^{n} B_k \in \mathcal{B}$, for $\mathcal{B}$ is convex, it follows that $\mathrm{dist}_h(A, \mathcal{B}) \le r$. Consequently $\mathcal{A} \subset N[\mathcal{B}, r]$ and (3) is proved.

Suppose now that $\mathcal{A}, \mathcal{B}, \mathcal{U} \in \mathcal{C}(\mathfrak{X}_{\mathbb{E}})$ and let $r > H(\mathcal{A} + \mathcal{U}, \mathcal{B} + \mathcal{U})$ be arbitrary. Clearly

$$\mathcal{A} + \mathcal{U} \subset N[\mathcal{B} + \mathcal{U}, r] \quad \text{and} \quad \mathcal{B} + \mathcal{U} \subset N[\mathcal{A} + \mathcal{U}, r]$$

and thus by (3)

$$\mathcal{A} \subset N[\mathcal{B}, r] \quad \text{and} \quad \mathcal{B} \subset N[\mathcal{A}, r],$$

which imply $H(\mathcal{A}, \mathcal{B}) \le r$. As $r > H(\mathcal{A} + \mathcal{U}, \mathcal{B} + \mathcal{U})$ is arbitrary it follows that

$$H(\mathcal{A}, \mathcal{B}) \le H(\mathcal{A} + \mathcal{U}, \mathcal{B} + \mathcal{U}).$$

Since on the other hand $H(\mathcal{A} + \mathcal{U}, \mathcal{B} + \mathcal{U}) \le H(\mathcal{A}, \mathcal{B})$, then (4) holds. $\square$

We are now ready to prove the following

Theorem 3.3 (Generalized Rådström embedding). *Let $\mathfrak{X}_{\mathbb{E}}$ be the space of all nonempty compact and convex sets contained in a real Banach space $\mathbb{E}$. Then there exists a Banach space $(\mathbb{F}, \|\cdot\|)$ and a map $J : \mathcal{C}(\mathfrak{X}_{\mathbb{E}}) \to \mathbb{V}$, where $\mathbb{V} = J[\mathcal{C}(\mathfrak{X}_{\mathbb{E}})]$, such that:*

(i) $J(\lambda\mathcal{A} + \mu\mathcal{B}) = \lambda J(\mathcal{A}) + \mu J(\mathcal{B})$ *for* $\mathcal{A}, \mathcal{B} \in \mathcal{C}(\mathfrak{X}_{\mathbb{E}})$ *and* $\lambda, \mu \geq 0$;
(ii) $\|J(\mathcal{A}) - J(\mathcal{B})\| = H(\mathcal{A}, \mathcal{B})$ *for* $\mathcal{A}, \mathcal{B} \in \mathcal{C}(\mathfrak{X}_{\mathbb{E}})$;
(iii) $\mathbb{V}$ *is a convex cone contained in* $\mathbb{F}$, *complete under the metric induced by the norm of* $\mathbb{F}$.

Proof. The space $(\mathcal{C}(\mathfrak{X}_{\mathbb{E}}), H)$ has the properties stated in Propositions 2.1, 2.2 and in addition it satisfies the algebraic and metric cancellation laws of Propositions 3.1, 3.2. By Rådström theorem [18], there exists a normed space $\mathbb{F}_0$ and a map $J_0 : \mathcal{C}(\mathfrak{X}_{\mathbb{E}}) \to \mathbb{V}_0$, where $\mathbb{V}_0 = J_0[\mathcal{C}(\mathfrak{X}_{\mathbb{E}})]$ is a convex cone in $\mathbb{F}_0$, satisfying (i) and (ii). Denote by $\mathbb{F}$ the Banach space obtained by completion of $\mathbb{F}_0$ with the corresponding linear isometric map $u : \mathbb{F}_0 \to \mathbb{F}$ having dense image $\overline{u[\mathbb{F}_0]} = \mathbb{F}$. Define $J : \mathcal{C}(\mathfrak{X}_{\mathbb{E}}) \to \mathbb{F}$ by $J(\mathcal{A}) = (u \circ J_0)(\mathcal{A})$, $\mathcal{A} \in \mathcal{C}(\mathfrak{X}_{\mathbb{E}})$, and set $\mathbb{V} = (u \circ J_0)[\mathcal{C}(\mathfrak{X}_{\mathbb{E}})]$. It is easily seen that J satisfies (i) and (ii) and that $\mathbb{V}$ is a convex cone in $\mathbb{F}$. Furthermore $\mathbb{V}$ is complete because $\mathcal{C}(\mathfrak{X}_{\mathbb{E}})$ is so and J is an isometry. □

Consider the space $\mathfrak{X}_{\mathbb{E}}$ of all nonempty compact convex subsets of a real Banach space $\mathbb{E}$. By Rådström's embedding theorem [18], [16] there exists a real Banach space $(\mathbb{F}, \|\cdot\|)$ and a map $i : \mathfrak{X}_{\mathbb{E}} \to \mathbb{W}$, where $\mathbb{W} = i[\mathfrak{X}_{\mathbb{E}}] \subset \mathbb{F}$, such that:

(i) $i(\lambda A + \mu B) = \lambda i(A) + \mu i(B)$ for $A, B \in \mathfrak{X}_{\mathbb{E}}$ and $\lambda, \mu \geq 0$;
(ii) $\|i(A) - i(B)\| = h(A, B)$ for $A, B \in \mathfrak{X}_{\mathbb{E}}$;
(iii) $\mathbb{W}$ is a convex cone contained in $\mathbb{F}$, complete under the metric induced by the norm of $\mathbb{F}$.

Set

$$\mathcal{K}(\mathbb{W}) = \{\Phi \subset \mathbb{W} : \Phi \text{ is compact nonempty}\},$$

$$\mathcal{C}(\mathbb{W}) = \{\Phi \subset \mathbb{W} : \Phi \text{ is compact convex nonempty}\}.$$

Each of these spaces is complete under the Pompeiu-Hausdorff metric H.

Define $I : \mathcal{P}(\mathfrak{X}_{\mathbb{E}}) \to \mathcal{P}(\mathbb{W})$ by

$$I(\mathcal{A}) = i[\mathcal{A}], \quad \mathcal{A} \in \mathcal{P}(\mathfrak{X}_{\mathbb{E}})$$

and observe that I is one-to-one and onto. Moreover, $\mathcal{A} \in \mathcal{K}(\mathfrak{X}_{\mathbb{E}})$ (resp. $\mathcal{A} \in \mathcal{C}(\mathfrak{X}_{\mathbb{E}})$) implies $I(\mathcal{A}) \in \mathcal{K}(\mathbb{W})$ (resp. $I(\mathcal{A}) \in \mathcal{C}(\mathbb{W})$) and thus each one of the following maps

$$I : \mathcal{K}(\mathfrak{X}_{\mathbb{E}}) \to \mathcal{K}(\mathbb{W}), \qquad I : \mathcal{C}(\mathfrak{X}_{\mathbb{E}}) \to \mathcal{C}(\mathbb{W})$$

is one-to-one and onto.

The proofs of the following Propositions 3.4 and 3.5 are easy and thus they are omitted.

Proposition 3.4. *For all $\mathcal{A}, \mathcal{B} \in \mathcal{K}(\mathfrak{X}_{\mathbb{E}})$ and $\lambda, \mu \geq 0$ we have:*

$$(i) \qquad I(\lambda\mathcal{A} + \mu\mathcal{B}) = \lambda I(\mathcal{A}) + \mu I(\mathcal{B})$$

$$(ii) \qquad H(I(\mathcal{A}), I(\mathcal{B})) = H(\mathcal{A}, \mathcal{B}).$$

Proposition 3.5. *For each $\mathcal{A} \in \mathcal{K}(\mathfrak{X}_{\mathbb{E}})$ we have*

$$\mathrm{co}_{\mathbb{F}} I(\mathcal{A}) = I(\mathrm{co}_{\mathfrak{X}_{\mathbb{E}}} \mathcal{A}), \qquad \overline{\mathrm{co}}_{\mathbb{F}} I(\mathcal{A}) = I(\overline{\mathrm{co}}_{\mathfrak{X}_{\mathbb{E}}} \mathcal{A}).$$

Proposition 3.6. *Let $\{\mathcal{U}_n\}$ be a sequence of compact sets $\mathcal{U}_n \in \mathcal{K}(\mathfrak{X}_{\mathbb{E}})$ and suppose that, for some set $\mathcal{A} \in \mathcal{C}(\mathfrak{X}_{\mathbb{E}})$,*

$$\lim_{n\to\infty} H\left(\frac{1}{n}\sum_{i=1}^{n} \overline{\mathrm{co}}_{\mathfrak{X}_{\mathbb{E}}} \mathcal{U}_i, \mathcal{A}\right) = 0.$$

Then

$$\lim_{n\to\infty} H\left(\frac{1}{n}\sum_{i=1}^{n} \mathcal{U}_i, \mathcal{A}\right) = 0. \tag{6}$$

Proof. Consider the corresponding sequences $\{I(\mathcal{U}_n)\} \subset \mathcal{K}(\mathbb{W})$ and $\{I(\overline{\mathrm{co}}_{\mathfrak{X}_{\mathbb{E}}} \mathcal{U}_n)\} \subset \mathcal{C}(\mathbb{W})$. We have

$$H\left(\frac{1}{n}\sum_{i=1}^{n} \overline{\mathrm{co}}_{\mathbb{F}} I(\mathcal{U}_i), I(\mathcal{A})\right) = H\left(I\left(\frac{1}{n}\sum_{i=1}^{n} \overline{\mathrm{co}}_{\mathfrak{X}_{\mathbb{E}}} \mathcal{U}_i\right), I(\mathcal{A})\right) = H\left(\frac{1}{n}\sum_{i=1}^{n} \overline{\mathrm{co}}_{\mathfrak{X}_{\mathbb{E}}} \mathcal{U}_i, \mathcal{A}\right)$$

and thus

$$\lim_{n\to\infty} H\left(\frac{1}{n}\sum_{i=1}^{n} \overline{\mathrm{co}}_{\mathbb{F}} I(\mathcal{U}_i), I(\mathcal{A})\right) = 0.$$

By virtue of Artstein and Hansen's lemma [1], it follows that

$$\lim_{n\to\infty} H\left(\frac{1}{n}\sum_{i=1}^{n} I(\mathcal{U}_i), I(\mathcal{A})\right) = 0.$$

From this we obtain (6), for

$$H\left(\frac{1}{n}\sum_{i=1}^{n} I(\mathcal{U}_i), I(\mathcal{A})\right) = H\left(I\left(\frac{1}{n}\sum_{i=1}^{n} \mathcal{U}_i\right), I(\mathcal{A})\right) = H\left(\frac{1}{n}\sum_{i=1}^{n} \mathcal{U}_i, \mathcal{A}\right).$$

This completes the proof. $\square$

4. Expectation of random sets

Throughout the rest of the paper (Ω, P) is a complete probability space without atoms. Let $\mathfrak{P}$ be any one of the spaces $\mathcal{C}(\mathfrak{X}_{\mathbb{E}})$, $\mathcal{K}(\mathfrak{X}_{\mathbb{E}})$, $\mathcal{K}(\mathfrak{Y}_{\mathbb{E}})$, where the underlying Banach space $\mathbb{E}$ is separable. By a random set we mean a map $\mathcal{U} \in L^1(\Omega, \mathfrak{P})$, that is a map $\mathcal{U} : \Omega \to \mathfrak{P}$ which is measurable [4] (weakly measurable with the terminology of [12]) and integrably bounded, i.e., $\omega \to H(\mathcal{U}(\omega), \{\{0\}\})$ is integrable.

By using the Aumann integral [3] we now introduce the expectation of a random set in the following

Definition 4.1. For $\mathcal{U} \in L^1(\Omega, \mathcal{K}(\mathfrak{Y}_{\mathbb{E}}))$, put

$$\int_\Omega \mathcal{U} dP = \left\{ \overline{\int_\Omega S dP} : \ S : \Omega \to \mathfrak{Y}_{\mathbb{E}} \text{ is a measurable selector of } \mathcal{U} \right\}.$$

Then the set

$$E\mathcal{U} = \overline{\int_\Omega \mathcal{U} dP}$$

is called the *expectation* of $\mathcal{U}$.

The expectation $E\mathcal{U}$ of a random set $\mathcal{U} \in L^1(\Omega, \mathcal{K}(\mathfrak{X}_{\mathbb{E}}))$ or $\mathcal{U} \in L^1(\Omega, \mathcal{C}(\mathfrak{X}_{\mathbb{E}}))$ is defined analogously.

The following theorem shows that the expectation $E\mathcal{U}$ of any random set $\mathcal{U} \in L^1(\Omega, \mathfrak{P})$ is a nonempty closed subset of $\mathfrak{X}_{\mathbb{E}}$. With the notation of Section 2, we recall that $\Gamma : \mathcal{K}(\mathfrak{Y}_{\mathbb{E}}) \to \mathcal{K}(\mathfrak{X}_{\mathbb{E}})$ denotes the map defined by

$$\Gamma(\mathcal{A}) = \gamma[\mathcal{A}], \quad \mathcal{A} \in \mathcal{K}(\mathfrak{Y}_{\mathbb{E}}),$$

where $\gamma : \mathfrak{Y}_{\mathbb{E}} \to \mathfrak{X}_{\mathbb{E}}$ is the map given by $\gamma(A) = \overline{\mathrm{co}} A$, $A \in \mathfrak{Y}_{\mathbb{E}}$.

Theorem 4.2. *For any random set* $\mathcal{U} \in L^1(\Omega, \mathcal{K}(\mathfrak{Y}_{\mathbb{E}}))$ *we have*

$$E\mathcal{U} = E\Gamma \circ \mathcal{U}. \tag{7}$$

Proof. It suffices to show that

$$\int_\Omega \mathcal{U} dP = \int_\Omega \Gamma \circ \mathcal{U} dP. \tag{8}$$

Let A be an element of the set on the left-hand side of (8). Then, for some measurable selector $S : \Omega \to \mathfrak{Y}_{\mathbb{E}}$ of $\mathcal{U}$, we have

$$A = \overline{\int_\Omega S dP}. \tag{9}$$

Since $\gamma \circ S$ is a measurable selector of $\Gamma \circ \mathcal{U}$ and, by virtue of [13, Vol. 1, p. 201, Prop. 5.11]

$$\overline{\int_\Omega S dP} = \overline{\int_\Omega \gamma \circ S dP}$$

it follows that A is in the set on the right-hand side of (8). Hence

$$E\mathcal{U} \subset E\Gamma \circ \mathcal{U}. \tag{10}$$

Conversely, let A be an element of the set on the right hand side of (8). Then (9) holds for some measurable selector $S : \Omega \to \mathfrak{X}_{\mathbb{E}}$ of $\Gamma \circ \mathcal{U} : \Omega \to \mathcal{K}(\mathfrak{X}_{\mathbb{E}})$. We will show that there exists a measurable selector $\Sigma : \Omega \to \mathfrak{Y}_{\mathbb{E}}$ of $\mathcal{U} : \Omega \to \mathcal{K}(\mathfrak{X}_{\mathbb{E}})$ such that $S = \gamma \circ \Sigma$. To this end, consider the map $f : \Omega \times \mathfrak{Y}_{\mathbb{E}} \to \mathbb{R}$ given by

$$f(\omega, X) = h(S(\omega), \gamma(X)), \quad (\omega, X) \in \Omega \times \mathfrak{Y}_{\mathbb{E}}$$

and observe that $f(\omega, X)$ is measurable in ω and continuous in X. Clearly $S(\omega) \in \gamma[\mathcal{U}(\omega)]$ and thus for each $\omega \in \Omega$ there exists some $X \in \mathcal{U}(\omega)$ such that $S(\omega) = \gamma(X)$, which implies that the closed set $\{X \in \mathfrak{Y}_{\mathbb{E}} : f(\omega, X) = 0\}$ is nonempty. On the other hand (Ω, P) is complete and hence by [12, Theorem 6.4] the map

$$\omega \to \{X \in \mathfrak{Y}_{\mathbb{E}} : f(\omega, X) = 0\}, \quad \omega \in \Omega$$

is measurable. Consequently also the map

$$\omega \to \{X \in \mathfrak{Y}_{\mathbb{E}} : f(\omega, X) = 0\} \cap \mathcal{U}(\omega), \quad \omega \in \Omega$$

whose values are nonempty compact subsets $\mathfrak{Y}_{\mathbb{E}}$ is measurable and hence, by the Kuratowski and Ryll-Nardzewski theorem, it admits a measurable selector $\Sigma : \Omega \to \mathfrak{Y}_{\mathbb{E}}$. Evidently Σ is a measurable selector of $\mathcal{U}$ satisfying $f(\omega, \Sigma(\omega)) = 0$ for each $\omega \in \Omega$. As $f(\omega, \Sigma(\omega)) = h(S(\omega), \gamma(\Sigma(\omega)))$, it follows that $S(\omega) = \gamma(\Sigma(\omega))$, for each $\omega \in \Omega$, and thus $S = \gamma \circ \Sigma$. Since

$$A = \overline{\int_\Omega S dP} = \overline{\int_\Omega \gamma \circ \Sigma dP} = \overline{\int_\Omega \Sigma dP},$$

we conclude that A is an element of the set on the left hand side of (8) and thus $E\Gamma \circ \mathcal{U} \subset E\mathcal{U}$. This and (10) imply (7), completing the proof. $\square$

Theorem 4.3. *For any random set* $\mathcal{U} \in L^1(\Omega, \mathcal{K}(\mathfrak{X}_{\mathbb{E}}))$ *we have*

$$E\mathcal{U} = E\overline{\mathrm{co}}_{\mathfrak{X}_{\mathbb{E}}}\mathcal{U}. \tag{11}$$

Proof. It suffices to show that, given $A \in \int_\Omega \overline{\mathrm{co}}_{\mathfrak{X}_{\mathbb{E}}}\mathcal{U} dP$ and $\epsilon > 0$, there exists $B \in \int_\Omega \mathcal{U} dP$ such that $h(B, A) < \epsilon$. For some measurable $S : \Omega \to \mathfrak{X}_{\mathbb{E}}$ satisfying $S(\omega) \in \overline{\mathrm{co}}_{\mathfrak{X}_{\mathbb{E}}}\mathcal{U}(\omega)$, $\omega \in \Omega$, we have

$$A = \int_\Omega S dP. \tag{12}$$

With the notation of Section 3, let $i : \mathfrak{X}_{\mathbb{E}} \to \mathbb{W}$ be the map occurring in the Rådström embedding, where $\mathbb{W} = i[\mathfrak{X}_{\mathbb{E}}] \subset \mathbb{F}$, and let $I : \mathcal{K}(\mathfrak{X}_{\mathbb{E}}) \to \mathcal{K}(\mathbb{W})$ be given by $I(\mathcal{A}) = i[\mathcal{A}]$, $\mathcal{A} \in \mathcal{K}(\mathfrak{X}_{\mathbb{E}})$. Since $i \circ S : \Omega \to \mathbb{W}$ is measurable and $i(S(\omega)) \in i[\overline{\mathrm{co}}_{\mathfrak{X}_{\mathbb{E}}}\mathcal{U}(\omega)] = \overline{\mathrm{co}}_{\mathbb{F}} i[\mathcal{U}(\omega)] = \overline{\mathrm{co}}_{\mathbb{F}} I \circ \mathcal{U}(\omega)$, we have

$$\int_\Omega i \circ S dP \in \int_\Omega \overline{\mathrm{co}}_{\mathbb{F}} I \circ \mathcal{U} dP \subset \overline{\int_\Omega I \circ \mathcal{U} dP}. \tag{13}$$

Here the inclusion holds by virtue of [11, Corollary 4.3], for the probability space is without atoms. From (13) it follows that, for some measurable $\Sigma : \Omega \to \mathbb{W}$ satisfying $\Sigma(\omega) \in I \circ \mathcal{U}(\omega) = i[\mathcal{U}(\omega)]$, $\omega \in \Omega$, we have

$$\left\| \int_\Omega \Sigma dP - \int_\Omega i \circ S dP \right\| < \epsilon. \tag{14}$$

Put $\tilde{S} = i^{-1} \circ \Sigma$. Clearly $\tilde{S} : \Omega \to \mathfrak{X}_{\mathbb{E}}$ is a measurable selector of $\mathcal{U}$ and thus

$$\int_\Omega \tilde{S} dP \in \int_\Omega \mathcal{U} dP.$$

Setting

$$B = \int_\Omega \tilde{S} dP, \tag{15}$$

in view of (12), (15) and (14) we have

$$h(B, A) = \left\| i\left(\int_\Omega \tilde{S} dP \right) - i\left(\int_\Omega S dP \right) \right\| = \left\| \int_\Omega \Sigma dP - \int_\Omega i \circ S dP \right\| < \epsilon.$$

Then (11) holds and the proof is complete. □

From Theorems 4.2 and 4.3 it follows that for any random set $\mathcal{U} \in L^1(\Omega, \mathcal{K}(\mathfrak{Y}_{\mathbb{E}}))$ we have

$$E\mathcal{U} = E\Gamma \circ \mathcal{U} = E\overline{\mathrm{co}}_{\mathfrak{X}_{\mathbb{E}}} \Gamma \circ \mathcal{U} \tag{16}$$

and thus in particular $E\mathcal{U} \in \mathcal{C}(\mathfrak{X}_{\mathbb{E}})$.

5. Convexification in the limit

The following proposition is an immediate consequence of the Shapley-Folkmann-Starr theorem [14, p. 407].

Proposition 5.1. *Let A be a nonempty compact subset of $\mathbb{R}^d$. Then, for each $\sigma > 0$ there exists $n^0 = n^0(A, \sigma)$ such that*

$$h\left(\frac{\overbrace{A + \cdots + A}^{m-times}}{n}, \frac{m}{n} coA \right) < \sigma, \quad m, n \in \mathbb{N},\ n > n^0.$$

Proof. By virtue of the Shapley-Folkmann-Starr theorem, for arbitrary $m, n \in \mathbb{N}$ we have

$$h\Bigg(\frac{\overbrace{A + \cdots + A}^{m-\text{times}}}{n}, \frac{m}{n} coA \Bigg) = \frac{1}{n} h(\overbrace{A + \cdots + A}^{m-\text{times}}, co(\overbrace{A + \cdots + A}^{m-\text{times}})) \leq \frac{1}{n}\sqrt{d} \cdot \|A\|$$

from which the statement follows. □

With the notation of Section 2, for $r > 0$ set

$$\mathfrak{Y}_r = \{A \in \mathfrak{Y} | h(A, \{0\}) \leq r\}, \qquad \mathfrak{X}_r = \{A \in \mathfrak{X} | h(A, \{0\}) \leq r\}$$

$$\mathcal{K}(\mathfrak{Y})_r = \{\mathcal{A} \in \mathcal{K}(\mathfrak{Y}) | H(\mathcal{A}, \{\{0\}\}) \leq r\}, \qquad \mathcal{K}(\mathfrak{X})_r = \{\mathcal{A} \in \mathcal{K}(\mathfrak{X}) | H(\mathcal{A}, \{\{0\}\}) \leq r\}$$

and

$$\mathcal{C}(\mathfrak{X})_r = \{\mathcal{A} \in \mathcal{K}(\mathfrak{X}) | H(\mathcal{A}, \{\{0\}\}) \leq r\}.$$

Moreover, we denote by B the closed unit ball in $\mathbb{R}^d$ centered at zero.

Theorem 5.2. *Let $\{\mathcal{U}_n\}$ be a sequence of sets $\mathcal{U}_n \in \mathcal{K}(\mathfrak{Y})_r$ and let $\{\Gamma(\mathcal{U}_n)\}$ correspond, where $\Gamma(\mathcal{U}_n) \in \mathcal{K}(\mathfrak{X})_r$, $n \in \mathbb{N}$. Suppose that for some $\mathcal{A} \in \mathcal{C}(\mathfrak{X})_r$,*

$$\lim_{n\to\infty} H\left(\frac{1}{n}\sum_{i=1}^{n} \overline{\text{co}}_{\mathfrak{X}}\Gamma(\mathcal{U}_i), \mathcal{A}\right) = 0. \tag{17}$$

Then

$$\lim_{n\to\infty} H\left(\frac{1}{n}\sum_{i=1}^{n} \mathcal{U}_i, \mathcal{A}\right) = 0. \tag{18}$$

Proof. From (17) by virtue of Proposition 3.6 we have

$$\lim_{n\to\infty} H\left(\frac{1}{n}\sum_{i=1}^{n} \Gamma(\mathcal{U}_i), \mathcal{A}\right) = 0.$$

To show that (18) holds it is sufficient to prove the following

Claim 1. We have

$$\lim_{n\to\infty} H\left(\frac{1}{n}\sum_{i=1}^{n} \mathcal{U}_i, \frac{1}{n}\sum_{i=1}^{n} \Gamma(\mathcal{U}_i)\right) = 0. \tag{19}$$

Let $\varepsilon > 0$. Since $\mathfrak{Y}_r$ is compact it admits a finite ε-net, say $\mathcal{N} = \{A_1, A_2, \ldots, A_N\}$ with $A_k \in \mathfrak{Y}_r$, $k = 1, \ldots, N$. Moreover, let $0 < \sigma < \varepsilon/N$. By Proposition 5.1, for each $k = 1, \ldots, N$ there exists $n_k^0 = n^0(\sigma, A_k)$ such that

$$h\left(\frac{\overbrace{A_k + \cdots + A_k}^{m-\text{times}}}{n}, \frac{m}{n}\text{co}A_k\right) < \sigma, \quad \text{for each } n \geq n_k^0,\ m \in \mathbb{N}. \tag{20}$$

Let $\bar{n} = \max\{n_1^0, n_2^0, \ldots, n_N^0\}$. Since $\mathcal{N}$ is an ε-net of $\mathfrak{Y}_r$, the set $\mathcal{P}$ of all nonempty subsets of $\mathcal{N}$, say $\mathcal{P} = \{\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_M\}$, is a finite ε-net of $\mathcal{K}(\mathfrak{Y}_r)$.

Let us associate to the given sequence $\{\mathcal{U}_n\}$ another sequence, say $\{\mathcal{V}_n'\}$, where $\mathcal{V}_n' \in \mathcal{P}$ satisfies $H(\mathcal{V}_n', \mathcal{U}_n) < \varepsilon$ for every $n \in \mathbb{N}$. Evidently,

$$H\left(\frac{1}{n}\sum_{i=1}^{n} \mathcal{V}_i', \frac{1}{n}\sum_{i=1}^{n} \mathcal{U}_i\right) < \varepsilon, \quad n \in \mathbb{N} \tag{21}$$

and

$$H\left(\frac{1}{n}\sum_{i=1}^{n}\Gamma(\mathcal{V}_i'),\frac{1}{n}\sum_{i=1}^{n}\Gamma(\mathcal{U}_i)\right)<\varepsilon,\quad n\in\mathbb{N} \tag{22}$$

where the latter holds, for $H(\Gamma(\mathcal{V}_i'),\Gamma(\mathcal{U}_i))\leq H(\mathcal{V}_i',\mathcal{U}_i)$.

Claim 2. For every $n\geq\bar{n}$ we have

$$H\left(\frac{1}{n}\sum_{i=1}^{n}\mathcal{V}_i',\frac{1}{n}\sum_{i=1}^{n}\Gamma(\mathcal{V}_i')\right)<\varepsilon. \tag{23}$$

In view of Claim *2* (the proof of which is postponed) it is readily seen that Claim *1* is valid. In fact

$$\begin{aligned}H\left(\frac{1}{n}\sum_{i=1}^{n}\mathcal{U}_i,\frac{1}{n}\sum_{i=1}^{n}\Gamma(\mathcal{U}_i)\right)\leq H\left(\frac{1}{n}\sum_{i=1}^{n}\mathcal{U}_i,\frac{1}{n}\sum_{i=1}^{n}\mathcal{V}_i'\right)+H\left(\frac{1}{n}\sum_{i=1}^{n}\mathcal{V}_i',\frac{1}{n}\sum_{i=1}^{n}\Gamma(\mathcal{V}_i')\right)\\+H\left(\frac{1}{n}\sum_{i=1}^{n}\Gamma(\mathcal{V}_i'),\frac{1}{n}\sum_{i=1}^{n}\Gamma(\mathcal{U}_i)\right)\end{aligned}$$

and thus, by virtue of (21)–(23), one has

$$H\left(\frac{1}{n}\sum_{i=1}^{n}\mathcal{U}_i,\frac{1}{n}\sum_{i=1}^{n}\Gamma(\mathcal{U}_i)\right)<3\varepsilon,\quad n\geq\bar{n}.$$

Hence (19) holds and Claim *1* is valid.

In order to prove Claim *2* we first show that

$$\frac{1}{n}\sum_{i=1}^{n}\mathcal{V}_i'\subset N\left[\frac{1}{n}\sum_{i=1}^{n}\Gamma(\mathcal{V}_i'),\varepsilon\right],\quad n\geq\bar{n}. \tag{24}$$

Let $X\in\frac{1}{n}\sum_{i=1}^{n}\mathcal{V}_i'$ be arbitrary. Thus for some $A_i'\in\mathcal{V}_i'$, $i=1,\dots,n$, we have $X=\frac{1}{n}\sum_{i=1}^{n}A_i'$. Set

$$Y=\frac{1}{n}\sum_{i=1}^{n}\mathrm{co}A_i' \tag{25}$$

and observe that $Y\in\frac{1}{n}\sum_{i=1}^{n}\Gamma(\mathcal{V}_i')$. For the validity of (24) it suffices to show that $h(X,Y)<\varepsilon$.

Indeed, for each $i=1,\dots,n$ we have $A_i'\in\mathcal{V}_i'\subset\mathcal{N}$, which implies that $\{A_1',A_2',\dots,A_n'\}\subset\{A_1,A_2,\dots,A_N\}$. Let us divide the set $\{A_1',A_2',\dots,A_n'\}$ into N subsets, some of which are possibly empty, in such a way that for each $k=1,2,\dots,N$, the k^{th}-set consists of the m_k elements A_i' which are equal to A_k. Thus

$$X=\frac{1}{n}[(\overbrace{A_1+\cdots+A_1}^{m_1-\text{times}})+(\overbrace{A_2+\cdots+A_2}^{m_2-\text{times}})+\cdots+(\overbrace{A_N+\cdots+A_N}^{m_N-\text{times}})] \tag{26}$$

where $m_1 + \cdots + m_N = n$, $0 \leq m_k \leq n$. Since $n \geq \bar{n}$, where $\bar{n}$ is given just after (20), it follows that (20) holds with $\bar{n}$ in the place of n_k^0. Therefore we have:

$$\begin{aligned} &\frac{1}{n}(\overbrace{A_k + \cdots + A_k}^{m_k-\text{times}}) \subset \frac{m_k}{n}\text{co}A_k + \sigma B, \\ &\frac{m_k}{n}\text{co}A_k \subset \frac{1}{n}(\overbrace{A_k + \cdots + A_k}^{m_k-\text{times}}) + \sigma B \end{aligned} \tag{27}$$

for each $n \geq \bar{n}$ and $k = 1, \ldots, N$. From (26), in view of (27) it follows that

$$\begin{aligned} X &\subset \frac{m_1}{n}\text{co}A_1 + \cdots + \frac{m_N}{n}\text{co}A_N + \sigma NB \\ &= \frac{1}{n}[(\overbrace{\text{co}A_1 + \cdots + \text{co}A_1}^{m_1-\text{times}}) + \cdots + (\overbrace{\text{co}A_N + \cdots + coA_N}^{m_N-\text{times}})] + \sigma NB \\ &= \frac{1}{n}(\text{co}A_1' + \cdots + \text{co}A_n') + \sigma NB \\ &\subset Y + \varepsilon B, \end{aligned} \tag{28}$$

for $\sigma N < \varepsilon$. Similarly from (25), in view of (27), we have

$$\begin{aligned} Y &= \frac{1}{n}(\text{co}A_1' + \cdots + \text{co}A_n') \\ &= \frac{1}{n}[(\overbrace{\text{co}A_1 + \cdots + \text{co}A_1}^{m_1-\text{times}}) + \cdots + (\overbrace{\text{co}A_N + \cdots + \text{co}A_N}^{m_N-\text{times}})] \\ &= \frac{m_1}{n}\text{co}A_1 + \cdots + \frac{m_N}{n}\text{co}A_N \\ &\subset \frac{1}{n}[(\overbrace{A_1 + \cdots + A_1}^{m_1-\text{times}}) + \cdots + (\overbrace{A_N + \cdots + A_N}^{m_N-\text{times}})] + \sigma NB \\ &= \frac{1}{n}(A_1' + \cdots + A_n') + \sigma NB \subset X + \varepsilon B. \end{aligned} \tag{29}$$

Now (28) and (29) imply $h(X, Y) < \varepsilon$ and thus (24) is valid.

By an analogous argument one can show that

$$\frac{1}{n}\sum_{i=1}^{n}\Gamma(\mathcal{V}_i') \subset N\left[\frac{1}{n}\sum_{i=1}^{n}\mathcal{V}_i', \varepsilon\right], \quad n \geq \bar{n}$$

which, combined with (24), yields (23) and thus Claim *2* is proved. This completes the proof. □

6. Strong law of large numbers

In this section we prove two theorems on the strong law of large numbers, the first one dealing with independent identically distributed (i.i.d.) random sets $\mathcal{U}_n \in L^1(\Omega, \mathcal{K}(\mathfrak{X}_{\mathbb{E}}))$, the second one with i.i.d. random sets $\mathcal{U}_n \in L^1(\Omega, \mathcal{K}(\mathfrak{Y}))$.

Theorem 6.1. *Let $\mathfrak{X}_{\mathbb{E}}$ be the space of all nonempty compact convex subsets of a separable real Banach space $\mathbb{E}$. If $\{\mathcal{U}_n\}$ is a sequence of i.i.d. random sets in $L^1(\Omega, \mathcal{K}(\mathfrak{X}_{\mathbb{E}}))$, then we have*

$$\lim_{n\to\infty} H\left(\frac{1}{n}\sum_{i=1}^{n}\mathcal{U}_i(\omega), E\mathcal{U}_1\right) = 0, \quad a.s. \tag{30}$$

Proof. The map $I : \mathcal{K}(\mathfrak{X}_{\mathbb{E}}) \to \mathcal{K}(\mathbb{W})$ introduced in Section 3 is a positively homogeneous isometry. Clearly $\{I \circ \mathcal{U}_n\}$ is a sequence of i.i.d. random sets in $L^1(\Omega, \mathcal{K}(\mathbb{W}))$ and hence the strong law of large numbers of Artstein and Hansen [1] (see also [10]) yields

$$\lim_{n\to\infty} H\left(\frac{1}{n}\sum_{i=1}^{n} I \circ \mathcal{U}_i(\omega), EI \circ \mathcal{U}_1\right) = 0, \quad \text{a.s.} \tag{31}$$

Furthermore, in view of Proposition 3.4,

$$I^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} I \circ \mathcal{U}_i(\omega)\right) = \frac{1}{n}\sum_{i=1}^{n}\mathcal{U}_i(\omega), \qquad I^{-1}(EI \circ \mathcal{U}_1) = E\mathcal{U}_1$$

and thus

$$\begin{aligned} H\left(\frac{1}{n}\sum_{i=1}^{n}\mathcal{U}_i(\omega), E\mathcal{U}_1\right) &= H\left(I^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} I \circ \mathcal{U}_i(\omega)\right), I^{-1}(EI \circ \mathcal{U}_1)\right) \\ &= H\left(\frac{1}{n}\sum_{i=1}^{n} I \circ \mathcal{U}_i(\omega), EI \circ \mathcal{U}_1\right). \end{aligned}$$

From this and (31) we obtain (30), completing the proof. □

Theorem 6.2. *Let $\mathfrak{Y}$ be the space of all nonempty compact subsets of $\mathbb{R}^d$. Suppose that $\{\mathcal{U}_n\}$ is a sequence of i.i.d. random sets in $L^1(\Omega, \mathcal{K}(\mathfrak{Y}))$ satisfying $H(\mathcal{U}_n(\omega), \{\{0\}\}) \le \phi(\omega)$ for some $\phi \in L^1(\Omega, \mathbb{R})$. Then we have*

$$\lim_{n\to\infty} H\left(\frac{1}{n}\sum_{i=1}^{n}\mathcal{U}_i(\omega), E\mathcal{U}_1\right) = 0, \quad a.s. \tag{32}$$

Proof. Clearly $\{\overline{\text{co}}_{\mathfrak{X}}\Gamma \circ \mathcal{U}_n\}$ is a sequence of i.i.d. random sets in $L^1(\Omega, \mathcal{C}(\mathfrak{X}))$. Let $\{\xi_n\}$, where $\xi_n = J \circ \overline{\text{co}}_{\mathfrak{X}}\Gamma \circ \mathcal{U}_n$, correspond according to Theorem 3.3. Now $\{\xi_n\}$ is a sequence of i.i.d. random elements taking their values in $\mathbb{V}$, a convex and complete cone of a Banach space, and so the strong law of large numbers for random elements in Banach spaces [15] yields

$$\lim_{n\to\infty}\left\|\frac{1}{n}\sum_{i=1}^{n}\xi_i(\omega) - E\xi_1\right\| = 0, \quad \text{a.s.} \tag{33}$$

Since J is an isometry and

$$J\left(\frac{1}{n}\sum_{i=1}^{n}\overline{\text{co}}_{\mathfrak{X}}\Gamma \circ \mathcal{U}_i(\omega)\right) = \frac{1}{n}\sum_{i=1}^{n}\xi_i(\omega), \qquad J(E\overline{\text{co}}_{\mathfrak{X}}\Gamma \circ \mathcal{U}_1) = E\xi_1$$

then from (33) we deduce that

$$\lim_{n\to\infty} H\left(\frac{1}{n}\sum_{i=1}^{n}\overline{\mathrm{co}}_{\mathfrak{X}}\Gamma\circ\mathcal{U}_i(\omega), E\overline{\mathrm{co}}_{\mathfrak{X}}\Gamma\circ\mathcal{U}_1\right)=0, \quad \text{a.s.}$$

Moreover, $H(\mathcal{U}_n(\omega), \{\{0\}\}) \leq \phi(\omega)$ and thus by Theorem 5.2 one has

$$\lim_{n\to\infty} H\left(\frac{1}{n}\sum_{i=1}^{n}\mathcal{U}_i(\omega), E\overline{\mathrm{co}}_{\mathfrak{X}}\Gamma\circ\mathcal{U}_1\right)=0, \quad \text{a.s.}$$

Consequently (32) holds, since $E\overline{\mathrm{co}}_{\mathfrak{X}}\Gamma\circ\mathcal{U}_1 = E\mathcal{U}_1$ by (16). This completes the proof. □

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