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Ball Proximinal Spaces

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The notion of ball proximinality and the strong ball proximinality were recently introduced in [2]. We prove that spaces with strong $1\frac{1}{2}$-ball property are ball proximinal and in particular M-ideals are ball proximinal. We show that the problem of ball proximinality of hyperplanes is related to the problem of proximinality of certain convex sets determined by them.

Keywords: Proximinal, ball proximinal, strongly ball proximinal

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1. Introduction and Notation

Let X be a normed linear space and C be any closed subset of X. We say C is proximinal in X if for every x in X, the set

$$P_C(x) = \{y \in C : \|x - y\| = d(x, C)\}$$

is a non-empty set.

The notion of ball proximinality of a closed subspace was introduced in [2], motivated by an example of Saidi given in [17].

Definition 1.1. A subspace Y of a normed linear space X is ball proximinal in X if Y_1, the closed unit ball of Y, is proximinal in X.

It is easily verified (see [17] and [2]) that if Y is ball proximinal in X, then Y is proximinal in X. That the converse is not true, was shown in [17] by a counter example. Thus, ball proximinality implies proximinality, while the converse is not true.

In this paper, we show that subspaces with strong $1\frac{1}{2}$-ball property are ball proximinal. This gives many new examples of ball proximinal subspaces, including M-ideals. Also, it turns out that subspaces of real Banach spaces with the $1\frac{1}{2}$-ball property but

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not having the strong $1\frac{1}{2}$-ball property are not ball proximinal. This indicates a way to produce further examples of proximinal but non-ball proximinal spaces.

We then consider ball proximinality of hyperplanes and show that the ball proximinality of a proximinal hyperplane $H = \ker f$ is related to the proximinality of the face of the closed unit ball of X, formed by the set of elements where f attains its norm. Finally, we study ball proximinality of hyperplanes in specific Banach spaces like the sequence space c_0 and $C(Q, \mathbb{R})$.

We use the following notation and definitions in this paper. Throughout this paper, by a subspace we mean a closed subspace. If X is a normed linear space, X^* and $X^{(2)}$ denote the dual and bidual of X respectively and

$$X_1 = \{x \in X : \|x\| \leq 1\},$$

denotes the closed unit ball of X. For x in X and $r > 0$, we set

$$B[x, r] = \{y \in X : \|x - y\| \leq r\},$$
$$B(x, r) = \{y \in X : \|x - y\| < r\}$$

and if A is a subset of X then the distance of x from the set A is denoted by $d(x, A)$. That is,

$$d(x, A) = \inf\{\|x - z\| : z \in A\}.$$

For any $\delta > 0$ we set

$$P_C(x, \delta) = \{z \in C : \|x - z\| < d(x, C) + \delta\}.$$

Following [7], we say a proximinal set C of a normed linear space X is *strongly proximinal* if for each x in X and $\epsilon > 0$, there exists $\delta > 0$ such that

$$s(x, \delta) = \sup\{d(z, P_C(x)) : z \in P_C(x, \delta)\} < \epsilon. \quad (1)$$

Definition 1.2. A ball proximinal subspace Y of X is called strongly ball proximinal if Y_1 is strongly proximinal in X.

It is easily verified that strongly ball proximinal spaces are strongly proximinal.

2. Main Results

Let X be a Banach space and Y be a subspace of X. We first list some well known intersection properties of balls.

Definition 2.1 ([9]). A subspace Y of a Banach space X is said to have the n-ball property if for all families $B[x_i, r_i]$, $i = 1, 2, 3, ..., n$ of n closed balls satisfying

$$B[x_i, r_i] \cap Y \neq \emptyset \text{ for all } i = 1, 2, 3, ..., n$$

and

$$\bigcap_{i=1}^{n} B[x_i, r_i] \neq \emptyset,$$

then

$$\bigcap_{i=1}^{n} B[x_i, r_i + \epsilon] \cap Y \neq \emptyset \text{ for all } \epsilon > 0.$$

The $1\frac{1}{2}$-ball property is a weakening of the 2-ball property, by allowing the center of one of the balls to be in the subspace.

Definition 2.2 ([16]). A subspace Y of a Banach space X is said to have the (strong)$1\frac{1}{2}$-ball property if, whenever $B[a, r], B[b, s]$ are closed balls in X with $B[a, r] \cap B[b, s] \neq \emptyset$, $Y \cap B[a, r] \neq \emptyset$ and b in Y, then $Y \cap B[a, r + \epsilon] \cap B[b, s + \epsilon] \neq \emptyset$ for every $(\epsilon \geq 0)\epsilon > 0$.

It can be shown that [9] the 3-ball property implies the n-ball property for any $n > 3$ and the Strict n-ball property (Definition 2.1 holds with $\epsilon = 0$). It also follows from $ii) \Rightarrow v)$ of Theorem 2.2 of [9] that 3-ball property implies the the strong $1\frac{1}{2}$-ball property. Clearly, the 3-ball property implies the 2-ball property and the strong $1\frac{1}{2}$-ball property implies the $1\frac{1}{2}$-ball property.

We also need the notion of L-proximinality in the discussion.

Definition 2.3 ([16]). A subspace Y of a Banach space X is said to be ***L*-proximinal** if it is proximinal and $\|x\| = d(x, Y) + d(0, P_Y(x))$ for any x in X.

The notion of L-proximinality was introduced in [14] and its equivalence to $1\frac{1}{2}$-ball property was shown in [6] and [16]. We quote the relevant result below.

Fact A ([16]). Let Y be a subspace of a Banach space X. Then

1. Y has the $1\frac{1}{2}$-ball property in X if and only if Y is L-proximinal in X.
2. Y has the strong $1\frac{1}{2}$-ball property in X if and only if it is L-proximinal in X and for each x in X, there exists y in $P_Y(x)$ such that $\|x\| = \|x - y\| + \|y\|$.

We now prove our main results. We now show that spaces with the strong $1\frac{1}{2}$-ball property are ball proximinal.

Theorem 2.4. *Let X be a Banach space and Y be a subspace of X with the strong $1\frac{1}{2}$-ball property . Then Y is ball proximinal in X.*

Proof. Select any x in X. First observe that $d(x, Y_1) \geq d(x, Y)$ and hence $P_Y(x) \cap Y_1$ is contained in $P_{Y_1}(x)$. In particular, $P_{Y_1}(x)$ is non-empty if $P_Y(x) \cap Y_1$ is non-empty.

Now Y has the strong $1\frac{1}{2}$-ball property and so by Fact A, there exists y in $P_Y(x)$ such that

$$\|x\| = \|x - y\| + \|y\| = d(x, Y) + \|y\|. \tag{2}$$

We now consider two cases.

Case 1. $\|x\| \leq 1$. In this case, using (2) we have $\|y\| \leq 1$. Clearly y is in $P_Y(x) \cap Y_1$ and hence y is in $P_{Y_1}(x)$.

Case 2. $\|x\| > 1$. If 0 is in $P_Y(x)$ then clearly 0 is in $P_{Y_1}(x)$. So assume that 0 does not belongs to a non-empty set $P_Y(x)$. Hence $\|x\| > d(x, Y)$ and consequently

$\|y\| > 0$. Let $y_0 = \frac{y}{\|y\|}$. Then using (2) we have

$$\begin{aligned} d(x, Y_1) &\leq \|x - y_0\| \\ &\leq \|x - y\| + \|y - y_0\| \\ &= \|x\| - \|y\| + \|y\| - 1 \\ &= \|x\| - 1 \\ &= d(x, X_1) \\ &\leq d(x, Y_1). \end{aligned}$$

Therefore $\|x - y_0\| = d(x, Y_1)$ and y_0 is in $P_{Y_1}(x)$. □

The above theorem gives numerous new examples of ball proximinal spaces. Many examples of spaces with 3-ball property are known and by Theorem 2.4 these spaces are ball proximinal. It is well known that M-ideals have the 3-ball property. Hence we have

Corollary 2.5. *Let X be a Banach space and Y be an M-ideal in X. Then Y is ball proximinal in X.*

We recall that Banach spaces which are M-ideals in their second duals are called M-embedded spaces. We now have

Corollary 2.6. *Let X be an M-embedded Banach space. Then X is ball proximinal in its bidual.*

Well known examples of M-embedded spaces include c_0 and $K(H)$ [9]. By the above Corollary 2.6, these are examples of proxbid spaces which are ball proximinal in their biduals. A list of the spaces with the strong $1\frac{1}{2}$-ball property, which includes subalgebras of $C(Q, \mathbb{R})$, is given in [18]. By Theorem 2.4, these spaces provide further examples of ball proximinal spaces.

Remark 2.7. We recall from [18] that a subalgebra of $C(X, \mathbb{C})$ does not have the strong $1\frac{1}{2}$-ball property, unless it is an ideal. However it can be shown (Theorem E in [13]) that if Y is a subalgebra of $C(X, \mathbb{C})$ then Y is indeed strongly ball proximinal and the metric projection from $C(X, \mathbb{C})$ onto Y_1 is Hausdorff metric continuous.

We now characterize the spaces with the strong $1\frac{1}{2}$-ball property in terms of ball proximinality.

Theorem 2.8. *Let Y be a subspace of a Banach space X. Then Y has the strong $1\frac{1}{2}$-ball property if and only if the following hold:*

1. *Y is ball proximinal in X*
2. *for any x in X, if $\|x\| \leq 1$, $P_Y(x) \cap P_{Y_1}(x) \neq \emptyset$ and if $\|x\| > 1$, $d(x, X_1) = d(x, Y_1)$.*

Proof. Suppose Y has the strong $1\frac{1}{2}$-ball property. Then by Theorem 2.4, Y is ball proximinal in X. Also it is clear from the proof of Theorem 2.4 that, for any x in X, $P_Y(x) \cap P_{Y_1}(x) \neq \emptyset$, if $\|x\| \leq 1$ and $\|x\| - 1 = d(x, X_1) = d(x, Y_1)$, if $\|x\| > 1$.

Conversely assume both these conditions. We will show that Y has the strong $1\frac{1}{2}$-ball property. Let x be in X and $r > 0$. Assume $Y \cap B[x,r] \neq \emptyset$ and $\|x\| \leq r+1$. It is enough to show that $Y \cap B[0,1] \cap B[x,r] \neq \emptyset$. We now consider two cases.

Case 1. $\|x\| \leq 1$. By our assumption $P_Y(x) \cap P_{Y_1}(x) \neq \emptyset$. Let y_0 be in $P_Y(x) \cap P_{Y_1}(x)$. We have $d(x,Y) = d(x,Y_1) \leq \|x\| \leq r+1$. Now $Y \cap B[x,r] \neq \emptyset$ implies $d(x,Y) \leq r$, which in turn implies $\|x - y_0\| = d(x,Y_1) = d(x,Y) \leq r$. That is, y_0 is in $Y \cap B[0,1] \cap B[x,r]$.

Case 2. $\|x\| > 1$. In this case, $d = d(x,Y_1) = d(x,X_1) = \|x\| - 1 \leq r$. We have Y is ball proximinal in X. Select y in $P_{Y_1}(x)$. Then $\|x - y\| = d \leq r$. So y is in $Y \cap B[0,1] \cap B[x,r]$. □

It turns out that the spaces with the strong $1\frac{1}{2}$-ball property satisfy a stronger ball proximinality condition at all points with norm less than or equal to one.

Theorem 2.9. *If a subspace Y of a Banach space X has the strong $1\frac{1}{2}$-ball property, then Y is strongly ball proximinal at each x in X_1.*

Proof. Let x be in $X_1 \setminus Y$ and $d = d(x,Y)$. Then $d > 0$ and $\|x\| - d < 1$. Hence

$$\|x\| - d = 1 - \eta, \quad \text{for some } \eta > 0. \tag{3}$$

Given $\epsilon > 0$, choose $0 < \delta < 1$ such that $\delta + \frac{3\delta}{\delta+\eta} < \epsilon$.

Now by Theorem 2.8, $d = d(x,Y_1)$. Let y be in Y_1 such that

$$\|x - y\| < d + \delta. \tag{4}$$

Now by the strong $1\frac{1}{2}$-ball property of Y, $\|x - y\| = d + \inf\{\|z - y\| \ : \ z \in P_Y(x)\}$. This with (4) implies $d(y, P_Y(x)) < d + \delta - d = \delta$. So there exists y_0 in $P_Y(x)$ such that $\|y_0 - y\| < \delta$. Clearly, $\|y_0\| < \|y\| + \delta \leq 1 + \delta$. Now we will show that there exists z in $P_Y(x) \cap Y_1$ such that $\|y - z\| < \epsilon$ and this will complete the proof.

Note that by Fact A, we have $\|x\| - d = 1 - \eta = d(0, P_Y(x))$ and there is a z_1 in $P_Y(x)$ with $\|z_1\| = 1 - \eta$. Let $w_\lambda = \lambda y_0 + (1-\lambda) z_1$. Then $\|w_\lambda\| \leq \lambda(1+\delta) + (1-\lambda)(1-\eta) = 1 + \delta\lambda - (1-\lambda)\eta$. Now

$$1 + \delta\lambda - (1-\lambda)\eta = 1 \Longleftrightarrow 1 - \lambda = \frac{\delta}{\delta+\eta} \Longleftrightarrow \lambda = \frac{\eta}{\delta+\eta}.$$

Let $\lambda = \frac{\eta}{\delta+\eta}$ and $z = w_\lambda$. Then $0 < \lambda < 1$ and

$$\|y_0 - z\| = (1-\lambda)\|y_0 - z_1\| \leq \frac{3\delta}{\delta+\eta},$$

since $\|y_0 - z_1\| \leq 2 + 1 = 3$. Also, z is in $P_Y(x)$ as $P_Y(x)$ is a convex set and $\|z\| \leq 1 + \delta\lambda - (1-\lambda)\eta = 1$. Clearly z is in $P_{Y_1}(x)$ and $\|y - z\| \leq \|y - y_0\| + \|y_0 - z\| \leq \delta + \frac{3\delta}{\delta+\eta} < \epsilon$. □

Before proceeding further, we begin with the following simple observation.

Proposition 2.10. *Let Y be a proximinal subspace of a Banach space X and x be in X. If $\inf\{\|y\| : y \in P_Y(x)\} \leq 1$, then $d(x, Y) = d(x, Y_1)$. If Y is a strongly proximinal subspace of X, then $\inf\{\|y\| : y \in P_Y(x)\} \leq 1$ if and only if $d(x, Y) = d(x, Y_1)$.*

Proof. Suppose $\inf\{\|y\| : y \in P_Y(x)\} \leq 1$. We will show that $d(x, Y) = d(x, Y_1)$. To see this, note that $d(x, Y) \leq d(x, Y_1)$. So it is sufficient to show that $d(x, Y_1) \leq d(x, Y)$. By our assumption there exists $(y_n) \subseteq P_Y(x)$ such that $\lim_{n\to\infty} \|y_n\| = 1$. Let $z_n = \frac{y_n}{\|y_n\|}$ for every $n \geq 1$. Then z_n is in Y_1 and

$$\|x - z_n\| \leq \|x - y_n\| + \|y_n - z_n\| = d(x, Y) + \|y_n\| - 1$$

for all $n \geq 1$. Now taking limit as $n \to \infty$, we have $\lim_{n\to\infty} \|x - z_n\| = d(x, Y)$. Since z_n is in Y_1 for all n, this implies $d(x, Y_1) = d(x, Y)$.

Now suppose that Y is a strongly proximinal subspace of X and $d(x, Y) = d(x, Y_1)$. We will show that $\inf\{\|y\| : y \in P_Y(x)\} \leq 1$. Let $d = d(x, Y) = d(x, Y_1)$. Then there exists $(y_n) \subseteq Y_1$ such that $\lim_{n\to\infty} \|x - y_n\| = d$. Since Y is strongly proximinal in X, this implies $\lim_{n\to\infty} d(y_n, P_Y(x)) = 0$. Thus there exists $(z_n) \subseteq P_Y(x)$ such that $\|y_n - z_n\| \leq 2d(y_n, P_Y(x))$, for every $n \geq 1$. Clearly $\|z_n\| \leq \|y_n\| + 2d(y_n, P_Y(x)) \leq 1 + 2d(y_n, P_Y(x))$ and so $\lim_{n\to\infty} \|z_n\| = 1$. This clearly implies $\inf\{\|y\| : y \in P_Y(x)\} \leq 1$. □

We have given above many examples of ball proximinal spaces. Now, the result below indicates a way to produce examples of spaces which are proximinal but not ball proximinal.

Theorem 2.11. *Let Y be a subspace of a Banach space X. If Y has the $1\frac{1}{2}$-ball property but does not have the strong $1\frac{1}{2}$-ball property, then Y is not ball proximinal in X.*

Proof. Suppose that Y has the $1\frac{1}{2}$-ball property but does not have the strong $1\frac{1}{2}$-ball property. Then by Fact A, there exists x in X such that $\|x\| = d(x, Y) + \alpha$, where $\alpha = \inf\{\|y\| : y \in P_Y(x)\}$ and this infimum is not attained. If $\alpha = 0$, then we must have $\|x\| = d(x, Y)$. Hence 0 is in $P_Y(x)$ and the infimum is attained. So $\alpha > 0$. Let $x_0 = \frac{x}{\alpha}$. Then $P_Y(x_0) = \frac{1}{\alpha} P_Y(x)$, $\inf\{\|y\| : y \in P_Y(x_0)\} = 1$ and clearly this infimum is not attained. Now by Proposition 2.10, $d(x_0, Y) = d(x_0, Y_1)$ and therefore $P_{Y_1}(x_0) = P_Y(x_0) \cap Y_1$. But $P_Y(x_0) \cap Y_1$ is empty as $\inf\{\|y\| : y \in P_Y(x_0)\}$ is not attained. Consequently $P_{Y_1}(x_0)$ is empty and Y is not ball proximinal in X. □

Spaces with the $1\frac{1}{2}$-ball property satisfy a stronger proximinality criteria known as the U-proximinality (See [10]), defined below.

Definition 2.12 ([12]). A subspace Y of a Banach space X is said to **U-proximinal** in X if there exists a positive function $\epsilon(\rho)$, $\rho > 0$, with $\epsilon(\rho)$ tends to 0 as ρ tends to 0 and satisfies

$$(1 + \rho)X_1 \cap (X_1 + Y) \subseteq X_1 + \epsilon(\rho)(X_1 \cap Y).$$

The notion of U-proximinal spaces was introduced by Ka-sing Lau in [12]. If Y is a U-proximinal subspace of a Banach space X, then the metric projection P_Y is Hausdorff metric continuous (see [12]). In particular, P_Y has a continuous selection by the Michael selection theorem.

In [5], Garkavi had shown that if X is a non-reflexive Banach space and Y is a hyperplane in X, then X can be equivalently renormed so that Y has the $1\frac{1}{2}$-ball property but not the strong $1\frac{1}{2}$-ball property in X, endowed with the new norm. Thus we have

Corollary 2.13. *There exists a Banach space X and a U-proximinal hyperplane H in X such that H is not ball proximinal in X.*

Corollary 2.14. *There exists a Banach space X and a proximinal hyperplane H in X such that the metric projection P_H is Hausdorff metric continuous on X but H is not ball proximinal in X.*

3. Ball proximinal hyperplanes

Let X be a Banach space, f in $X^* \setminus \{0\}$ and let $H = \ker f$. We recall that for any x in X, we have $d(x, H) = \frac{|f(x)|}{\|f\|}$ and $P_H(x) = \{x - f(x)\, z \ : \ z \in J_X(f)\}$, when $\|f\| = 1$. In what follows, we derive a necessary condition satisfied by ball proximinal hyperplanes. To begin with, we have the following simple observation.

Proposition 3.1. *Let X be a Banach space, f in X^* with $\|f\| = 1$ and $H = \ker f$ be a proximinal hyperplane. Let x be an element in X satisfying $d(x, H) = d(x, H_1)$ and let $\alpha_x = \inf\{\|y\| : y \in P_H(x)\}$. Then we have the following.*

1. *If $\alpha_x < 1$, then $P_{H_1}(x) \neq \emptyset$.*
2. *If $\alpha_x > 1$, then $P_{H_1}(x) = \emptyset$.*
3. *If $\alpha_x = 1$, then $P_{H_1}(x) \neq \emptyset$ if and only if $P_{J_X(f)}(\frac{x}{f(x)}) \neq \emptyset$.*

Proof. Let $d = d(x, H) = d(x, H_1)$. In this case, clearly $P_{H_1}(x) \neq \emptyset$ if and only if $P_H(x) \cap H_1 \neq \emptyset$. If $\alpha_x < 1$, then $P_H(x) \cap H_1 \neq \emptyset$ and so $P_{H_1}(x) \neq \emptyset$. If $\alpha_x > 1$, then clearly $P_H(x) \cap H_1 = \emptyset$ and so $P_{H_1}(x) = \emptyset$. If $\alpha_x = 1$, then

$$\begin{aligned} P_H(x) \cap H_1 \neq \emptyset &\iff \text{there exists y in } \mathrm{P_H(x)} \text{ such that } \|\mathrm{y}\| = \alpha_\mathrm{x} = 1 \\ &\iff \inf\{\|y\| : y \in P_H(x)\} \text{ is attained} \\ &\iff \inf\{\|x - f(x)z\| : z \in J_X(f)\} \text{ is attained} \\ &\iff \inf\left\{\left\|\frac{x}{f(x)} - z\right\| : z \in J_X(f)\right\} \text{ is attained} \\ &\iff P_{J_X(f)}\left(\frac{x}{f(x)}\right) \neq \emptyset \end{aligned}$$

□

We now give a necessary condition for ball proximinality of a hyperplane. This result also shows that the ball proximinality of a hyperplane ker f is related to proximinality of the face $J_X(f)$, determined by the linear functional f in X^*.

Theorem 3.2. *Let X be a Banach space, f in X^* with $\|f\| = 1$ and $H = \ker f$ be a ball proximinal hyperplane. Then $P_{J_X(f)}(x) \neq \emptyset$ for all x in X with $f(x) = 1$.*

Proof. Let x be an element in X such that $f(x) = 1$. Without loss of generality, assume that $d(x, J_X(f)) = \beta > 0$. Now

$$\inf\{\|y\| : y \in P_H(x)\} = \inf\{\|x - z\| : z \in J_X(f)\} = \beta.$$

Let $w = \frac{x}{\beta}$. Then $P_H(w) = \frac{1}{\beta} P_H(x)$ and $f(w) = \frac{1}{\beta}$. So

$$\inf\{\|y\| : y \in P_H(w)\} = \frac{1}{\beta}\beta = 1.$$

Now by Proposition 2.10, $d(w, H) = d(w, H_1)$ and by Proposition 3.1,

$$P_{H_1}(w) \neq \emptyset \iff P_{J_X(f)}\left(\frac{w}{f(w)}\right) \neq \emptyset \iff P_{J_X(f)}(x) \neq \emptyset.$$

Since H is ball proximinal in X, we have $P_{H_1}(w) \neq \emptyset$. So $P_{J_X(f)}(x) \neq \emptyset$. Since x in X with $f(x) = 1$ was chosen arbitrarily, this proves our claim. □

We recall that a norm $\|.\|$ on a Banach space X is said to be strongly sub-differentiable (SSD) at x in X if the one-sided limit

$$\lim_{t \to 0^+} \frac{1}{t}(\|x + th\| - \|x\|)$$

exists uniformly in $h \in S_X$. The following characterization from [3] of functionals at which the dual norm is strongly sub differentiable, is needed in our discussion.

Theorem B ([7]). *Let X be a Banach space and f in X^* with $\|f\| = 1$. Then the following are equivalent.*

1. *The dual norm $\|.\|_{X^*}$ is SSD at f.*
2. *We have f in $NA_1(X)$ and for all $\epsilon > 0$, there exists $\delta > 0$ such that*

$$x \in X_1 \text{ and } f(x) > 1 - \delta \Longrightarrow d(x, J_X(f)) < \epsilon.$$

Further if 1. holds, then for any x in X,

$$d(x, J_X(f)) = d(x, J_{X^{(2)}}(f)). \tag{5}$$

Remark 3.3. It is stated in [7] that (5) holds for all x in X_1. However it is clear from the proof given therein that (5) holds for all x in X.

Theorem 3.4. *Let X be a Banach space, f in X^* with $\|f\| = 1$. If the proximinal set $J_{X^{(2)}}(f)$ is strongly proximinal in $X^{(2)}$ and $\|.\|_{X^*}$ is SSD at f, then $J_X(f)$ is strongly proximinal in X.*

Proof. Note that $\|.\|_{X^*}$ is SSD at f which implies $J_X(f)$ is a non-empty set. Also $\|.\|_{X^*}$ is SSD at f. So given $\eta > 0$, there exists $\delta_1 > 0$ such that

$$y \in X_1 \text{ and } f(y) > 1 - \delta_1 \Longrightarrow d(y, J_X(f)) < \eta. \tag{6}$$

Now $J_{X^{(2)}}(f)$ is strongly proximinal in $X^{(2)}$. So given $\epsilon > 0$, there exists $\delta = \delta_\epsilon > 0$ such that for any g in $X^{(2)}$ and ϕ is in $J_{X^{(2)}}(f)$, we have

$$\|g - \phi\| \leq d + \delta \Rightarrow \exists\ t \text{ in } J_{X^{(2)}}(f) \text{ with } \|g - t\| = d \text{ and } \|\phi - \mathrm{t}\| < \epsilon, \tag{7}$$

where $d = d(g, J_{X^{(2)}}(f))$. First we prove the following claim.

Claim. If x is in X, then given $\epsilon > 0$, there exists $\delta_\epsilon > 0$ (δ_ϵ tends to 0 as ϵ tends to 0) such that if y is in $J_X(f), \|x - y\| \leq d(x, J_X(f)) + \delta_\epsilon$ and k is in $\mathbb{N}$, there is a y_1 in $J_X(f)$ such that $\|x - y_1\| < d(x, J_X(f)) + \frac{\delta_\epsilon}{k}$ and $\|y_1 - y\| < \epsilon$.

Proof of the Claim. For $\epsilon > 0$, let $\delta = \delta_\epsilon$ be given by (7). We have $d = d(x, J_X(f)) = d(x, J_{X^{(2)}}(f))$. If y is in $J_X(f) \subseteq J_{X^{(2)}}(f)$ and $\|x - y\| \leq d + \delta$, then there exists t in $J_{X^{(2)}}(f)$ such that $\|x - t\| = d$ and $\|t - y\| < \epsilon$. Choose $0 < \eta < \frac{\delta}{2k}$ such that $\|t - y\| + 2\eta < \epsilon$. By the Principle of local reflexivity, there exists x_η in X_1 such that $\|x - x_\eta\| < d + \eta$, $\|x_\eta - y\| < \|t - y\| + \eta$ and $f(x_\eta) > 1 - \delta_1$. By (6), there exists y_1 in $J_X(f)$ such that $\|x_\eta - y_1\| < \eta$. Also $\|x - y_1\| \leq \|x - x_\eta\| + \|x_\eta - y_1\| < d + \eta + \eta < d + \frac{\delta}{k}$ and $\|y - y_1\| \leq \|y - x_\eta\| + \|x_\eta - y_1\| \leq \|t - y\| + \eta + \eta < \epsilon$. Hence the Claim.

We now show that the set $P_{J_X(f)}(x)$ is non-empty, if x is in X. Let x be an element in X and $\epsilon_n = \frac{\epsilon}{2^n}$ for $n \geq 1$. Choose $(k_n) \subseteq \mathbb{N}$ such that $\frac{\delta_{\epsilon_n}}{k_n} < \delta_{\epsilon_{n+1}}$ for $n \geq 1$. Select z_1 in $J_X(f)$ such that $\|x - z_1\| \leq d + \delta_{\epsilon_1}$. Then there exists z_2 in $J_X(f)$ such that $\|z_1 - z_2\| < \epsilon_1$ and $\|x - z_2\| < d + \frac{\delta_{\epsilon_1}}{k_1} < d + \delta_{\epsilon_2}$. Assume $\{z_1, z_2, ...z_n\} \subseteq J_X(f)$ have been constructed so that $\|z_i - z_{i+1}\| < \epsilon_i$ for $1 \leq i \leq n-1$ and $\|x - z_i\| < d + \delta_{\epsilon_i}$ for $1 \leq i \leq n$. By the above claim, there exists z_{n+1} in $J_X(f)$ such that $\|z_n - z_{n+1}\| < \epsilon_n$ and $\|x - z_{n+1}\| < d + \delta_{\epsilon_{n+1}}$. This completes the induction. If $z_\infty = \lim_{n\to\infty} z_n$, then z_∞ is in $J_X(f)$ and $\|x - z_\infty\| = d$. So z_∞ is in $P_{J_X(f)}(x)$. Further for $n \geq 1$, we have

$$\begin{aligned} \|z_1 - z_n\| &\leq \sum_{i=1}^{n-1} \|z_i - z_{i+1}\| \\ &< \sum_{i=1}^{n} \epsilon_i \\ &\leq \epsilon. \end{aligned}$$

Now taking limit n tends to ∞, we have $\|z_1 - z_\infty\| \leq \epsilon$ and hence $J_X(f)$ is strongly proximinal at x. Since x in X was arbitrarily chosen, this implies $J_X(f)$ is strongly proximinal in X. □

4. Results from specific Banach spaces

In this section we present few results related to the ball proximinality in the real Banach spaces c_0 and $C(Q, \mathbb{R})$.

Here we recall that the sequence space c_0 is an M-ideal in l_∞ and hence by Corollary 2.5, c_0 is ball proximinal in l_∞. However the simple direct proof for the fact that the (real) sequence space c_0 is ball proximinal in l_∞ is given below.

Let $X = c_0$, $x = (x_1, x_2, x_3, ...)$ be in l_∞, $\overline{\alpha} = \limsup |x_n|$ and $\underline{\alpha} = \liminf |x_n|$. Then $d(x, X_1) = \max\{\|x\| - 1, \limsup |x_n|, \liminf |x_n|\}$. For, choose $N_2 < N_3 < ... < N_k < ...$ such that $\underline{\alpha} + \frac{1}{k} < x_n < \overline{\alpha} - \frac{1}{k}$, for all $n \geq N_k$. Now choose $|z_n| \leq \frac{1}{k}$ and $|x_n - z_n| < \max\{|\overline{\alpha}|, |\underline{\alpha}|\}$, where $N_k \leq n < N_{k+1}$ and

$$z_n = \begin{cases} -1, & \text{if } x_n < -1; \\ x_n, & \text{if } |x_n| \leq 1; \\ 1, & \text{if } x_n > 1 \end{cases}$$

for $1 \leq n \leq N_2$. Now let $z = (z_n)$. Then z is in X_1 and $\|x - z\| = \max\{\|x\| - 1, \limsup |x_n|, \liminf |x_n|\}$. Hence c_0 is ball proximinal in l_∞.

We now show that if Y is a proximinal subspace of finite codimension in c_0, then Y is ball proximinal in $l_\infty \cong (c_0)^{(2)}$. Our proof is similar to that of Theorem 4.1 in [11]. We need the following result from [2] in this proof.

Proposition C ([2]). *Let $\{X^i : i \in \mathbb{N}\}$ be a family of Banach spaces and Y^i be a ball proximinal subspace in X^i for each $i \in \mathbb{N}$. Consider the following direct sums $X = (\oplus_{c_0} X^i)_{i\in\mathbb{N}}$ and $Y = (\oplus_{c_0} Y^i)_{i\in\mathbb{N}}$. Then Y is a ball proximinal subspace of X.*

Theorem 4.1. *A finite co-dimensional, proximinal subspace of c_0 is ball proximinal in l_∞ and hence ball proximinal in c_0.*

Proof. Let Y be a finite co-dimensional proximinal subspace of c_0. Since $NA(c_0)$ is the set of all finite sequences in l_1 and $Y^\perp$ is a finite dimensional subspace of X^*, there exists a positive integer N such that for any $f = (f_n)$ in $Y^\perp$, f_n is zero, for all $n \geq N$.

Let $\{e_n : n \geq 1\}$ denote the natural basis of c_0. For any sequence $x = (x_n)$ of scalars, we set $x' = \sum_{n=1}^{N} x_n e_n$. Also we set

$$X' = \mathrm{sp}\{e_1, e_2, ...e_N\},$$
$$X'' = \{(x_n) \in l_\infty : x_n = 0, 1 \leq n \leq N\},$$
$$Y' = \left\{x' : x \in Y\right\}$$

and finally

$$Y'' = \{(x_n) \in c_0 : x_n = 0, 1 \leq n \leq N\}.$$

Recall that c_0 is an M-ideal in l_∞ and so it follows that Y'' is an M-ideal in X''. Now by the Corollary 2.5, Y'' is ball proximinal in X''. Since Y' is a subspace of the finite dimensional space X', Y' is ball proximinal in X'. Now $X = X' \oplus_\infty X'' = l_\infty$ and $Y = Y' \oplus_\infty Y''$. Then by Proposition C, Y is ball proximinal in l_∞. □

We now consider the Banach space $C(Q, \mathbb{R})$. We show that if $H = \ker \mu$ is a proximinal hyperplane in $C(Q, \mathbb{R})$, then $J_X(\mu)$ is a proximinal subset of $C(Q, \mathbb{R})$. Thus the necessary condition for ball proximinality given by Theorem 3.2 is satisfied by all the proximinal hyperplanes in $C(Q,\mathbb{R})$.

Before we proceed with the proof, we quote the following well known fact and theorem that are needed.

Fact D. Let $X = C(Q,\mathbb{R})$ and μ in $(C(Q,\mathbb{R})^*$. Then μ is in $NA(X)$ if and only if $S(\mu^+) \cap S(\mu^-) = \emptyset$.

Theorem E (Interposition Theorem) ([4]). *Let S be a normal topological space. If g and h are real valued functions on S, g is u.s.c., h is l.s.c. and $g \leq h$, then there exists $f \in C(S,\mathbb{R})$ such that $g \leq f \leq h$.*

Theorem 4.2. *Let $X = C(Q,\mathbb{R})$ with sup norm and μ in $NA(X)$. Then $J_X(\mu)$ is proximinal in X.*

Proof. Pick any f in X. Let $\alpha = \max\left\{\sup_{q\in S(\mu^+)} |f(q)-1|, \sup_{q\in S(\mu^-)} |f(q)+1|\right\}$.

Case 1. $\sup_{q\in Q\setminus S(\mu)} d(f(q),[-1,1]) \leq \alpha$. Note that an element g is in $J_X(\mu)$ if and only if $\|g\| = 1$, $g \equiv 1$ on $S(\mu^+)$, $g \equiv -1$ on $S(\mu^-)$. So $d(f, J_X(\mu)) \geq \alpha$. We will now construct g in $J_X(\mu)$ such that $\|f - g\| \leq \alpha$. This will complete the proof. Define g_1 and g_2 on Q as follows.

$$g_1(q) = g_2(q) = 1, \quad \text{if q} \in \text{S}(\mu^+), \tag{8}$$

$$g_1(q) = g_2(q) = -1, \quad \text{if q} \in \text{S}(\mu^-). \tag{9}$$

If q is in $Q \setminus S(\mu)$, set

$$g_1(q) = \begin{cases} 1 & \text{if } f(q) \geq 1 \\ f(q) + \min\{\alpha, 1 - f(q)\} & \text{if } f(q) < 1 \end{cases}$$

and

$$g_2(q) = \begin{cases} -1 & \text{if } f(q) \leq -1 \\ f(q) - \min\{\alpha, 1 + f(q)\} & \text{if } f(q) > -1 \end{cases}$$

Clearly $g_2 \leq g_1$ on Q and

$$\sup_{q\in Q} |f(q) - g_i(q)| = \alpha, \quad i = 1, 2 \tag{10}$$

and

$$\sup_{q\in Q} |g_i(q)| \leq 1, \quad i = 1, 2. \tag{11}$$

If g_1 is l.s.c. on Q and g_2 is u.s.c. on Q, then by Theorem E, there exists g in $C(Q)$ such that $g_2 \leq g \leq g_1$ on Q. Now (10) and (11) would imply $\|g\| \leq 1$ and $\sup_{q\in Q} |f(q) - g(q)| \leq \alpha$. It is clear from (8) and (9) that g is in $J_X(\mu)$ and hence g is a nearest element to f from $J_X(\mu)$. So it suffices to show that g_1 is l.s.c. on Q and g_2 is u.s.c. on Q.

Note that since $S(\mu^+)$ and $S(\mu^-)$ are disjoint closed sets, $g_{i|S(\mu)}$ is continuous for each $i = 1, 2$. It is easily verified that g_i restricted to the set $Q \setminus S(\mu)$ is continuous for each $i = 1, 2$. Thus it is enough to verify l.s.c. (u.s.c.) of $g_1(g_2)$ at all points of $S(\mu^+) \cap \overline{Q \setminus S(\mu)}$ and $S(\mu^-) \cap \overline{Q \setminus S(\mu)}$.

We now show that g_1 is l.s.c. at all points of $S(\mu) \cap \overline{Q \setminus S(\mu)}$. Pick any q_0 in $S(\mu^+) \cap \overline{Q \setminus S(\mu)}$. Then $g_1(q_0) = 1$. Let $(q_n) \subseteq Q \setminus S(\mu)$ be a sequence which converges to q_0. We will show that $\lim_{n\to\infty} g_1(q_n) = 1$. If $\lim_{n\longrightarrow\infty} f(q_n) > 1$, then $g_1(q_n) = 1$ eventually and $\lim_{n\to\infty} g_1(q_n) = 1$. Let $\lim_{n\to\infty} f(q_n) \leq 1$. Then $\lim_{n\to\infty} 1 - f(q_n) = 1 - f(q_0) \leq \alpha$. So there exists a sequence (ϵ_n) of non-negative numbers such that $\lim_{n\to\infty} \epsilon_n = 0$ and $1 - f(q_n) < \alpha + \epsilon_n$ for all $n \geq 1$. It is now easy to verify that either $g_1(q_n) = 1$ or $g_1(q_n) = f(q_n) + \alpha \geq f(q_n) + 1 - f(q_n) - \epsilon_n = 1 - \epsilon_n$ for all $n \geq 1$. In either case, $\lim_{n\to\infty} g_1(q_n) = 1$.

Let q_0 be an element in $S(\mu^-) \cap \overline{Q \setminus S(\mu)}$. Then $g_1(q_0) = -1$. If $f(q_0) > 1$, then there exists an open neighbourhood U of q_0 such that $g_1(q) = 1 > -1 = g_1(q_0)$, for every q in U. If $f(q_0) \leq 1$ and $1 - f(q_0) < \alpha$, then there exists an open neighbourhood U of q_0 such that $1 - f(q) < \alpha$ and $g_1(q) = f(q) + 1 - f(q) = 1$ for all q in U. If $1 - f(q_0) = \alpha$, then for any $0 < \epsilon < \frac{1}{2}$, there exists an open neighbourhood U of q_0 such that $|1 - f(q) - \alpha| < \epsilon$, for every q in U. Thus for q in U,

$$g_1(q) = \begin{cases} 1 & \text{if } 1 - f(q) \leq \alpha \\ f(q) + \alpha & \text{if } 1 - f(q) > \alpha \end{cases}$$

Now $g_1(q) = f(q) + \alpha > f(q) + 1 - f(q) - \epsilon = 1 - \epsilon$, if $1 - f(q) > \alpha$. That is, $g_1(q) > 1 - \epsilon$, for every q in U. In each case, there exists an open neighbourhood U of q_0 such that $g_1(q) \geq 1 - \epsilon > \frac{1}{2} > -1 = g_1(q_0)$, for every q in U. So g_1 is l.s.c. at q_0. This complete the proof for g_1 is l.s.c. on Q. A similar proof shows that g_2 is u.s.c. on Q.

Case 2. $\beta = \sup_{q\in Q\setminus S(\mu)} d(f(q), [-1, 1]) > \alpha$. Clearly $d(f, J_X(\mu)) \geq \beta$. Let $f_1 = \min\{f, 1 + \alpha\}$ and $f_2 = \max\{f, -1 - \alpha\}$. Then $f_2 \equiv f$ on $S(\mu)$ and $\alpha = \sup_{q\in Q\setminus S(\mu)} d(f_2(q), [-1, 1])$. By *Case 1*, there is a g in $J_X(\mu)$ such that $\|g - f_2\| = \alpha$. Then $|g(q) - f_2(q)| = \alpha < \beta$, for every q in Q. Note that

$$A = \{q \in Q : f(q) \neq f_2(q)\} \subseteq \{q \in Q : f(q) > 1 + \alpha\} \cap \{q \in Q : f(q) < -1 + \alpha\}.$$

It is enough to show that $|g(q) - f(q)| \leq \beta$ for q in A. If $f(q) > 1 + \alpha$, then $f_2(q) = 1 + \alpha$ and if $f(q) < -1 - \alpha$, then $f_2(q) = -1 + \alpha$. Now $\|f_2 - g\| \leq \alpha$ and $\|g\| \leq 1$ implies that $g(q) = 1$ if $f_2(q) = 1 + \alpha$ and $g(q) = -1$ if $f_2(q) = -1 - \alpha$. In either case, we have $|f(q) - g(q)| \leq \beta$ and $\|f - g\| \leq \beta$. Clearly g is a nearest element to f from $J_X(\mu)$. □

In [15], it has been shown that if X is a Banach space and μ is in S_{X^*} such that μ is an SSD point, then $S(\mu)$ is finite.

Theorem 4.3. *Let $X = C(Q, \mathbb{R})$, μ in X^* with $S(\mu)$ be a finite set. Then $H = \ker \mu$ is ball poximinal in X.*

Proof. Let $S(\mu) = \{q_i : 1 \leq i \leq k\}$ and $\mu = \sum_{i=1}^{k} \beta_i \delta_{q_i}$ where β_i is in $\mathbb{R}$, $1 \leq i \leq k$.

Pick any f in X. Set

$$\alpha = \inf\left\{ \max_{1\le i\le k} |\alpha_i - f(q_i)| : \alpha_i \in [-1,1],\ 1 \le i \le k \text{ and } \sum_{i=1}^{k} \alpha_i\beta_i = 0 \right\}.$$

Note that this infimum is attained. For, the set

$$A = \left\{ (\alpha_1, \alpha_2, ...\alpha_k) \in [-1,1]^k : \sum_{i=1}^{k} \alpha_i\beta_i = 0 \right\}$$

is a closed subset of the compact set $[-1,1]^k$ and the map $(\alpha_1, \alpha_2, ...\alpha_k) \longmapsto \max_{1\le i\le k} |\alpha_i - f(q_i)|$ is continuous on $\mathbb{R}^k$. Pick an element $(\alpha_1, \alpha_2, ...\alpha_k)$ in A, where the infimum is attained.

Case 1. $\sup_{q\in Q\setminus S(\mu)} d(f(q), [-1,1]) \le \alpha$. We observe that $H_1 = \{h \in C(Q,\mathbb{R}) : \|h\| \le 1 \text{ and } \sum_{i=1}^{k} h(q_i)\beta_i = 0\}$ and so $d(f, H_1) \ge \alpha$ in this case.

Let $h_1 = \min\{1, f\}$ and $h_2 = \max\{-1, h_1\}$. Then h_2 is in $C(Q,\mathbb{R})$. Let $\{U_i\}_1^k$ be pairwise disjoint open neighbourhoods of $\{q_i\}_1^k$ respectively. Let $U = \bigcup_{i=1}^{k} U_i$. Define $g(q_i) = \alpha_i, 1 \le i \le k$ and $g(q) = h_2(q)$ for q in $Q \setminus U$. Extend g continuously to Q with $\|g\| \le 1$. Let $g_1 = \min\{g, f + \alpha\}$ and $g_2 = \max\{g_1, f - \alpha\}$. Since $f + \alpha \ge -1$ on Q, $-1 \le g_1(q) \le 1$ for every q in Q and since $f - \alpha \le 1$ on Q, $-1 \le g_2(q) \le 1$ for every q in Q. Thus $|g_2| \le 1$ on Q. Now $g_1 \le f + \alpha$ and $f - \alpha \le f + \alpha$. So $g_2 \le f + \alpha$. Also $g_2 \ge f - \alpha$. Hence $\|f - g_2\| \le \alpha$ and g_2 is a nearest element to f from H_1.

Case 2. $\beta = \sup_{q\in Q\setminus S(\mu)} d(f(q), [-1,1]) > \alpha$. Clearly $d(f, H_1) \ge \beta$ in this case. Define $f_1 = \min\{f, 1 + \alpha\}$ and $f_2 = \max\{f, -1 - \alpha\}$. Then $f_2(q_i) = f(q_i), 1 \le i \le k$. Then by *Case 1*, there is a g in H_1 such that $\|f_2 - g\| \le \alpha$. We now claim that $\|f - g\| \le \beta$. Clearly $\max_{1\le i\le k} |f(q_i) - g(q_i)| \le \alpha < \beta$. Pick any q in $Q \setminus S(\mu)$. If $f_2(q) = f(q)$, clearly $|f(q) - g(q)| = |f_2(q) - g(q)| \le \alpha < \beta$. If $f_2(q) \ne f(q)$, then either $f(q) > 1 + \alpha$ or $f(q) < -1 - \alpha$. If $f(q) > 1 + \alpha$, then $f_2(q) = 1 + \alpha$ and consequently $g(q) = 1$. Thus $|f(q) - g(q)| = d(f(q), [-1,1]) \le \beta$. If $f(q) < -1 - \alpha$, then $f_2(q) = -1 - \alpha$ and $g(q) = -1$. Clearly $d(f(q), [-1,1]) = |f(q) - g(q)| \le \beta$ in this case. Thus $\|f - g\| \le \beta$ and g is a nearest element to f from H_1. This implies H_1 is proximinal and H is ball proximinal in $C(Q,\mathbb{R})$. □

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