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Characterization of Weakly Efficient Solutions for Non-Regular Multiobjective Programming Problems with Inequality-Type Constraints*

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Necessary conditions of optimality are presented for weakly efficient solutions to multiobjective minimization problems with inequality-type constraints. These conditions are applied when the constraints do not necessarily satisfy any regularity assumptions and they are based on the concept of 2-regularity introduced by Izmailov. In general, the optimality conditions do not provide the complete weak Pareto optimal set, so 2-KKT-pseudoinvex problems are defined. This new concept of generalized convexity is both necessary and sufficient to guarantee the characterization of all weakly efficient solutions based on the optimality conditions and it is the weakest one.

Keywords: Convexity, regularity, constraints qualifications, optimality conditions

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1. Introduction

In many real-life optimization problems, multiple objectives must be taken into account that may be related to the economical, technical, social and environmental aspects of optimization problems.

In this paper we study the general Multiobjective Optimization Problem:

$$\begin{array}{ll} \text{Min} & f(x) = (f_1(x), \ldots, f_p(x)), \\ \text{s.t.} & x \in A, \end{array} \qquad \text{(MOP)}$$

where the feasible region A is expressed as a finite number of inequality constraints, that is, $A = \{x \in \mathbb{R}^n : g_i(x) \leq 0,\ i = 1, \ldots, m\}$. Let $f_j : \mathbb{R}^n \longrightarrow \mathbb{R}\ j = 1, ..., p$ and $g_i : \mathbb{R}^n \longrightarrow \mathbb{R},\ i = 1, \ldots, m$.

In this paper, we assume that all the functions are Fréchet differentiable and that $g_i,\ i = 1, \ldots, m$ is twice Fréchet differentiable when necessary. We denote the first- and second-order derivatives for a scalar function by ∇g_i, $\nabla^2 g_i$ and the first-order derivative for a vector function by f'.

We say that a function is twice Fréchet differentiable at a point x, when it is Fréchet differentiable on a neighborhood of x, and its derivative is Fréchet differentiable at x.

All vectors are row vectors. We use the superscript t to denote transposition.

The multiple objectives in an optimization problem are usually incommensurate and in conflict one another. This means that in general, a multiple objective optimization problem does not have a single solution that can optimize all objectives simultaneously. Because of this, the goal in multiple optimization is not search for optimal solutions but instead to find efficient (non-inferior, non-dominated or Pareto-optimal) solutions that can best attain the prioritized multiple objectives.

The weakly efficient solution is an important concept in mathematical modeling, economics, decision theory, optimal control and game theory. However, finding the weakly efficient solution set of (MOP) is not an easy task. Many authors have studied sufficient and necessary optimality conditions of the Karush-Kuhn-Tucker type (hereafter KKT conditions) involving weakly efficient solutions for a multiobjective programming problem ([19], [13], [10]).

But, as it is well known, the KKT conditions are necessary for optimality if the problem (MOP) is *regular* and are sufficient if (MOP) is a convex problem.

In the past few years, numerous papers have appeared in the literature reflecting further generalizations in these categories. Kaul *et al.* [9] considered a differentiable multiobjective optimization problem involving generalized Type I functions. They investigated KKT type necessary and sufficient conditions under generalized Type I assumptions. Combining the concepts of Type I and uninvex functions, Rueda *et al.* [18] gave optimality conditions and duality in various settings (real valued, fractional, multiobjective). Suneja and Srivastava [20] introduced generalized *d*-Type I functions, which are defined in terms of the directional derivative for a multiobjective optimization problem. In [16] and [17], it was proved that the equivalence between minima and Karush-Kuhn-Tucker points, established by Martin [12] for scalar differentiable optimization problems when the problem verifies the KKT-invexity, remains true for differentiable vector optimization problems. To this end, vector Karush-Kuhn-Tucker points and KKT-pseudoinvex vector problems were accurately defined. Moreover, it was proved that these new generalized convexity assumptions are the weakest necessary conditions to characterize the weak Pareto solution set completely.

In some approaches to optimization problems, the necessary optimality conditions are derived under the same constraint qualifications as in nonlinear programming, with a scalar-valued objective function. Constraint qualifications are often assumed, but do not always hold. A problem whose constraints do not verify any regularity conditions is called a non-regular problem. The relevance of these scalar problems and important references that contain meaningful practical examples are given in [2] and [5].

In [8], Izmailov presented a constructive description of the tangent cone to the admissible set at a particular feasible point for mathematical programming problems with non-regular inequality-type constraints, and meaningful necessary optimality conditions were provided on the basis of the above description of the tangent cone. Later, Avakov *et al.* [3] proved first- and second-order necessary conditions for a local extremum for a scalar-valued minimization problems with equality and inequality constraints when the constraints do not satisfy any regularity assumptions.

For the non-regular scalar case, convexity notions are scarce or absent, only one work [6] presents meaningful generalized convexity notions. This concept is the weakest possible that ensures that the necessary optimality condition given by Izmailov in [8] is sufficient, and thus the solution set for the scalar case is characterized.

But for the multiobjective non-regular case, there are no optimality conditions for weakly efficient solutions and there are no generalized convexity notions to characterize the optimal solutions set.

In this paper, we give necessary optimality conditions for a multiobjective problem whose constraints do not necessarily verify any constraint qualification, based on Izmailov's description of its tangent cone in the 2-regular case. Furthermore, we present a suitable definition of Karush-Kuhn-Tucker points and, to ensure that the optimality conditions obtained are also sufficient to characterize the complete weakly efficient solutions set, we define a new concept of generalized convexity since the existing ones in the literature are not valid in the non-regular case. Finally, taking into account the ideas of Martin and Osuna-Gómez ([12], [17]) we prove that the concept of generalized convexity given here is the weakest possible to ensure the sufficiency of the necessary optimality conditions presented.

This work is organized as follows. In Section 1, we recall some classic results and definitions on optimality conditions and convexity for multiobjective regular problems. In Section 2, we recall some definitions and results from Izmailov's paper [8] for non-regular problems. In Section 3, we present necessary optimality conditions for weakly efficient solutions of (MOP). In Section 4, we show some illustrative examples and in Section 5 we define a proper generalized convexity notion for the non-regular case, which makes the necessary optimality conditions also sufficient. Finally, in Section 6, we present our conclusions.

2. Optimality conditions for multiobjective regular or non-degenerate problems

The following convention for vectors of $\mathbb{R}^n$ will be followed throughout this paper:

- $x < y \Leftrightarrow x_i < y_i, \forall\, i \in \{1, \ldots, n\}$;
- $x \leqq y \Leftrightarrow x_i \leq y_i, \forall\, i \in \{1, \ldots, n\}$;
- $x \leq y \Leftrightarrow x_i \leq y_i$ and $x \neq y$.
- $x > y \Leftrightarrow x_i > y_i, \forall\, i \in \{1, \ldots, n\}$;
- $x \geqq y \Leftrightarrow x_i \geq y_i, \forall\, i \in \{1, \ldots, n\}$;
- $x \geq y \Leftrightarrow x_i \geq y_i$ and $x \neq y$.

We first recall some definitions and results that are needed for the main results of this work.

Definition 2.1. A point $x^* \in A$ is said to be an efficient (Pareto) solution of (MOP) if and only if there exists no $x \in A$ such that $f(x) \leq f(x^*)$.

At times, locating efficient points is quite costly. As a result, a more general concept appears, namely the weakly efficient solution, which, under certain conditions, has better topological properties than the efficient solution [15].

Definition 2.2. A point $x^* \in A$ is said to be a weakly efficient (weak Pareto) solution of (MOP) if and only if there exists no $x \in A$ such that $f(x) < f(x^*)$.

Obviously every efficient solution is a weakly efficient solution.

Definition 2.3. A feasible point $x^* \in A$ is said to be a vector Fritz-John point (hereafter VFJP) for (MOP), if there exists $(\lambda^*, \mu^*) \in \mathbb{R}^{p+m}$, $(\lambda^*, \mu^*) \geqq 0$, such that

$$\begin{cases} \lambda^* f'(x^*) + \mu^* g'(x^*) = 0, \\ \mu^* g(x^*)^t = 0. \end{cases} \tag{1}$$

Definition 2.4. A feasible point $x^* \in A$ is said to be a vector Karush-Kuhn-Tucker point (hereafter VKKTP) for (MOP) if there exists $(\lambda^*,\ \mu^*) \in \mathbb{R}^{p+m}, (\lambda^*, \mu^*) \geqq 0$, $\lambda^* \neq 0$ such that

$$\begin{cases} \lambda^* f'(x^*) + \mu^* g'(x^*) = 0, \\ \mu^* g(x^*)^t = 0. \end{cases} \tag{2}$$

The following necessary optimality conditions for weakly efficiency are well known (see, e.g. [7] and [17]).

Theorem 2.5. *Let $x^* \in A$ be a weakly efficient solution for* (MOP) *then x^* is a VFJP for* (MOP).

If we add a constraint qualification, we can ensure that $\lambda^* \neq 0$, so we have a VKKTP for (MOP).

Theorem 2.6. *Assume that $x^* \in A$ is a weakly efficient solution for* (MOP) *and that a constraint qualification is satisfied at x^*. Then x^* is a VKKTP for* (MOP).

The reverse result is not true in general, so a generalized convexity notion is introduced to completely characterize the weakly efficient solution set, that is, to ensure that every VKKTP is a weakly efficient solution. The weakest convexity notion, is the KKT-pseudoinvexity [17].

Definition 2.7. The problem (MOP) is said to be a vector KKT-pseudoinvex problem on A, if there exists $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that for all $x,\ x^* \in A$, it is verified

$$\begin{cases} f(x) - f(x^*) < 0 \Rightarrow \eta(x, x^*) f'(x^*)^t < 0, \\ -\eta(x, x^*) \nabla g_i(x^*)^t \geq 0, \quad \forall i \in I(x^*), \end{cases}$$

where $I(x^*) = \{i = 1, ..., m : g_i(x^*) = 0\}$.

Theorem 2.8. *Every VKKTP is a weakly efficient solution if and only if* (MOP) *is a vector KKT-pseudoinvex problem on A.*

3. Description of the tangent cone for non-regular problems

The results presented in Section 2 do not make sense if the problem does not satisfy any constraint qualifications to ensure that the weakly efficient solutions can be characterize by the VKKTP. Therefore, it is of great interest to find results that extend those presented above for the non-regular case.

If the constraints of the problem do not satisfy any regularity assumptions, the tangent cone cannot be characterized by the linear approximations of the active constraints at a feasible point, and in consequence, meaningful optimality conditions cannot be established. Therefore, Izmailov proposed in [8] to use the second-order approximations of the active constraints at a feasible point in order to characterize the tangent cone in the 2-regular case and to use it to establish optimality conditions that will not be linear.

Based on the following scalar optimization problem,

$$\begin{array}{ll} \text{Min} & \phi(x), \\ \text{s.t.} & x \in A, \end{array} \quad \text{(SOP)}$$

Izmailov characterized a regular problem, and we shall prove that this characterization implies that its constraints satisfy a constraint qualification.

Proposition 3.1 ([8]). *The constraints of* (SOP) *are regular at $x^* \in A$ on the element $h \in \mathbb{R}^n$ satisfying the condition $h \in G'_A(x^*) = \{h \in \mathbb{R}^n : \nabla g_i(x^*) h^t \leq 0, \forall i \in I(x^*)\}$ if and only if*

$$\exists\ \tilde{h} \in \mathbb{R}^n : \nabla g_i(x^*) \tilde{h}^t < 0, \quad \forall\ i \in I(x^*).$$

Note that the regularity condition can be defined regardless of the specific element h, and it is valid for the multiobjective problem we are studying in this paper (MOP) since its constraints are the same.

Definition 3.2. The vector $h \in \mathbb{R}^n$ is said to be tangent to the set A at the point x^* if there exist a number $\epsilon > 0$ and a mapping $r(\cdot) : [0, \epsilon) \longrightarrow \mathbb{R}^n$ such that

$x^* + th + r(t) \in A \; \forall \; t \in [0, \epsilon)$ with $\frac{\|r(t)\|}{t} \to 0$ as $t \to 0^+$. The set of all vectors that are tangent to the set A at the point x^* forms a cone, called the tangent cone to the set A at the point x^*, and it is denoted by $T_A(x^*)$.

We shall prove that the regularity characterization given in Proposition 3.1 implies the Abadie constraint qualification ($G'_A(x^*) = T_A(x^*)$), which is the weakest possible in the multiobjective case, as was shown in [1]. First, however, we need the following results:

Lemma 3.3. *If $h \in \mathbb{R}^n$, g_i, $i \in I(x^*)$, are Fréchet differentiable at $x^* \in A$ satisfying $\nabla g_i(x^*)h^t < 0$, then, for any mapping $r(\cdot) : \mathbb{R}_+ \to \mathbb{R}^n$ with the property that $\|r(\delta)\| = o(\delta)$, as $\delta \to 0_+$, there exists $\epsilon > 0$ such that*

$$g_i(x^* + \delta h + r(\delta)) < 0, \quad \forall \; \delta \in (0, \epsilon).$$

Proof. Using Taylor's formula

$$\begin{aligned} & g_i(x^* + \delta h + r(\delta)) \\ &= g_i(x^*) + \delta \nabla g_i(x^*)h^t + \nabla g_i(x^*)r(\delta)^t + \|\delta h + r(\delta)\| \alpha(x^*, \delta h + r(\delta)). \end{aligned}$$

Dividing by $\delta > 0$ and taking the limit as $\delta \to 0_+$, we obtain

$$\lim_{\delta \to 0_+} \frac{g_i(x^* + \delta h + r(\delta))}{\delta} = \nabla g_i(x^*)h^t < 0,$$

then there exists $\epsilon > 0$ such that $g_i(x^* + \delta h + r(\delta)) < 0, \; \forall \; \delta \in (0, \epsilon)$. □

Proposition 3.4. *The vector $h \in \mathbb{R}^n$ is tangent to the set A at the point x^* if and only if it is tangent to the set $\{x \in \mathbb{R}^n : g_i(x) \leq 0, \; i \in I_h(x^*)\}$ at this point x^*, where $I_h(x^*) = \{i \in I(x^*) : \nabla g_i(x^*)h^t = 0\}$.*

Proof. Let us show that $T_{\{x \in \mathbb{R}^n : g_i(x) \leq 0, \; i \in I_h(x^*)\}}(x^*) \subseteq T_A(x^*)$, since the other inclusion is trivial.

If $h \in T_{\{x \in \mathbb{R}^n : g_i(x) \leq 0, \; i \in I_h(x^*)\}}(x^*)$, by definition there exist $\epsilon > 0$ and $r : [0, \epsilon) \to \mathbb{R}^n$, such that $g_i(x^* + \delta h + r(\delta)) \leq 0$, for all $\delta \in [0, \epsilon), \; i \in I_h(x^*)$.

As $I = I(x^*) \cup I^c(x^*)$:

- For $i \in I^c(x^*)$ we have $g_i(x^*) < 0$ and using the continuity property for g_i , we obtain $g_i(x^* + \delta h + r(\delta)) \leq 0$, for $\delta > 0$ that is sufficiently small.
- As $I(x^*) = I_h(x^*) \cup I_h^c(x^*)$:
  - For $i \in I_h(x^*)$ it is verified.
  - For $i \in I_h^c(x^*)$, we know $g_i(x^*) = 0$ and $\nabla g_i(x^*)h^t < 0$. By Lemma 3.3, it follows $g_i(x^* + \delta h + r(\delta)) < 0$, for $\delta > 0$ sufficiently small.

Then $h \in T_A(x^*)$ since there are $\epsilon > 0$ and $r : [0, \epsilon) \to \mathbb{R}^n$ such that $x^* + \delta h + r(\delta) \in A$, for all $t \in [0, \epsilon)$ and $\frac{\|r(t)\|}{t} \to 0$ as $t \to 0^+$. □

Remark 3.5. Proposition 3.4 implies that the tangent cone at a feasible point is characterized by constraints whose index set is $I_h(x^*) = \{i \in I(x^*) : \nabla g_i(x^*)h^t = 0\}$, that is,

$$h \in T_A(x^*) \Leftrightarrow \; h \in T_{\{x\in\mathbb{R}^n : g_i(x)\leq 0,\; i\in I_h(x^*)\}}(x^*).$$

Remark 3.6. Proposition 3.4 guarantees that if $I_h(x^*) = \emptyset$ and $h \in G'_A(x^*)$, then $h \in T_A(x^*)$ because $\nabla g_i(x^*)h^t < 0,\ \forall\, i \in I(x^*)$.

We recall now the necessary definitions and results about feasible and admissible directions which can be found in any book on mathematical programming, for instance in [14]:

Definition 3.7. In the conditions of our problem, $d \in \mathbb{R}^n$ is a feasible direction at $x^* \in A$ if $\nabla g_i(x^*)d^t < 0,\ \forall\, i \in I(x^*)$.

The set of all feasible directions at $x^* \in A$ is a cone.

Definition 3.8. $d \in \mathbb{R}^n$ is said to be an admissible direction for (SOP) at x^* if $d = \nabla k(0) = (\nabla k_i(0))_{i=1,2,\dots,n}$ is tangent to the curve arc $k(\theta)$ at x^*, where $k : \mathbb{R}_+ \to \mathbb{R}^n$ is a mapping satisfying the conditions:

a) $k(0) = x^*$.
b) for $\theta > 0$ sufficiently small, $k(0) \in A$.

The set of all admissible directions is a cone.

Through the above definitions we have the following result concerning regular problems:

Proposition 3.9. *The characterization of the regular problem given in Proposition* 3.1 *implies the Abadie constraint qualification* $T_A(x^*) = G'_A(x^*)$.

Proof. By Definition 3.7, $\widetilde{h}$ in Proposition 3.1 is a feasible direction, therefore, the cone of feasible directions R is nonempty, and its closure coincides with the linearized cone ($cl(R) = G'_A(x^*)$[14]).

As the cone of feasible directions is contained in the cone of admissible directions, and their closures verify the same relation, it follows that the closure of the admissible directions cone is the tangent cone; this implies the Abadie constraint qualification $T_A(x^*) = G'_A(x^*)$ [4]. □

In general $T_A(x^*) \subseteq G'_A(x^*)$, but in non-regular problems the last inclusion is strict, that is, the Abadie constraint qualification does not hold. This is also the weakest sufficient condition in the multiobjective case, as was shown in [1].

Thus, the tangent cone cannot be characterized by the linear approximations of the active constraints at a feasible point, so Izmailov [8] proposed to use the 2-regularity theory in order to characterize the tangent cone. He defined the following sets with $h \in G'_A(x^*)$:

$$H_h(x^*) = \left\{ y \in \mathbb{R}^n : \nabla g_i(x^*)y^t + \frac{1}{2}\nabla^2 g_i(x^*)[h,h] \leq 0\ \forall\, i \in I_h(x^*) \right\},$$

and

$$H(x^*) = \{h \in \mathbb{R}^n : \nabla g_i(x^*)h^t \leq 0, \ \forall\, i \in I(x^*) \text{ and } H_h(x^*) \neq \emptyset\}.$$

Izmailov proved that the set $H(x^*)$ is the set of vectors satisfying the first- and second-order necessary conditions of tangency at x^*. For each element $y \in H_h(x^*)$, let

$$J_{hy}(x^*) = \left\{ i \in I_h(x^*) : \nabla g_i(x^*)y^t + \frac{1}{2}\nabla^2 g_i(x^*)[h,h] = 0 \right\}$$

and

$$J_h(x^*) = \bigcap_{y \in H_h(x^*)} J_{hy}(x^*),$$

i.e., $J_h(x^*)$ is the index set of the active constraints that satisfy as an equality the first-and second-order necessary condition for tangency at x^*.

Using the above sets, Izmailov gave the following definition of 2-regularity.

Definition 3.10. The constraints of a scalar problem are said to be 2-regular at x^* on the element $h \in H(x^*)$ if $g_i, i \in I_h(x^*)$ are twice Fréchet differentiable at $x^* \in A$, and there exist $\widetilde{\xi},\ \widetilde{\eta} \in \mathbb{R}^n$ such that

i) $\nabla g_i(x^*)\widetilde{\xi}^t \leq 0, \ \forall\, i \in I_h(x^*)$,

ii) $\nabla g_i(x^*)\widetilde{\eta}^t + \nabla^2 g_i(x^*)[h, \widetilde{\xi}] < 0, \ \forall\, i \in J_h(x^*)$.

We say that a problem is 2-regular at $x^* \in A$ on the element $h \in H(x^*)$ if the constraints are 2-regular at x^* on the element $h \in H(x^*)$.

Remark 3.11. Note that the 2-regularity property is a local one since it depends on the element h and that it can also be used for a multiobjective problem with inequality-type constraints since its constraints are the same.

Therefore with the last definition, Izmailov gave a second-order sufficient condition for tangency to characterize the tangent cone in the non-regular case, using the 2-regularity property.

Theorem 3.12. *Let $x^* \in A$ and g_i, $i \in I(x^*)$ be Fréchet differentiable at x^*. Given an element $h \in H(x^*)$, let g_i, $i \in I_h(x^*)$ be twice Fréchet differentiable at x^* and let the constraints of the problem be 2-regular at the point x^* on the element $h \in H(x^*)$. Then $h \in T_A(x^*)$.*

Let the set $\overline{H}(x^*)$ of elements $h \in H(x^*)$, such that the constraints of the problem are 2-regular at x^* on the element h.

Up to now, there were no results about optimality conditions and generalized convexity notions for the multiobjective non-regular case. The scalar non-regular case was studied by Izmailov in [8] and by Hernández-Jiménez et al. in [6]. The multiobjective regular case was studied by Osuna-Gómez et al. in [17]. Our purpose is to generalize the results given in [6] and [17] for the multiobjective non-regular case.

4. Necessary Optimality conditions for non-regular multiobjective optimization problems

We now give the necessary definitions and results in order to establish necessary and sufficient optimality conditions based on the 2-regularity of the problem.

First, we need to define the analogues of the vector Karush-Kuhn-Tucker points for non-regular multiobjective programming problems:

Definition 4.1. A feasible point $x^* \in A$ for (MOP) is said to be a Vector Generalized Karush-Kuhn-Tucker point (hereafter VGKKTP) for (MOP) with respect to an element $h \in \overline{H}(x^*)$ if there exists $(\nu^*(h), \lambda^*(h)) \in \mathbb{R}_+^{p+m}$, $\lambda^*(h) = (\lambda_i^*(h))_{i \in I}, \mu_i^*(h) \geq 0,\ i \in J_h(x^*),\ \nu^*(h) \neq 0$, satisfying:

$$\begin{cases} \nu^*(h) f'(x^*) + \sum_{i \in I} \lambda_i^*(h) \nabla g_i(x^*) + \sum_{i \in J_h(x^*)} \mu_i^*(h) h \nabla^2 g_i(x^*) = 0, \\ \sum_{i \in J_h(x^*)} \mu_i^*(h) \nabla g_i(x^*) = 0, \\ \lambda_i^*(h) g_i(x^*) = 0, \quad i \in I, \end{cases} \tag{3}$$

where $\lambda_i^*(h) g_i(x^*) = 0,\ i \in I$ are the complementary slackness conditions.

Remark 4.2. Using the above complementary slackness conditions, (3) can be rewritten as

$$\begin{cases} \nu^*(h) f'(x^*) + \sum_{i \in I(x^*)} \lambda_i^*(h) \nabla g_i(x^*) + \sum_{i \in J_h(x^*)} \mu_i^*(h) h \nabla^2 g_i(x^*) = 0, \\ \sum_{i \in J_h(x^*)} \mu_i^*(h) \nabla g_i(x^*) = 0. \end{cases} \tag{4}$$

Remark 4.3. Observe that in Definition 4.1 it is not necessary for $\nu^*(h)$ to be strictly positive; it is sufficient that $\nu^*(h) \neq 0$.

The definition of GKKTP given in [6] in the scalar case, and the definition of VKKTP given in [17] in the regular case when ($J_h(x^*) = \emptyset$ [8]) are particular instances of Definition 4.1 given here.

We give now a geometric form of the necessary condition of optimality for weakly efficient solutions, which we then use to give another, more practical form.

Theorem 4.4. *If $x^* \in A$ is a weakly efficient solution for* (MOP)*, then*

$$\mathcal{D}(f, x^*) \cap T_A(x^*) = \emptyset,$$

where $\mathcal{D}(f, x^*) = \{h \in \mathbb{R}^n : \ f'(x^*) h^t < 0\}$ is the cone of the descent directions for the objective function.

Proof. Let $h \in T_A(x^*)$; then there exist $\epsilon_1 > 0$ and $r(\cdot) : [0, \epsilon) \to \mathbb{R}^n$ such that $x^* + \delta h + r(\delta) \in A,\ \forall\ \delta \in [0, \epsilon_1)$ with $\|r(\delta)\| = o(\delta)$, as $\delta \to 0^+$.

As f_i is Fréchet differentiable at $x^* \in A$, for every $i = 1, \cdots, p$, by definition $f_i(x^* + \delta h + r(\delta)) - f_i(x^*) = \nabla f_i(x^*)(\delta h + r(\delta))^t + \alpha(x^*; \delta h + r(\delta)) \|\delta h + r(\delta)\|$ with $\alpha(x^*; \delta h + r(\delta)) \to 0$, for all $\delta \in [0, \epsilon_2)$ and $\delta \to 0^+$.

Taking $\epsilon' < \epsilon = \min\{\epsilon_1, \epsilon_1\}$ and by the assumption that x^* is a weakly efficient solution and $x^* + \delta h + r(\delta) \in A,\ \forall\ \delta \in [0, \epsilon')$, then there exists $i \in \{1, \cdots, p\}$ such that $f_i(x^* + \delta h + r(\delta)) - f_i(x^*) \geq 0$.

So, $\delta \nabla f_i(x^*)h^t + \nabla f_i(x^*)r(\delta)^t + \alpha(x^*; \delta h + r(\delta))\|\delta h + r(\delta)\| \geq 0$; dividing by $\delta > 0$ and taking the limit as $\delta \to 0^+$,

$$\lim_{\delta \to 0^+} \frac{f_i(x^* + \delta h + r(\delta)) - f_i(x^*)}{\delta} = \nabla f_i(x^*)h^t \geq 0, \quad \forall\ \delta \in [0, \epsilon')$$

for some $i \in \{1, \cdots, p\}$ and then $h \notin \mathcal{D}(f, x^*)$. Therefore, we obtain $\mathcal{D}(f, x^*) \cap T_A(x^*) = \emptyset$. □

Remark 4.5. The necessary optimality condition can be interpreted as follows: if $x^* \in A$ is a weakly efficient solution for (MOP), then the system $f'(x^*)h^t < 0$ has no solution $h \in T_A(x^*)$.

Considering the definitions and results given above we prove the following necessary condition of optimality. It is not necessary for the constraints of the problem to satisfy any constraint qualification.

Theorem 4.6. *Let $x^* \in A$ be a weakly efficient solution for* (MOP). *Assume that the constraints are 2-regular at x^* on an element $h \in H(x^*)$ and that $f'(x^*)h^t \leqq 0$. Then x^* is a VGKKTP with respect to $h \in H(x^*)$.*

Proof. Consider the following additional functions:

$$\Psi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p, \qquad \Psi(\sigma) = f'(x^*)\xi^t;$$

$$G_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \qquad G_i(\sigma) = \nabla g_i(x^*)\xi^t, \quad i \in I_h(x^*);$$

$$K_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \qquad K_i(\sigma) = \nabla g_i(x^*)\eta^t + \frac{1}{2}\nabla^2 g_i(x^*)[\xi]^2, \quad i \in J_h(x^*),$$

where $\sigma = (\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n$.

We will choose the element $\bar{h} \in H_h(x^*)$ such that $J_{h\bar{h}}(x^*) = J_h(x^*)$, i.e.,

$$\nabla g_i(x^*)\overline{h}^t + \frac{1}{2}\nabla^2 g_i(x^*)[h]^2 = 0, \quad \forall\ i \in J_h(x^*).$$

Let $\sigma^* = (h, \bar{h}) \in \mathbb{R}^n \times \mathbb{R}^n$.

Moreover, let $\bar{\xi}$ and $\bar{\eta} \in \mathbb{R}^n$ be solutions of the system equations in Definition 3.10, which definitely exist because the constraints are 2-regular at x^* on $h \in H(x^*)$, and let $\bar{\sigma} = (\bar{\xi}, \bar{\eta})$ be one that satisfies $\begin{cases} \text{i)}\ \nabla g_i(x^*)\bar{\xi}^t \leq 0, & \forall\ i \in I_h(x^*), \\ \text{ii)}\ \nabla g_i(x^*)\bar{\eta}^t + \nabla^2 g_i(x^*)[\bar{\xi}, h] < 0, & \forall\ i \in J_h(x^*). \end{cases}$

Consider the following auxiliary vector optimization problem

$$\begin{array}{ll} \text{Min} & \Psi(\sigma) = f'(x^*)\xi^t \\ \text{s.t:} & \sigma \in \mathbf{B} = \{\sigma \in \mathbb{R}^n \times \mathbb{R}^n / G_i(\sigma) \leq 0, i \in I_h(x^*), K_i(\sigma) \leq 0, i \in J_h(x^*)\}. \end{array} \qquad \text{(MOP')}$$

Obviously, $\sigma^* \in \mathbf{B}$, and all the constraints of problem (MOP') are active at the point σ^*. For any feasible solution $\sigma = (\xi, \eta) \in \mathbf{B}$ sufficiently close to σ^*, $(\sigma \in E(\sigma^*) \cap \mathbf{B})$ we have

$$\nabla g_i(x^*)\xi^t < 0, \quad \forall i \in I(x^*) \backslash I_h(x^*), \tag{5}$$

that is,

$$I_\xi(x^*) \subseteq I_h(x^*). \tag{6}$$

Also,

$$\nabla g_i(x^*)\xi^t \leq 0, \quad \forall\, i \in I_h(x^*), \tag{7}$$

and

$$\nabla g_i(x^*)\eta^t + \frac{1}{2}\nabla^2 g_i(x^*)[\xi]^2 < 0, \quad \forall\, i \in I_h(x^*) \backslash J_h(x^*), \tag{8}$$

and thus by (6) we have

$$J_\xi(x^*) \subseteq J_h(x^*). \tag{9}$$

Finally, for such σ,

$$\nabla g_i(x^*)\eta^t + \frac{1}{2}\nabla^2 g_i(x^*)[\xi]^2 \leq 0, \quad \forall\, i \in J_h(x^*), \tag{10}$$

and thus from (7) and (10) for any $\sigma = (\xi, \eta) \in \mathbf{B} \cap E(\sigma^*)$, we obtain $\xi \in H(x^*)$.

Moreover by condition ii) in Definition 3.10 and (9), for any such σ,

$$\nabla g_i(x^*)\bar{\eta}^t + \nabla^2 g_i(x^*)[\bar{\xi}, h] < 0, \quad \forall\, i \in J_\xi(x^*), \tag{11}$$

and it follows from the first condition in Definition 3.10 and from (6) that

$$\nabla g_i(x^*)\bar{\xi}^t \leq 0, \quad \forall\, i \in I_\xi(x^*). \tag{12}$$

Let us show that the constraints of the original vector problem (MOP) are 2-regular at x^* on $\xi \in H(x^*)$. For this, we must find $\bar{\xi}, \overline{\eta} \in \mathbb{R}^n$ such that

$$\begin{cases} \nabla g_i(x^*)\bar{\xi}^t \leq 0, & \forall\, i \in I_\xi(x^*), \\ \nabla g_i(x^*)\bar{\eta}^t + \nabla^2 g_i(x^*)[\bar{\xi}, \xi] < 0, & \forall\, i \in J_\xi(x^*). \end{cases}$$

By (12) $\bar{\xi} \in \mathbb{R}^n$ is guaranteed to exist satisfying the first inequality. The second one is also guaranteed by (11) and because $\sigma = (\xi, \eta)$ is sufficiently close to $\sigma^* = (h, \overline{h})$.

Thus, the constraints of (MOP) are 2-regular at x^* on $\xi \in H(x^*)$. Then, by the sufficient second-order tangency condition, Theorem 3.12, we get $\xi \in T_A(x^*)$. By Theorem 4.4, ξ is not a solution of the system $f'(x^*)\xi^t < 0$.

Thus, the system $f'(x^*)\xi^t < 0$ has no solution in $\mathbf{B}' = E(\sigma) \cap \mathbf{B}$.

As by hypothesis $f'(x^*)h^t \leqq 0$ and there is no $\xi \in \mathbf{B}'$ such that $f'(x^*)\xi^t < 0$, then σ^* is a local weakly efficient solution for the vector auxiliary problem (MOP').

Applying Theorem 4.4, we have $T_{\mathbf{B}}(\sigma^*) \cap \mathcal{D}(\Psi, \sigma^*) = \emptyset$. Thus, $\nabla\Psi(\sigma^*)\sigma^t = f'(x^*)\xi^t < 0$ has no solution $\sigma = (\xi, \eta) \in T_{\mathbf{B}}(\sigma^*)$.

We now consider any element $\sigma = (\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n$ satisfying the conditions

$$\nabla G_i(\sigma^*)\sigma^t = \nabla g_i(x^*)\xi^t \leq 0, \quad \forall\, i \in I_h(x^*), \tag{13}$$

$$\nabla K_i(\sigma^*)\sigma^t = \nabla g_i(x^*)\eta^t + \nabla^2 g_i(x^*)[\xi, h] \leq 0, \quad \forall\ i \in J_h(x^*). \tag{14}$$

By virtue of the tangency conditions, the definition of 2-regularity and (13)–(14), for $\sigma^* = (h, \overline{h}),\ \overline{\sigma} = (\overline{\xi}, \overline{\eta})$ and $\sigma = (\xi, \eta)$, we have

$$\begin{aligned} G_i(\sigma^* + s\sigma + s^2\tau\bar{\sigma}) &= \nabla g_i(x^*)(h + s\xi + s^2\tau\bar{\xi})^t \\ &= s\nabla g_i(x^*)\xi^t + s^2\tau\nabla g_i(x^*)\bar{\xi}^t \leq 0, \quad \forall\ i \in I_h(x^*), \end{aligned} \tag{15}$$

for any $\tau \in \mathbb{R}_+$ and any $s \in \mathbb{R}_+$.

Moreover,

$$\begin{aligned} &K_i(\sigma^* + s\sigma + s^2\tau\bar{\sigma}) \\ &= \nabla g_i(x^*)(\bar{h} + s\eta + s^2\tau\bar{\eta})^t + \frac{1}{2}\nabla^2 g_i(x^*)[h + s\xi + s^2\tau\bar{\xi}]^2 \\ &= \nabla g_i(x^*)\bar{h}^t + \frac{1}{2}\nabla^2 g_i(x^*)[h]^2 + s(\nabla g_i(x^*)\eta^t + \nabla^2 g_i(x^*)[\xi, h]) \\ &\qquad + s^2\tau(\nabla g_i(x^*)\bar{\eta}^t + \nabla^2 g_i(x^*)[\xi, h]) + \frac{s^2}{2}\nabla^2 g_i(x^*)[\xi]^2 + o(s^2) \\ &\leq 0, \quad \forall\ i \in J_h(x^*), \end{aligned} \tag{16}$$

for any sufficiently small $s \in \mathbb{R}_+$, if the number $\tau > 0$ is sufficiently large.

By the definition of the tangent cone, (15) and (16) imply that any $\sigma = (\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n$, satisfying (13) and (14) is in the cone generated by the constraints of $\mathbf{B}$, that is, $\sigma \in T_{\mathbf{B}}(\sigma^*)$.

As $\sigma^* = (h, \overline{h}) \in \mathbf{B}$ is a local weakly efficient solution of the vector auxiliary problem (MOP') and we have just shown that $\sigma \in T_{\mathbf{B}}(\sigma^*)$ satisfies (13) and (14), $\nabla\Psi(\sigma^*)\sigma^t \not< 0$ for any σ satisfying (13) and (14).

Thus, the system

$$\begin{cases} \nabla\Psi(\sigma^*)\sigma^t < 0, \\ \nabla G_i(\sigma^*)\sigma^t \leq 0, \quad \forall\ i \in I_h(x^*), \\ \nabla K_i(\sigma^*)\sigma^t \leq 0, \quad \forall\ i \in J_h(x^*), \end{cases}$$

has no solution $\sigma \in \mathbb{R}^n \times \mathbb{R}^n$.

By Motzkin's Theorem [11], there exist $\nu^*(h) \geq 0,\ \nu^*(h) \in \mathbb{R}^p,\ \lambda_i^*(h) \geq 0,\ i \in I_h(x^*),\ \mu_i^*(h) \geq 0,\ i \in J_h(x^*)$ such that

$$\nu^*(h)\nabla\Psi(\sigma^*) + \lambda^*(h)G'_{I_h}(\sigma^*) + \mu^*(h)K'_{J_h}(\sigma^*) = 0,$$

that is,

$$\nu^*(h)[f'(x^*) \ \ 0] + \lambda^*(h)[g'_{I_h}(x^*) \ \ 0] + \mu^*(h)[g''_{J_h}(x^*)[h] \ \ g'_{J_h}(x^*)] = [0, 0],$$

and so

$$\begin{cases} \nu^*(h)f'(x^*) + \displaystyle\sum_{i \in I_h(x^*)} \lambda_i^*(h)\nabla g_i(x^*) + \sum_{i \in J_h(x^*)} \mu_i^*(h)h\nabla^2 g_i(x^*) = 0, \\ \displaystyle\sum_{i \in J_h(x^*)} \mu_i^*(h)\nabla g_i(x^*) = 0, \end{cases} \tag{17}$$

where $G'_{I_h}(\sigma^*) = [g'_{I_h}(x^*)\ 0] \in \mathcal{M}_{card(I_h(x^*))\times n}$, $K'_{J_h}(\sigma^*) = [g''_{J_h(x^*)}\ 0] \in \mathcal{M}_{card(I_h(x^*))\times n}$, $g'_{I_h}(x^*) \in \mathcal{M}_{card(I_h(x^*))\times n}$, $g'_{J_h}(x^*) \in \mathcal{M}_{card(J_h(x^*))\times n}$ and the corresponding rows of $g'_{I_h}(x^*)$ and $g'_{J_h}(x^*)$ represent the first-order derivatives of $g_i,\ i \in I_h(x^*)$ and $g_i,\ i \in J_h(x^*)$. The rows of $g''_{J_h}(x^*)$ are the second-order derivatives of $g_i,\ i \in J_h(x^*)$, that is, $\nabla^2 g_i(x^*) \in \mathcal{M}_{n\times n}$.

Taking $\lambda_i^*(h) = 0,\ \forall\, i \in I(x^*) \setminus I_h(x^*)$ and using the Definition 4.1, x^* is a VGKKTP with respect to the element $h \in \overline{H}(x^*)$. □

5. Examples

We show that the VKKTP do not completely characterize the set of the weakly efficient solutions for the non-regular case.

Example 5.1. Consider the following multiobjective problem, where $f(x) = (x_1 + x_2 - x_3, x_2 - x_3)$, $x^* = (0,0,0)$, and the constraints are defined by the functions $g_1(x) = -x_1 + x_2^2 + 2x_3^2,\ g_2(x) = x_1 - 2x_2^2 - x_3^2$ and $g_3(x) = -x_2$. The feasible point x^* is a weakly efficient solution of the problem, since it is a minimum for $f_1(x^*)$. The problem is non-regular at that point, but the constraints are 2-regular at x^* on $h \in \{(0,\alpha,\beta) \in \mathbb{R}^3,\ \alpha^2 \geq \beta^2,\ \alpha > 0\}$. Then $\overline{H}(x^*) = H(x^*)\setminus\{0\}$ (see Example 4.4 in [6]).

We can show that x^* is not a VKKTP, there do not exist $\nu \in \mathbb{R}^2_+ \setminus \{0\},\ \lambda \in \mathbb{R}^3_+$ such that

$$\nu f'(x^*) + \sum_{i\in I(x^*)} \lambda_i \nabla g_i(x^*) = 0 \Leftrightarrow \begin{cases} \nu_1 - \lambda_1 + \lambda_2 = 0, \\ \nu_1 + \nu_2 - \lambda_3 = 0, \\ -(\nu_1 + \nu_2) = 0. \end{cases}$$

Let us prove that x^* is a VGKKTP point with respect to some $h \in \overline{H}(x^*)$. Considering $h \in \overline{H}(x^*)$, we have

$$f'(x^*)h^t = \begin{pmatrix} \alpha - \beta \\ \alpha - \beta \end{pmatrix}.$$

If $\alpha = \beta,\ h = (0,\alpha,\alpha),\ \alpha > 0$, then $I(x^*) = \{1,2,3\}$, $J_h(x^*) = \{1,2\}$ and $f'(x^*)h^t = 0$.

The conditions for a VGKKTP in matrix form are

$$\begin{pmatrix} \nu_1^*(h) \\ \nu_1^*(h) + \nu_2^*(h) \\ -(\nu_1^*(h) + \nu_2^*(h)) \end{pmatrix} + \lambda_1^*(h) \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2^*(h) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_3^*(h) \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$
$$+ \mu_1^*(h) \begin{pmatrix} 0 \\ 2\alpha \\ 4\alpha \end{pmatrix} + \mu_2^*(h) \begin{pmatrix} 0 \\ -4\alpha \\ -2\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\mu_1^*(h) \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \mu_2^*(h) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus,

$$\begin{cases} \nu_1^*(h) = \lambda_1^*(h) - \lambda_2^*(h), \\ \mu_1^*(h) = \mu_2^*(h) = \dfrac{\nu_1^*(h) + \nu_2^*(h)}{2\alpha}, \quad \alpha > 0. \\ \lambda_3^*(h) = 0. \end{cases}$$

A solution is, $\nu^*(h) = (1, 0)$, $\lambda^*(h) = (1, 0, 0)$, $\mu^*(h) = \left(\frac{1}{2\alpha}, \frac{1}{2\alpha}\right)$, $\alpha > 0$.

So x^* is not a VKKTP; it is a VGKKTP. □

Let us show by examples that the hypothesis $f'(x^*)h^t \leqq 0$ is necessary to ensure the thesis of Theorem 4.6.

Example 5.2. Considering the problem in the last example, if $\alpha = -\beta$, $h = (0, \alpha, -\alpha)$, $\alpha > 0$, then $I(x^*) = \{1, 2, 3\}$, $J_h(x^*) = \{1, 2\}$ and $f'(x^*)h^t = \binom{2\alpha}{2\alpha} > 0$. The conditions for a VGKKTP are the same as in Example 5.1, so $x^* = (0, 0, 0)$ is a VGKKTP.

Likewise if $\alpha > \beta = 0$, or if $\alpha > \beta > 0$, then $I(x^*) = \{1, 2, 3\}$, $J_h(x^*) = \emptyset$ and $f'(x^*)h^t > 0$. In the two cases, the conditions for a VGKKTP are

$$\nu^*(h)f'(x^*) + \sum_{i \in I(x^*)} \lambda_i^*(h) \nabla g_i(x^*) = 0,$$

which has no solution $\nu^*(h) \neq 0$. Therefore, in this case ($f'(x^*)h^t > 0$), x^* is not a VGKKTP with respect to any h.

Taking the same feasible set as in Example 5.1 and constructing different vector problems with the objective functions $(x_1 + x_2 - x_3, -x_3)$, $(x_1 + x_2 - x_3, 2x_2 - x_3)$ and $(x_1 + 2x_2 - x_3, -x_3)$, we find that $x^* = (0, 0, 0)$ is a weakly efficient solution, and the constraints are 2-regular at x^* on any element $h \in H(x^*)\backslash\{0\}$. We conclude that if $f'(x^*)h^t \geq 0$ or $f'(x^*)h^t \nleq 0$ is verified, then we cannot ensure that x^* is a VGKKTP with respect to the element h. □

6. Generalized convexity and characterization of weakly efficient solutions for non-regular multiobjective optimization problems

Taking into account the ideas presented in [6] and [17] we study the multiobjective non-regular case. Moreover, we prove that the non-regular scalar case studied in [6] and the regular multiobjective case studied in [17] are particular instances of the one presented here. We have seen in Section 4 that a weakly efficient solution of a multiobjective problem (regular or non-regular) always satisfies a KKT type optimality condition. However, we cannot ensure that of all the points that satisfy a KKT type optimality condition are weakly efficient solutions, so we cannot completely characterize the set of efficient solutions. Therefore to establish the converse of Theorem 4.6, it is necessary to introduce new notions of generalized convexity, since the existing ones in the literature are not valid for non-regular problems.

Definition 6.1. (MOP) is said to be a vector 2-KKT-pseudoinvex problem at $x^* \in A$ with respect to an element $h \in \overline{H}(x^*)$, if there exist $\eta_h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and

$\gamma_h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that for all $x \in A$, the following conditions are satisfied:

$$\begin{cases} f(x) - f(x^*) < 0 \Rightarrow \eta_h(x,x^*) f'(x^*)^t < 0, \\ -\eta_h(x,x^*) \nabla g_i(x^*)^t \geq 0, \quad \forall\, i \in I(x^*), \\ -\nabla^2 g_i(x^*)[\eta_h(x,x^*), h] - \gamma_h(x,x^*) \nabla g_i(x^*)^t \geq 0, \quad \forall\, i \in J_h(x^*). \end{cases} \tag{18}$$

(MOP) is said to be a vector 2-KKT-pseudoinvex problem at $x^* \in A$, if it is a vector 2-KKT-pseudoinvex problem at x^* with respect to all $h \in \overline{H}(x^*)$.

(MOP) is said to be a vector 2-KKT-pseudoinvex problem on A, if it is a vector 2-KKT-pseudoinvex problem at all elements of A.

Remark 6.2. The definition of a vector KKT-pseudoinvex problem (Definition 2.7) is a particular case of the definition of a vector 2-KKT-pseudoinvex problem (Definition 6.1) when the problem is regular at x^* ($J_h(x^*) = \emptyset$).

We prove now the last main result of this paper:

Theorem 6.3. *Let $x^* \in A$. Every VGKKTP with respect to some element $h \in \overline{H}(x^*)$ is a weakly efficient solution for* (MOP) *if and only if* (MOP) *is a vector 2-KKT-pseudoinvex problem on A.*

Proof. Suppose that x^* is a VGKKTP with respect to some element $h \in \overline{H}(x^*)$ that is not a weakly efficient solution for the problem. Then, there exists $x \in A$ such that $f(x) < f(x^*)$.

As x^* is a VGKKTP with respect to $h \in \overline{H}(x^*)$, there are $\nu^*(h) \in \mathbb{R}^p_+$, $\nu^*(h) \neq 0$, $\lambda_i^*(h) \geq 0$, $i \in I(x^*)$, $\mu_i^*(h) \geq 0$, $i \in J_h(x^*)$, such that

$$\begin{cases} \nu^*(h) f'(x^*) + \lambda^*(h) g'_{I(x^*)}(x^*) + \mu^*(h) h g''_{J_h(x^*)}(x^*) = 0, \\ \mu^*(h) g'_{J_h(x^*)}(x^*) = 0. \end{cases} \tag{19}$$

where the corresponding rows of $g'_I(x^*)$ and $g'_{J_h}(x^*)$ represent the first-order derivatives of g_i, $i \in I(x^*)$ and g_i, $i \in J_h(x^*)$. The rows of $g''_{J_h}(x^*)$ are the second-order derivatives of g_i, $i \in J_h(x^*)$.

The problem is a vector 2-KKT-pseudoinvex problem at x^* with respect to $h \in \overline{H}(x^*)$, so $f(x) - f(x^*) < 0 \Rightarrow f'(x^*)\eta_h(x,x^*) < 0$, and therefore

$$\eta_h(x,x^*)(\nu^*(h) f'(x^*))^t < 0. \tag{20}$$

Multiplying (19_1) and (19_2) by $\eta_h(x,x^*)$ and $\gamma_h(x,x^*)$, respectively, summing and using (19_2)

$$\eta_h(x,x^*)[\nu^*(h) f'(x^*) + \lambda^*(h) g'_{I(x^*)}(x^*) + \mu^*(h) h g''_{J_h(x^*)}(x^*)]^t = 0. \tag{21}$$

Using (20) and (21), we have:

$$-\eta_h(x,x^*)[\lambda^*(h) g'_{I(x^*)}(x^*) + \mu^*(h) h g''_{J_h(x^*)}(x^*)]^t < 0. \tag{22}$$

As (MOP) is a vector 2-KKT-pseudoinvex problem and $\lambda^*(h), \mu^*(h) \geqq 0$,

$$\begin{cases} -\eta_h(x,x^*)(\lambda^*(h)g'_{I(x^*)}(x^*))^t \geq 0, \\ -\mu^*(h)g''_{J_h(x^*)}(x^*)[\eta_h(x,x^*),h] + \gamma_h(x,x^*)(\mu^*(h)g'_{J_h(x^*)}(x^*))^t \geq 0. \end{cases} \tag{23}$$

Summing the latest inequalities and by virtue of (19_2), we have

$$-\eta_h(x,x^*)[\lambda^*(h)g'_{I(x^*)}(x^*) + \mu^*(h)hg''_{J_h(x^*)}(x^*)]^t \geq 0, \tag{24}$$

which stands in contradiction to (22). Therefore, x^* is a weakly efficient solution for (MOP).

Conversely, we now suppose that every VGKKTP is a weakly efficient solution and we have to prove that (MOP) is a vector 2-KKT-pseudoinvex problem.

It would be sufficient to prove that for a fixed $x^* \in A$, if the system

$$\begin{cases} f(x) - f(x^*) < 0 \\ x \in A \end{cases} \tag{25}$$

has a solution, then the system

$$\begin{cases} \eta_h(x,x^*)f'(x^*)^t < 0, \\ -\eta_h(x,x^*)\nabla g_i(x^*)^t \geq 0, \quad \forall\, i \in I(x^*), \\ -\nabla^2 g_i(x^*)[\eta_h(x,x^*),h] - \gamma_h(x,x^*)\nabla g_i(x^*)^t \geq 0, \quad \forall\, i \in J_h(x^*). \end{cases} \tag{26}$$

has a solution $\eta_h(x,x^*),\ \gamma_h(x,x^*) \in \mathbb{R}^n$.

If (25) has a solution, then x^* is not a weakly efficient solution for (MOP), and by hypothesis it is also not a VGKKTP either with respect to any $h \in \overline{H}(x^*)$. Therefore, there are no $\nu^*(h) \in \mathbb{R}^p_+,\ \nu^*(h) \neq 0;\ \lambda^*_i(h) \in \mathbb{R}_+,\ i \in I(x^*)$; and no $\mu^*_i(h) \in \mathbb{R}_+,\ i \in J_h(x^*)$ such that the system

$$\begin{cases} \nu^*(h)f'(x^*) + \lambda^*(h)g'_{I(x^*)}(x^*) + \mu^*(h)hg''_{J_h(x^*)}(x^*) = 0, \\ \mu^*(h)g'_{J_h(x^*)}(x^*) = 0. \end{cases} \tag{27}$$

has a solution for any $h \in \overline{H}(x^*)$.

In matrix form,

$$-\nu^*(h)[f'(x^*)\ \ 0] - \lambda^*(h)[g'_I(x^*)\ \ 0] - \mu^*(h)[hg''_{J_h}(x^*)\ \ g'_{J_h}(x^*)] = 0$$

has no solution for any $h \in \overline{H}(x^*)$.

By Slater's alternative theorem [11], for each $h \in \overline{H}(x^*)$ there exist

$$\eta_h(x^*),\ \gamma_h(x^*) \in \mathbb{R}^n,$$

such that the following system has a solution:

$$\begin{cases} [f'(x^*) \quad 0] \begin{bmatrix} \eta_h(x^*) \\ \gamma_h(x^*) \end{bmatrix} < 0, \\ [g'_I(x^*) \quad 0] \begin{bmatrix} \eta_h(x^*) \\ \gamma_h(x^*) \end{bmatrix} \leqq 0, \\ [hg''_{J_h}(x^*) \quad g'_{J_h}(x^*)] \begin{bmatrix} \eta_h(x^*) \\ \gamma_h(x^*) \end{bmatrix} \leqq 0. \end{cases} \tag{28}$$

Equivalently,

$$\begin{cases} \eta_h(x^*)f(x^*)^t < 0, \\ -\eta_h(x^*)g'_I(x^*)^t \geqq 0, \\ -g''_{J_h}(x^*)[\eta_h(x^*), h] + \gamma_h(x^*)g'_{J_h}(x^*)^t \geqq 0 \end{cases} \tag{29}$$

has a solution.

Therefore, for all $x \in A$ with $f(x) - f(x^*) < 0$, there exist $\eta_h(x, x^*) := \eta_h(x^*) \in \mathbb{R}^n$ and $\gamma_h(x, x^*) := \gamma_h(x^*) \in \mathbb{R}^n$ for each $h \in \overline{H}(x^*)$ such that the system (26) has a solution. So (MOP) is a vector 2-KKT-pseudoinvex problem at x^* with respect to all elements $h \in \overline{H}(x^*)$. This proves that (MOP) is a vector 2-KKT-pseudoinvex problem on A. □

Remark 6.4.

- The previous proof provides a constructive method for finding vectors $\eta_h(x, x^*)$, $\gamma_h(x, x^*) \in \mathbb{R}^n$, $\forall\ h \in \overline{H}(x^*)$ that appear in the definition of a vector 2-KKT-pseudoinvex problem.
- The Theorem 2.8 in [17], which generalizes Martin's result in [12], is a particular case of the Theorem 6.3 when the problem is regular.

Let us show with the following example that the 2-KKT-pseudoinvexity is the weakest condition to characterize the weakly efficient solutions for a non-regular constrained multiobjective programming problem.

Example 6.5. Consider the following multiobjective problem, where $f(x) = (-x_1, -(x_1 + x_2))$, $x^* = (0, 0, 0)$, and the constraints are defined by the functions $g_1(x) = -x_1 + x_2^2 + 2x_3^2$, $g_2(x) = x_1 - 2x_2^2 - x_3^2$, $g_3(x) = -x_2$, $g_4(x) = x_1 - 1$ and $g_5(x) = x_2 - 1$. This problem is not a regular problem at the feasible point $x^* = (0, 0, 0)$, because $I(x^*) = \{1, 2, 3\}$, and so the active constraints are the same as in Example 5.1. Therefore, $\bar{H}(x^*) = \{(0, \alpha, \beta) : \alpha^2 \geq \beta^2, \alpha > 0\}$.

x^* is not a weakly efficient solution because there exists $\hat{x} = (1, 1, 0)$ another feasible point, such that $f(\hat{x}) = (-1, -2) \leq (0, 0) = f(x^*)$.

We prove that x^* is a VGKKTP for all $h \in \bar{H}(x^*)$:

1. Let $h = (0, \alpha, \alpha)$, $\alpha > 0$, $(\alpha = \beta)$ or $h = (0, \alpha, -\alpha)$, $\alpha > 0$, $(\alpha = -\beta)$. Then

$J_h(x^*) = \{1, 2\}$ and the VGKKTP conditions are:

$$\begin{cases} \nu_1^*(h) = \lambda_3^*(h) + \lambda_2^*(h) - \lambda_1^*(h), \\ \nu_2^*(h) = -\lambda_3^*(h), \\ \mu_1^*(h) = \mu_2^*(h) = 0. \end{cases}$$

For example, $\lambda^*(h) = (1, 2, 0)$, $\nu^*(h) = (1, 0)$ and $\mu^*(h) = (0, 0)$ is a solution for the above system, so x^* is a VGKKTP with respect to $h = (0, \alpha, \alpha)$ or $h = (0, \alpha, -\alpha), \alpha > 0$.

2. Let $h = (0, \alpha, \beta)$ with $\alpha > \beta = 0$ or $\alpha > \beta > 0$; then $J_h(x^*) = \emptyset$, and the VGKKTP conditions are

$$\begin{cases} -\nu_1^*(h) - \nu_2^*(h) = \lambda_1^*(h) - \lambda_2^*(h), \\ \nu_2^*(h) = -\lambda_3^*(h). \end{cases}$$

Taking $\nu^*(h) = (1, 0)$ and $\lambda^*(h) = (1, 2, 0)$, x^* is a VGKKTP with respect to h.

Finally, we will check that the problem is not a vector 2-KKT-pseudoinvex problem on the feasible set, since it is not a vector 2-KKT-pseudoinvex problem at x^* with respect to all $h \in \bar{H}(x^*)$. By example, let $x = \hat{x} = (1, 1, 0)$ a feasible point; then the second condition to be a 2-KKT-pseudoinvex problem at x^* with respect to all $h \in \bar{H}(x^*)$ is:

$$-(\eta_{h_1}(x, x^*), \eta_{h_2}(x, x^*), \eta_{h_3}(x, x^*)) \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \geq 0 \Leftrightarrow \begin{cases} \eta_{h_1}(x, x^*) = 0, \\ \eta_{h_2}(x, x^*) \geq 0. \end{cases}$$

Therefore, the first condition stays:

$$(-1, -2) \geqq (0, \eta_{h_2}(x, x^*), \eta_{h_3}(x, x^*)) \begin{pmatrix} -1 & -1 \\ 0 & -1 \\ 0 & 0 \end{pmatrix} = (0, -\eta_{h_2}(x, x^*)).$$

$-1 \geq 0$ is a contradiction, so the problem is not a vector 2-KKT-pseudoinvex problem on the feasible set. □

7. Conclusions

Up to now, there were no results on generalized convexity notions and optimality conditions for non-regular vector problems in the literature. Our paper covers them, and generalizes the results and definitions in the scalar non-regular case [6]. In the particular case that the problem is a regular one, the definitions and results existing in the literature are particular instances of the ones presented in this paper.

In the scalar case, the definition of Generalized Karush-Kuhn-Tucker point (GKKTP) given in [6] is a particular case of the definition of VGKKTP (Definition 4.1).

If (MOP) is a regular problem, $J_h(x^*) = \emptyset$ [8], then we have verified that

- The definition of a VKKTP (Definition 2.4) is a particular case of the definition of a VGKKTP (Definition 4.1).
- The definition of a vector KKT-pseudoinvex problem (Definition 2.7) is a particular instance of the definition of a vector 2-KKT-pseudoinvex problem (Definition 6.1).
- The Theorem 2.8 in [17], which generalizes Martin's result in [12], is a particular case of the Theorem 6.3.

Taking into account the 2-regularity theory and the description of the tangent cone based on the second-order derivatives proposed by Izmailov, we give a necessary condition for optimality that is sufficient under a generalized convexity notion for vector problems whose constraints do not necessarily satisfy a constraint qualification. We completely characterize the set of weakly efficient solutions for those problems, and the generalized convexity notion defined here is the weakest to characterize it.

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