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Legendre-Fenchel Transform of the Spectral Exponent of Analytic Functions of Weighted Composition Operators*

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Let f be an analytic function with nonnegative coefficients. We derive the Legendre-Fenchel transform of the composition $\ln \circ f \circ \exp$ as a function depending on coefficients of f. We apply it to obtain the variational principle for the spectral exponent of operators that can be written as analytic functions of the weighted composition operators.

Keywords: Weighted composition operators, spectral radius, Legendre-Fenchel transform

47A10, 47B37, 44A15 Mathematics Subject Classification:

1. Introduction

In operator algebras functions of operators are considered. It is interesting to relate properties of a given operator and functions of it. One of the most important characteristics of bounded operator is the spectral radius, one that for many classes of operator may be a question of independent interest. It turns out that the logarithm of the spectral radius (the spectral exponent) of weighted composition operators convexly depends on the logarithms of their weights (see [3]). We investigate the Legendre-Fenchel transform of the spectral exponent of analytic functions of weighted composition operators acting in L^p-spaces.

We recall a general result obtained for the spectral radius of weighted composition operators. Let X be a Hausdorff compact space with Borel measure μ, $\alpha : X \mapsto X$ a continuous mapping preserving μ (i.e. $\mu \circ \alpha^{-1} = \mu$) and a be a continuous function on X. Antonevich, Bakhtin and Lebedev constructed a functional τ_α, called T-entropy, on the set of probability and α-invariant measures M_α^1 such that for the spectral radius

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of the weighted composition operator $(aT_\alpha)u(x) = a(x)u(\alpha(x))$ acting in L^p-spaces the following variational principle holds

$$\ln r(aT_\alpha) = \max_{\nu \in M_\alpha^1} \left\{ \int_X \ln |a| d\nu - \frac{\tau_\alpha(\nu)}{p} \right\}. \tag{1}$$

The above result was announced in [1, 2] and its proof is inserted in [3]. Therein one can find more detailed history of the spectral radius of weighted composition operator investigations.

For positive $a \in C(X)$ let $\varphi = \ln a$ then a functional $\lambda(\varphi) = \ln r(e^\varphi T_\alpha)$ possesses several properties among others continuity and convexity on $C(X)$ (see [3]). The formula (1) states that λ is the Legendre-Fenchel transform of the function $\frac{\tau_\alpha}{p}$, i.e.

$$\lambda(\varphi) = \max_{\nu \in M_\alpha^1} \left\{ \int_X \varphi d\nu - \lambda^*(\nu) \right\}, \tag{2}$$

where

$$\lambda^*(\nu) = \begin{cases} \frac{\tau_\alpha(\nu)}{p}, & \nu \in M_\alpha^1 \\ +\infty, & \text{otherwise.} \end{cases}$$

It means that the effective domain $D(\lambda^*)$ is contained in M_α^1.

In this paper we will study the variational principle for operators that can be written as analytic functions of the weighted composition operators $\sum_{n=0}^\infty a_n (e^\varphi T_\alpha)^n$. We derive a relationship between the convex conjugate of $\ln r(\sum_{n=0}^\infty a_n (e^\varphi T_\alpha)^n)$ and T-entropy. This result is the generalization of the variational principle, obtained by authors in [7], for the spectral exponent of polynomials of weighted composition operators.

2. Spectral exponent of analytic functions of weighted composition operators

In the paper [8] we proved that for the polynomials w with nonnegative coefficients the spectral radius of $w(e^\varphi T_\alpha)$ is equal to polynomials of the spectral radius of $e^\varphi T_\alpha$, i.e. $r(w(e^\varphi T_\alpha)) = w(r(e^\varphi T_\alpha))$ (see therein Remark 2.7). Generalization of this fact on analytic functions with nonnegative coefficients will be a beginning point of our considerations.

Let $f(z) = \sum_{n=0}^\infty a_n z^n$ be an analytic function with nonnegative coefficients a_n for which the radius of convergence is greater than $r(e^\varphi T_\alpha)$. For this reason the operator $f(e^\varphi T_\alpha) = \sum_{n=0}^\infty a_n (e^\varphi T_\alpha)^n$ is a well defined bounded operator.

Theorem 2.1. *For any analytic function $f(z) = \sum_{n=0}^\infty a_n z^n$ with nonnegative coefficients a_n which radius of convergence is greater than $r(e^\varphi T_\alpha)$ the following equality holds*

$$f(r(e^\varphi T_\alpha)) = r(f(e^\varphi T_\alpha)).$$

Proof. By the relation $r(w(e^{\varphi}T_{\alpha})) = w(r(e^{\varphi}T_{\alpha}))$ that is satisfied for any polynomial w with nonnegative coefficients we have

$$\sum_{n=0}^{N} a_n(r(e^{\varphi}T_{\alpha}))^n = r\left(\sum_{n=0}^{N} a_n(e^{\varphi}T_{\alpha})^n\right).$$

Tending with N to infinity we obtain the following equality

$$\sum_{n=0}^{\infty} a_n(r(e^{\varphi}T_{\alpha}))^n = f(r(e^{\varphi}T_{\alpha})) = \lim_{N\to\infty} r\left(\sum_{n=0}^{N} a_n(e^{\varphi}T_{\alpha})^n\right).$$

Upper semicontinuity of the spectral radius on the algebra of linear and bounded operators implies the following inequality

$$\begin{aligned} f(r(e^{\varphi}T_{\alpha})) &\le r\left(\lim_{N\to+\infty}\sum_{n=0}^{N} a_n(e^{\varphi}T_{\alpha})^n\right) \\ &= r\left(\sum_{n=0}^{\infty} a_n(e^{\varphi}T_{\alpha})^n\right) = r(f(e^{\varphi}T_{\alpha})). \end{aligned} \tag{3}$$

Let now R_N denote the sum $\sum_{n=N+1}^{\infty} a_n(e^{\varphi}T_{\alpha})^n$. Notice that for any n each $a_n(e^{\varphi}T_{\alpha})^n$ commutes with R_N. For this reason

$$\begin{aligned} r(f(e^{\varphi}T_{\alpha})) &= r\left(\sum_{n=0}^{N} a_n(e^{\varphi}T_{\alpha})^n + R_N\right) \\ &\le \sum_{n=0}^{N} a_n r(e^{\varphi}T_{\alpha})^n + r(R_N). \end{aligned} \tag{4}$$

Continuity of the spectral radius at zero implies that $\lim_{N\to\infty} r(R_N) = 0$ and in consequence we obtain as well the opposite inequality

$$r(f(e^{\varphi}T_{\alpha})) \le f(r(e^{\varphi}T_{\alpha})).$$

□

By Theorem 2.1 the spectral exponent of the operator $f(e^{\varphi}T_{\alpha})$ we can rewrite in the following way

$$\begin{aligned} \ln r(f(e^{\varphi}T_{\alpha})) &= \ln f(r(e^{\varphi}T_{\alpha})) \\ &= \ln \sum_{n=0}^{\infty} a_n(r(e^{\varphi}T_{\alpha}))^n = \ln \sum_{n=0}^{\infty} a_n e^{n\lambda(\varphi)}. \end{aligned}$$

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be given analytic function with nonnegative coefficients a_n and the convergence radius $R \in (0, +\infty]$. Additionally we assume that $f(R) = +\infty$.

Let $\mathcal{D}$ denote the set $\{\varphi \in C(X) : r(e^{\varphi}T_{\alpha}) < R\}$. Because the spectral exponent λ is convex and continuous on $C(X)$ then $r(e^{\varphi}T_{\alpha}) = e^{\lambda(\varphi)}$ is also convex and continuous on this space. For these reasons $\mathcal{D}$ is an open and convex subset of $C(X)$. Let us emphasize that for φ belonging to the boundary of $\mathcal{D}$ the spectral radius $r(e^{\varphi}T_{\alpha}) = R$ and $f(r(e^{\varphi}T_{\alpha})) = +\infty$.

Define the functional $\widetilde{\lambda}$ on $C(X)$ as follows

$$\widetilde{\lambda}(\varphi) = \begin{cases} \ln \sum_{n=0}^{\infty} a_n e^{n\lambda(\varphi)}, & \varphi \in \mathcal{D} \\ +\infty, & \text{otherwise.} \end{cases} \tag{5}$$

Before we proceed to the variational principle for $\widetilde{\lambda} = \ln \circ f \circ \exp \circ \lambda$ first we derive the formula on the Legendre-Fenchel transform of the function $\ln \circ f \circ \exp$ of real variable λ. Convexity of $\ln \circ f \circ \exp$ is proved below.

Lemma 2.2. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be given analytic function with nonnegative coefficients a_n and the convergence radius $R \in (0, +\infty]$. Then the composition* $\ln \circ f \circ \exp$ *is strictly convex function on $(-\infty, \ln R)$.*

Proof. Let $\lambda_1, \lambda_2 \in (-\infty, \ln R)$ and $s \in (0,1)$ then

$$\begin{aligned} [\ln \circ f \circ \exp](s\lambda_1 + (1-s)\lambda_2) &= \ln \sum_{n=0}^{\infty} a_n e^{n(s\lambda_1 + (1-s)\lambda_2)} \\ &= \ln \sum_{n=0}^{\infty} \left[a_n e^{n\lambda_1}\right]^s \left[a_n e^{n\lambda_2}\right]^{1-s}. \end{aligned} \tag{6}$$

Applying to the series $\sum_{n=0}^{\infty} (a_n e^{n\lambda(\varphi_1)})^s (a_n e^{n\lambda(\varphi_2)})^{1-s}$ Hölder inequality for exponents $p = \frac{1}{s}$ and $q = \frac{1}{1-s}$ we get that

$$\ln \sum_{n=0}^{\infty} \left[a_n e^{n\lambda_1}\right]^s \left[a_n e^{n\lambda_2}\right]^{1-s} \le s \ln \sum_{n=0}^{\infty} a_n e^{n\lambda_1} + (1-s) \ln \sum_{n=0}^{\infty} a_n e^{n\lambda_2}.$$

In the above the equality holds when $\lambda_1 = \lambda_2$. It means that the function $\ln \circ f \circ \exp$ is strictly convex on $(-\infty, \ln R)$. □

To obtain the Legendre-Fenchel transform of $\ln \circ f \circ \exp$ we will need also a formula on the logarithm of series.

Proposition 2.3. *For the sequence $\mathbf{b} = (b_n)_{n=0}^{\infty}$ such that $b_n > 0$ and $\sum_{n=0}^{\infty} b_n < \infty$ the following holds*

$$\ln \sum_{n=0}^{\infty} b_n = \max_{t_n \ge 0,\, \sum t_n = 1} \limsup_{N \to \infty} \sum_{n=0}^{N} t_n (\ln b_n - \ln t_n).$$

This maximum is attained for $t_n = \frac{b_n}{\sum_{n=0}^{\infty} b_n}$.

Proof. Concavity of the logarithm function implies

$$\ln\left(\sum_{n=0}^{N} p_n x_n\right) \geq \sum_{n=0}^{N} p_n \ln x_n,$$

for any sequence $(p_n)_{n=0}^{N}$ of probability weights and $x_n > 0$. Substituting $p_n = \frac{t_n}{\sum_{n=0}^{N} t_n}$ and $x_n = \frac{b_n}{t_n}$ in the above inequality we obtain the following

$$\ln \sum_{n=0}^{N} b_n - \ln \sum_{n=0}^{N} t_n \geq \sum_{n=0}^{N} \frac{t_n}{\sum_{n=0}^{N} t_n} \ln \frac{b_n}{t_n}$$

and, equivalently,

$$\left(\ln \sum_{n=0}^{N} b_n - \ln \sum_{n=0}^{N} t_n\right) \sum_{n=0}^{N} t_n \geq \sum_{n=0}^{N} (t_n \ln b_n - t_n \ln t_n).$$

Observe that under assumption that $\sum_{n=0}^{\infty} t_n = 1$ the limit of the left-hand side, by N tending to $+\infty$, exists and is equal to $\ln \sum_{n=0}^{\infty} b_n$. For this reason we get that

$$\ln \sum_{n=0}^{\infty} b_n \geq \limsup_{N\to\infty} \sum_{n=0}^{N} (t_n \ln b_n - t_n \ln t_n).$$

Because for $t_n = \frac{b_n}{\sum_{n=0}^{\infty} b_n}$, in the above, the equality holds then the proof is complete. $\square$

Let S denote the set $\{(t_n)_{n=0}^{\infty} : \ t_n \geq 0, \ \sum_{n=0}^{\infty} t_n = 1\}$ and I be some subset of $\mathbb{N}$. Writing $(t_k)_{k\in I} \in S$ we understand that the sequence $(t_k)_{k\in I}$ is equivalent to this one $(t_n)_{n=0}^{\infty} \in S$ in which $t_n = t_k$ for $n \in I$ and $t_n = 0$ otherwise. Let now $\mathbf{b} = (b_n)_{n=0}^{\infty}$ be a sequence of nonnegative numbers and $I = \{n \in \mathbb{N} : \ b_n > 0\}$. Under the assumption $\sum_{k\in I} b_k < \infty$ we can rewrite Proposition 2.1 as follows

$$\ln \sum_{k\in I} b_k = \max_{(t_k)_{k\in I}\in S} \limsup_{N\to\infty} \sum_{k\in I,\, k\leq N} (t_k \ln b_k - t_k \ln t_k). \tag{7}$$

Remark 2.4. Taking $I = \{1, 2, ..., n\}$ and substituting e^{x_k} instead of b_k, for $k \in I$, we get the variational principle for the log-exponential function

$$\ln \sum_{k=1}^{n} e^{x_k} = \max_{(t_k)\in S} \left\{ \sum_{k=1}^{n} t_k x_k - \sum_{k=1}^{n} t_k \ln t_k \right\},$$

see for instance Example 11.12 in [9].

The below theorem presents the formula on $[\ln \circ f \circ \exp]^*$ as a function depending on coefficients of f.

Theorem 2.5. *Let $f(z) = \sum_{k\in I} a_k z^k$ be an analytic function with $a_k > 0$ and $l = \min I$. Let R denote the convergent radius of f, if $\lim_{r\to(\ln R)^-} f(r) = +\infty$ then the Legendre-Fenchel transform of* $\ln\circ f\exp$ *is given by the following formula*

$$[\ln\circ f\circ\exp]^*(c) = \begin{cases} \min\limits_{(t_k)\in S_c} \liminf\limits_{N\to\infty} \sum\limits_{k\in I,\ k\le N} t_k \ln\frac{t_k}{a_k} & c > l, \\ -\ln a_l & c = l, \\ +\infty & c < l, \end{cases} \tag{8}$$

where $S_c = \{(t_k)_{k\in I} \in S:\ \sum_{k\in I} kt_k = c\}$.

Proof. Substituting in (7) $b_k = a_k e^{k\lambda}$ we obtain that for the function $[\ln\circ f\circ\exp](\lambda) = \ln\sum_{k\in I} a_k e^{k\lambda}$ the following formula holds

$$[\ln\circ f\circ\exp](\lambda) = \ln\sum_{k\in I} a_k e^{k\lambda} = \max_{(t_k)\in S} \limsup_{N\to\infty} \sum_{k\in I,\ k\le N} \left(kt_k\lambda - t_k\ln\frac{t_k}{a_k}\right). \tag{9}$$

By Proposition 2.3 the above maximum is attained for the sequence $\left(\frac{a_k e^{k\lambda}}{f(e^\lambda)}\right)_{k\in I}$. Notice that for such sequence the series

$$\sum_{k\in I} kt_k = \frac{1}{f(e^\lambda)}\sum_{k\in I} k a_k e^{k\lambda} = [\ln\circ f\circ\exp]'(\lambda),$$

i.e. it is convergent to the value of derivation of $\ln\circ f\circ\exp$ at λ. For this reason we can search for the maximum in (9) over a subset of S defined by the restricted condition $\sum_{k\in I} kt_k < \infty$. It follows that we can rewrite (9) in the form

$$[\ln\circ f\circ\exp](\lambda) = \max_{\substack{(t_k)\in S \\ \sum_{k\in I} kt_k<\infty}} \left\{\left(\sum_{k\in I} kt_k\right)\lambda - \liminf_{N\to\infty} \sum_{k\in I,\ k\le N} t_k\ln\frac{t_k}{a_k}\right\}. \tag{10}$$

Observe now that $\sum_{k\in I} kt_k$ can take any value from the interval $[l, +\infty)$; the value l is attained for $(t_k)_{k\in I} = (\delta_{l,k})_{k\in I}$. Let S_c denote the set $\{(t_k)_{k\in I} \in S:\ \sum_{k\in I} kt_k = c\}$ for $c \in [l, +\infty)$. Hence we can divide the maximum over the set $\{(t_k)_{k\in I} \in S:\ \sum_{k\in I} kt_k < +\infty\}$ on two maximums over the set S_c and the interval $[l, +\infty)$

$$\begin{aligned} [\ln\circ f\circ\exp](\lambda) &= \max_{c\in[l,+\infty)} \max_{(t_k)\in S_c,} \left\{\left(\sum_{k\in I} kt_k\right)\lambda - \liminf_{N\to\infty} \sum_{k\in I,\ k\le N} t_k\ln\frac{t_k}{a_k}\right\} \\ &= \max_{c\in[l,+\infty)} \left\{c\lambda - \min_{(t_k)\in S_c,} \liminf_{N\to\infty} \sum_{k\in I,\ k\le N} t_k\ln\frac{t_k}{a_k}\right\}. \end{aligned} \tag{11}$$

It means that $\ln\circ f\circ\exp$ is the convex conjugate of the expression

$$\min_{(t_k)\in S_c,} \liminf_{N\to\infty} \sum_{k\in I,\ k\le N} t_k\ln\frac{t_k}{a_k}$$

as the function depending on c. So, immediate we obtain that

$$[\ln\circ f\circ\exp]^*(c)\le \min_{(t_k)\in S_c}\liminf_{N\to\infty}\sum_{k\in I,\,k\le N} t_k\ln\frac{t_k}{a_k}, \tag{12}$$

since $[\ln\circ f\circ\exp]^*$ is the largest convex and lower semicontinuous minorant of

$$\min_{(t_k)\in S_c}\liminf_{N\to\infty}\sum_{k\in I,\,k\le N} t_k\ln\frac{t_k}{a_k}.$$

Now we show that this expression at the sequence $\left(\frac{a_k e^{k\lambda}}{f(e^\lambda)}\right)_{k\in I}$ is equal to $[\ln\circ f\circ\exp]^*(c)$. We have seen that the condition $\sum_{k\in I} kt_k = c$, for this sequence, is equivalent to

$$[\ln\circ f\circ\exp]'(\lambda)=c. \tag{13}$$

Notice that

$$\lim_{\lambda\to-\infty}[\ln\circ f\circ\exp]'(\lambda)=\lim_{\lambda\to-\infty}\frac{\sum_{k\in I} ka_k e^{k\lambda}}{\sum_{k\in I} a_k e^{k\lambda}}=l,$$

where $l=\min I$. For $R<+\infty$ and $\lambda_0<\lambda<\ln R$, by Mean Value Theorem, we have

$$[\ln\circ f\circ\exp](\lambda)-[\ln\circ f\circ\exp](\lambda_0)=[\ln\circ f\circ\exp]'(\xi_\lambda)(\lambda-\lambda_0),$$

for some $\xi_\lambda\in(\lambda_0,\lambda)$. Since $f(R)=+\infty$ the above left-hand side tends to $+\infty$ by λ tending to $\ln R$. It follows that also $[\ln\circ f\circ\exp]'(\xi_\lambda)\to+\infty$ by $\lambda\to\ln R$ and, in consequence,

$$\lim_{\lambda\to(\ln R)^-}[\ln\circ f\circ\exp]'(\lambda)=+\infty.$$

Consider now the case $R=+\infty$. For any $N\in I$ let

$$f_N(e^\lambda)=\sum_{k\in I,k\le N} a_k e^{k\lambda}=\sum_{n=0}^{N} a_n e^{n\lambda},$$

where $a_n=a_k$ if $n\in I$ and $a_n=0$ otherwise. We show that

$$[\ln\circ f_N\circ\exp]'(\lambda)<[\ln\circ f\circ\exp]'(\lambda), \tag{14}$$

for any $\lambda\in\mathbb{R}$; that is we show that

$$f_N'(e^\lambda)f(e^\lambda)<f'(e^\lambda)f_N(e^\lambda).$$

Using Cauchy product of series we get that for $n\ge N$ the n-th coefficient of the left-hand side equals $\sum_{i=1}^{N} ia_ia_{n-i+1}$ and corresponding one on the right-hand side is greater and is equal $\sum_{i=0}^{N}(n-i+1)a_ia_{n-i+1}$. For this reason inequality (14) is satisfied. Because the limit of the left-hand side of (14), by $\lambda\to+\infty$, is equal to N, then, if I is infinite subset of $\mathbb{N}$, the right-hand side must tends to $+\infty$ by $\lambda\to+\infty$. Thus we obtained that the equation (13) possesses the unique solution for any $c>l$.

For the sequence $\left(\frac{a_k e^{k\lambda}}{f(e^\lambda)}\right)_{k\in I}$, $c > l$ and λ satisfying (13), the following expression

$$\liminf_{N\to\infty} \sum_{k\in I,\, k\le N} t_k \ln \frac{t_k}{a_k}$$

is equal to

$$\begin{aligned}\sum_{k\in I} \frac{a_k e^{k\lambda}}{f(e^\lambda)} \ln \frac{e^{k\lambda}}{f(e^\lambda)} &= \lambda[\ln\circ f\circ \exp]'(\lambda) - [\ln\circ f\circ\exp](\lambda)\\ &= c\lambda - [\ln\circ f\circ\exp](\lambda)\end{aligned}$$

and it gives the classical Legendre transform of the function $\ln\circ f\circ\exp$ at the point c. It means that the minimum in (12) is attained and is equal to $[\ln\circ f\circ\exp]^*(c)$. For $c = l$, by the definition of the Legendre-Fenchel transform, we get

$$\begin{aligned}[\ln\circ f\circ\exp]^*(l) &= \sup_{\lambda\in(-\infty,\ln R)} \left\{ l\lambda - \ln\sum_{k\in I} a_k e^{k\lambda}\right\}\\ &= -\inf_{\lambda\in(-\infty,\ln R)} \left\{\ln\sum_{k\in I} a_k e^{(k-l)\lambda}\right\} = -\ln a_l.\end{aligned}$$

□

If I is a finite subset of $\mathbb{N}$ with $l = \min I$ and $N = \max I$ then f is a polynomial of degree N and the Legendre-Fenchel transform of $\ln\circ f\circ\exp$ takes the form

$$[\ln\circ f\circ\exp]^*(c) = \min_{(t_k)\in S_c} \sum_{k\in I} t_k \ln\frac{t_k}{a_k},$$

for $c \in [l, N]$ and $+\infty$ otherwise.

The convex conjugate of $\widetilde{\lambda}$ is presented below.

Theorem 2.6. *For the functional $\widetilde{\lambda}$, defined by (5), the following variational principle holds*

$$\widetilde{\lambda}(\varphi) = \sup_{m\in M_\alpha^l} \{< m, \varphi > - \widetilde{\lambda}^*(m)\},$$

where $M_\alpha^l = \{m \in C(X)^* : m \in M_\alpha$ and $m(X) \in [l, +\infty)\}$,

$$\widetilde{\lambda}^*(m) = \frac{1}{p} m(X)\tau_\alpha\left(\frac{m}{m(X)}\right) + \min_{(t_k)\in S_{m(X)}} \liminf_{N\to\infty} \sum_{k\in I,\, k\le N} t_k \ln\frac{t_k}{a_k} \quad \text{for } m(X)\neq 0$$

and $S_{m(X)} = \{(t_k)_{k\in I} \in S : \sum_{k\in I} kt_k < +\infty$ and $m(X) = \sum_{k\in I} kt_k\}$; if $m(X) = 0$ then $\widetilde{\lambda}^*(0) = -\ln a_0$.

Proof. Because λ is convex and continuous on $C(X)$ then by Lemma 2.2 the functional $\widetilde{\lambda}$, as the composition $\ln \circ f \circ \exp \circ \lambda$, is also convex and continuous on $C(X)$ in the extended system of real numbers. By virtue of Theorem 2.5 we have

$$\widetilde{\lambda}(\varphi) = \max_{c\in[l,+\infty]} \left\{ c\lambda(\varphi) - \min_{(t_k)\in S_c,} \liminf_{N\to\infty} \sum_{k\in I,\, k\le N} t_k \ln \frac{t_k}{a_k} \right\}. \tag{15}$$

Substituting (2) into (15) we obtain that

$$\begin{aligned} \widetilde{\lambda}(\varphi) &= \max_{c\in[l,+\infty]} \left\{ c \max_{\nu\in M_\alpha^1} \left\{ \int_X \varphi d\nu - \frac{1}{p}\tau(\nu) \right\} - \min_{(t_k)\in S_c,} \liminf_{N\to\infty} \sum_{k\in I,\, k\le N} t_k \ln \frac{t_k}{a_k} \right\} \\ &= \max_{c\in[l,+\infty]} \max_{\nu\in M_\alpha^1} \left\{ \int_X \varphi d[c\nu] - \frac{c}{p}\tau(\nu) - \min_{(t_k)\in S_c,} \liminf_{N\to\infty} \sum_{k\in I,\, k\le N} t_k \ln \frac{t_k}{a_k} \right\}. \end{aligned} \tag{16}$$

If m denote the measure $c\nu$ then $c = m(X)$ and $\nu = \frac{m}{m(X)}$. Let M_α^l denote the set of α-invariant measures m such that $m(X) \ge l$. Now we can change the above two maximums on one over the set M_α^l

$$\widetilde{\lambda}(\varphi) = \max_{m\in M_\alpha^l} \left\{ \int_X \varphi dm - \frac{m(X)}{p}\tau_\alpha\left(\frac{m}{m(X)}\right) - \min_{(t_k)\in S_{m(X)}} \liminf_{N\to\infty} \sum_{k\in I,\, k\le N} t_k \ln \frac{t_k}{a_k} \right\}. \tag{17}$$

Notice now that

$$\min_{(t_k)\in S_{m(X)}} \liminf_{N\to\infty} \sum_{k\in I,\, k\le N} t_k \ln \frac{t_k}{a_k} = [\ln \circ f \circ \exp]^*(m(X))$$

is lower semicontinuous and convex on M_α^l. Moreover by convexity of τ_α, for $s \in [0,1]$, we get

$$\begin{aligned} &[s\bar{\nu}_1(X) + (1-s)\bar{\nu}_2(X)]\tau_\alpha\left(\frac{s\bar{\nu}_1 + (1-s)\bar{\nu}_2}{s\bar{\nu}_1(X) + (1-s)\bar{\nu}_2(X)}\right) \\ &= [s\bar{\nu}_1(X) + (1-s)\bar{\nu}_2(X)]\tau_\alpha\left(\frac{s\bar{\nu}_1(X)}{s\bar{\nu}_1(X) + (1-s)\bar{\nu}_2(X)} \cdot \frac{\bar{\nu}_1}{\bar{\nu}_1(X)} \right. \\ &\qquad \left. + \frac{(1-s)\bar{\nu}_2(X)}{s\bar{\nu}_1(X) + (1-s)\bar{\nu}_2(X)} \cdot \frac{\bar{\nu}_2}{\bar{\nu}_2(X)}\right) \\ &\le s\bar{\nu}_1(X)\tau_\alpha\left(\frac{\bar{\nu}_1}{\bar{\nu}_1(X)}\right) + (1-s)\bar{\nu}_2(X)\tau_\alpha\left(\frac{\bar{\nu}_2}{\bar{\nu}_2(X)}\right). \end{aligned}$$

Because τ_α is lower semicontinuous on M_α^1 then the functional $\frac{m(X)}{p}\tau_\alpha\left(\frac{m}{m(X)}\right)$ is lower semicontinuous on M_α^l. Thus for $m \in M_\alpha^l$

$$\widetilde{\lambda}^*(m) = \frac{1}{p} m(X)\tau_\alpha\left(\frac{m}{m(X)}\right) + \min_{(t_k)\in S_{m(X)}} \liminf_{N\to\infty} \sum_{k\in I,\, k\le N} t_k \ln \frac{t_k}{a_k}.$$

In this case the effective domain $D(\widetilde{\lambda}^*)$ is contained in the set M_α^l of all nonnegative, α-invariant measures m such that $m(X) \geq l$.

To calculate the value of $\widetilde{\lambda}^*$ at $m = 0$ we use the Legendre-Fenchel transform, i.e.

$$\widetilde{\lambda}^*(\mathbf{0}) = \sup_{\varphi \in C(X)} \{0 - \ln r(f(e^\varphi T_\alpha))\}.$$

Notice that $m(X)$ may be 0 when $l = 0$ that is when $0 \in I$ $(a_0 > 0)$. By virtue of Theorem 2.1 we obtain

$$\widetilde{\lambda}^*(\mathbf{0}) = -\inf_{\varphi \in C(X)} \ln \left(a_0 + \sum_{k \in I - \{0\}} a_k (r(e^\varphi T_\alpha))^k \right).$$

Because the spectral radius $r(e^\varphi T_\alpha))$ can be arbitrary small positive number then the expression $\sum_{k \in I - \{0\}} a_k (r(e^\varphi T_\alpha))^k$ may take also arbitrary small value. Thus we obtain that

$$\widetilde{\lambda}^*(\mathbf{0}) = -\ln a_0.$$

□

If I is a finite subset of $\mathbb{N}$ and $N = \max I$, that is f is a polynomial of degree N, then the Legendre-Fenchel transform of $\widetilde{\lambda}$ takes the form

$$\widetilde{\lambda}^*(m) = \frac{1}{p} m(X) \tau_\alpha \left(\frac{m}{m(X)} \right) + \min_{(t_k) \in S_{m(X)}} \sum_{k \in I} t_k \ln \frac{t_k}{a_k},$$

for α-invariant measure m such that $m(X) \in [l, N]$ (see [7, Th. 3.3]).

When the evident form of analytic function f is known then we can sometimes present the convex conjugate of the spectral exponent of $f(e^\varphi T_\alpha)$ as a function depending on τ_α and the form of $[\ln \circ f \circ exp]^*$.

Example 2.7. Take the function $f(z) = e^z$. Notice that $\ln f(e^\lambda) = e^\lambda$ and $e^{\lambda(\varphi)} = r(e^\varphi T_\alpha)$. In this case we obtain the variational principle for the spectral radius of $e^\varphi T_\alpha$ as it is. Recall that

$$\exp^*(c) = \begin{cases} c \ln c - c, & c > 0 \\ 0, & c = 0 \\ +\infty, & c < 0. \end{cases}$$

By the above

$$r(e^\varphi T_\alpha) = \max_{m \in M_\alpha} \left\{ \int_X \varphi dm - \frac{m(X)}{p} \tau_\alpha \left(\frac{m}{m(X)} \right) - m(X) \ln m(X) - m(X) \right\};$$

in other words

$$[\exp \circ \lambda]^*(m) = \frac{m(X)}{p} \tau_\alpha \left(\frac{m}{m(X)} \right) + m(X) \ln m(X) + m(X),$$

for $m \in M_\alpha$; $[\exp \circ \lambda]^*(\mathbf{0}) = -\ln 1 = 0$.

Remark 2.8. For finite set I, for instance $I = \{1, 2, \ldots, n\}$, the functional $\widetilde{\lambda}$ is the composition of the log-exponential function $g(\mathbf{x}) = \ln \sum_{k=1}^{n} e^{x_k}$ with functionals $x_k(\lambda) = c_k + k\lambda$, where $c_k = \ln a_k$. It is possible to derive $\widetilde{\lambda}^*$ applying the general rules of convex conjugate calculus obtained in [4, 5] to the composition with the function g (see [6, Cor. 4 of Th. 2]). Let us stress that these rules do not range considered by us the case of infinite number of variables x_k.

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